MATH 4321 – Game Theory

Final Examination, 2018

[**points**]

1. You and your $n-1$ roommates each have five hours of free time you could spend cleaning your apartment. You all dislike cleaning, but you all like having a clean apartment: each person's payoff is the total hours spent (by everyone) cleaning, minus a number c times the hours spent (individually) cleaning. The payoff of player i is given by

$$u_i(s_1, s_2, \ldots, s_n) = -cs_i + \sum_{j=1}^{n} s_j.$$

   Assume everyone chooses simultaneously how much time to spend cleaning.

   (a) Find the Nash equilibrium when (i) $c < 1$, (ii) $c > 1$.   [5]

   (b) Set $n = 5$ and $c = 2$. Is the Nash equilibrium Pareto efficient? If not, can you find an outcome in which everyone is better off than the Nash equilibrium outcome?   [5]

2. Suppose there are n firms in the Cournot oligopoly model. Let q_i denote the quantity produced by firm i, $i = 1, 2, \ldots, n$, and let $Q = q_1 + \cdots + q_n$ denote the aggregate production. Let $P(Q)$ denote the price, where $P(Q) = a - Q$, $Q \leq a$. Assume that firms have no fixed cost and that the cost of producing quantity q_i is cq_i, where $a > c$.

   (a) Find the Nash equilibrium of the game in which firms choose their quantities simultaneously.   [6]

   (b) What happens to the equilibrium price as n approaches infinity? Give your interpretation of the answer.   [2]

3. Consider a third-price sealed-bid auction, the winner is the highest bidder who pays the third highest price (assuming that there are more than three bidders). Recall that truthful bid means the bid price is the same as the personal valuation. Show that the truthful bid weakly dominates any lower bid, but does not weakly dominate any higher bid.   [7]

   Hint: Consider the three possible cases (i) $b_i < v_i$, (ii) $b_i = v_i$, and (iii) $b_i > v_i$, where b_i is the bid price and v_i is the personal valuation of bidder i.

4. Consider the continuous time noisy duel, where one bullet is allocated to each duelist. Let the strategy of the Row player is to fire the pistol when the duelist are x units apart, where $0 \leq x \leq D$, D is the original distance apart. We use similar definition for y as the strategy of the Column player, $0 \leq y \leq D$. Let $P_1(x)$ and $P_2(y)$ be the accuracy functions of the Row player and Column player, respectively.

   (a) Explain why the payoff to the Column player is given by
$$N(x, y) = \begin{cases} 1 - 2P_1(x) & \text{if } x > y \\ P_2(y) - P_1(y) & \text{if } x = y \\ 2P_2(y) - 1 & \text{if } x < y \end{cases}.$$
   [2]

   Hint: $x < y$ means the Column player shoots first.

(b) Suppose the continuous duel game becomes silent, where the players do not hear the firing of the pistol by the opponent. Find the corresponding payoff to the Column player in terms of $P_1(x)$ and $P_2(y)$. [3]

5. (a) Consider the voting game vector $[18; 6, 6, 6, 6, 2, 2]$.

    (i) Explain why the two voters with 2 votes are dummy. [2]

    (ii) Find the Shapley-Shubik indices and Banzhaf indexes for the voters. [2]

  (b) Show that a collection of dummies remains to be a dummy coalition. [3]

  (c) Consider the majority-minority voters system with 5 voters, 3 of them are in the majority group and the remaining 2 voters are in the minority group. The passage of a bill requires at least 3 votes and at least 1 vote from the minority group.

    (i) Express the above voting game using a voting vector of the form: $[q; w_1, w_2, \ldots, w_5]$. Find the possible values for the quota q and w_i for the votes held by player i, $i = 1, 2, \ldots, 5$. [4]

    (ii) Suppose the 3 voters in the majority group is homogeneous with homogeneous voting probability p, the 2 voters in the minority group are independent with voting probabilities, q_1 and q_2. Find the probability that a majority player's vote decides the passage of a bill. [6]

    Hint: $\displaystyle \int_0^1 x^m (1-x)^n \, dx = \frac{m! n!}{(m+n+1)!}$, m and n are non-negative integers.

6. (a) Consider the Nash model of bargaining with security point (u^*, v^*). The non-linear programming formulation is given by

$$\text{maximize } g(u, v) = (u - u^*)(v - v^*)$$
$$\text{subject to } (u, v) \in S, \ u \geq u^*, \ v \geq v^*,$$

where S is the feasible set. Let $(\bar{u}, \bar{v})$ maximizes $g(u, v)$, show that $(\bar{u}, \bar{v})$ is Pareto-optimal. [4]

  (b) Consider the following bimatrix game

$$\begin{pmatrix} (4, 2) & (2, -1) \\ (-1, 2) & (2, 4) \end{pmatrix}.$$

Let $A = \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$.

    (i) Show that the safety values are given by

$$\text{value}(A) = 2 \quad \text{and} \quad \text{value}(B^T) = 2.$$ [2]

    Hint: Use the row min column max property to identify the saddlepoint.

    (ii) Consider the formulation of the Nash model with the security point: $(u^*, v^*) = (2, 2)$. Show that the Pareto-optimal boundary is given by

$$u + v = 6, \quad 2 \leq u \leq 4 \quad \text{and} \quad 2 \leq v \leq 4.$$

    Solve for the bargaining solution based on the Nash bargaining model. [4]

    Hint: Consider the maximum of $g(u, v) = (u - 2)(v - 2)$ along the Pareto optimal boundary.

(iii) Suppose Player 1 threatens to play row 1 and Player 2 threatens to play column 1, reformulate the Nash bargaining model and find the new bargaining solution. **[3]**

— *End* —