MATH 4321 – Game Theory

Final Examination, 2019

Time allowed: 120 minutes *Course instructor: Prof. Y. K. Kwok*

[points]

1. Two candidates are competing in a political race. Each candidate i, $i = 1, 2$, can spend $s_i \geq 0$ on ads that reach out to voters, which in turn increases the probability that candidate i wins the race. Given a pair of spending choices (s_1, s_2), the probability that candidate i wins is given by $\frac{s_i}{s_1+s_2}$. If neither spends any resources then each wins with probability $\frac{1}{2}$. Each candidate values winning at a payoff of $v > 0$, and the cost of spending s_i is just s_i.

   (a) Given two spending levels (s_1, s_2), find the expected payoff of a candidate i. Find candidate i's best-response function. Be careful to consider $s_j = 0$ and $s_j > 0$ separately, $j \neq i$. **[3]**

   (b) Explain why we cannot find any Nash equilibrium that corresponds to either $s_1 = 0$ or $s_2 = 0$. Find the Nash equilibrium spending levels. **[5]**

   (c) What happens to the Nash equilibrium levels if player 1 still values winning at v but player 2 values winning at kv, where $k > 1$? **[3]**

2. We generalize the Cournot model to N firms, $N \geq 2$. The profit function for firm i is given by

$$u_i(q_1, \cdots, q_i, \cdots, q_N) = q_i \left[\max\left(\Gamma - \sum_{j=1}^{N} q_j, 0 \right) - c_i \right], \quad i = 1, 2, \cdots, N,$$

   where q_i is the quantity of product produced and c_i is the cost of producing one unit. Also, Γ is a sufficiently large constant that is larger than the sum of all feasible production quantities; that is $\Gamma > \sum_{j=1}^{N} q_j$.

   (a) Find the optimal quantity produced by each firm. Find and discuss the nature of Nash equilibrium. **[6]**

   Hint: Given the $N \times N$ matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix},$$

   its inverse is given by

$$A^{-1} = \frac{1}{N+1} \begin{pmatrix} N & -1 & -1 & \cdots & -1 \\ -1 & N & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & N \end{pmatrix}.$$

(b) Assuming the same cost c for all firms, find the optimal quantity produced by each firm when $N \to \infty$. **[2]**

3. Let $D(v)$ denote the expected payment made by one of the bidders (say bidder 1) and $F(v)$ denote the cummulative distribution function of the random valuation of the item on sale (common to all bidders). The lecture note derives the following relation:

$$D(v) = vF^{N-1}(v) - \int_{v_{min}}^{v} F^{N-1}(u)\,\mathrm{d}u,$$

where N is the total number of bidders and $F(v)$ assumes value over the interval $[v_{min}, v_{max}]$. We consider the charity (all-pay) auction. Let $\beta(v)$ denote the optimal bidding rule of bidder 1. Suppose $F(v)$ is uniform over $[v_{min}, v_{max}]$.

(a) Find the optimal bidding rule under the charity auction rule. **[3]**

Hint:

$$F(v) = \begin{cases} 0, & v \le v_{min} \\ \dfrac{v - v_{min}}{v_{max} - v_{min}}, & v_{min} < v < v_{max} \\ 1, & v \ge v_{max} \end{cases}.$$

(b) Verify that the choice of this optimal bidding rule for all bidders [under symmetric common value assumption of $F(v)$] is a Nash equilibrium. **[6]**

Hint: Bidder 1's expected payoff is

$$\Pi(x; v) = vF^{N-1}(x) - D(x), \quad \text{where } x = \beta^{-1}(b).$$

Recall that $x = v$ at $b = b^*$. Check that

$$\left.\frac{\mathrm{d}\Pi}{\mathrm{d}b}\right|_{b^* = \beta(v)} = 0.$$

4. Consider the noisy duel between two duelists with accuracy functions $P_1(x)$ and $P_2(y)$ over the interval $[0, D]$. A strategy for player 1 is to fire his bullet when the two duelists are x units apart, $0 \le x \le D$; and similarly player 2 when they are y units apart, $0 \le y \le D$. Let the payoff be 1 to the surviving duelist and -1 to the non-surviving duelist.

(a) Find the expected payoff $M(x, y)$ to player 1 under (*i*) $x > y$, (*ii*) $x = y$ and (*iii*) $x < y$. **[3]**

(b) Let x^* be the distance at which

$$P_1(x^*) + P_2(x^*) = 1,$$

and similarly,

$$P_1(y^*) + P_2(y^*) = 1.$$

Show that (x^*, y^*) is a Nash equilibrium. **[5]**

Hint: Verify that

$$M(x, y^*) \leq M(x^*, y^*) \leq M(x^*, y).$$

5. Consider the United Nations Security Council with 5 permanent members and 10 non-permanent members. It takes 9 votes, the "big five" plus at least 4 others to pass a bill. First, we assume homogenity in the voting probabilities p for all 15 countries.

   (a) Find the conditional probability that a non-permanent member can make a difference (pivotal) in terms of the homogeneous voting probability p. **[1]**

   (b) Compute the Shapley-Shubik index for a non-permanent country. **[3]**
   Express your answers in terms of C_k^n and factorials.

   (c) Suppose one permanent member votes independently from all other 14 members. Under this new assumption, find the conditional probability that a non-permanent member can be pivotal. Compute the absolute Banzhaf index for a non-permanent country. **[6]**

6. In the two-person Nash bargaining model, the objective function to be maximized is given by

$$f(S, u^*, v^*) = (u - u^*)(v - v^*),$$

   where (u^*, v^*) is the security point.

   (a) Consider the threat strategies $(u_0, v_0) = (X_t A Y_t^T, X_t B Y_t^T)$, where A and B are the payoff matrices of player 1 and 2, respectively, and (X_t, Y_t) is the pair of threat strategies. Let m_p denote the slope of the Pareto-optimal boundary line. Show that X_t and Y_t are the optimal strategies of the two players of the zero-sum game with matrix $-m_p A - B$. **[6]**

   (b) Find the Nash bargaining solution and the threat solution to the battle of sexes game with matrix

$$\begin{pmatrix} (4,2) & (2,-1) \\ (-1,2) & (2,4) \end{pmatrix}.$$

   [8]

   Hint: Here, the payoff matrices are

$$A = \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}.$$

   The security point can be easily identified by finding the saddlepoints of the game matrices A and B. It is easy to identify the threat strategy of the row player to be row 1.

— *End* —