# MATH 4321 – Game Theory

## Mid-term Test, 2018

*Time allowed: 80 minutes* *Instructor: Prof. Y. K. Kwok*

**[points]**

1. We consider saddle point and value of a zero sum game under pure strategies. Given the game matrix $A = (a_{ij})$, we define

$$\begin{aligned} v^- &= \max_{i=1,\ldots,n} \min_{j=1,\ldots,m} a_{ij} \\ v^+ &= \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} a_{ij}. \end{aligned}$$

   (a) Explain why
$$v^- \leq v^+.$$ **[2]**

   (b) We call a particular row $i^*$ and column $j^*$ a saddle point in pure strategies of the zero sum game if
$$a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j}$$
   for all rows and columns. Show that if a pure strategy saddle point exists, then $v^+ = v^-$. **[3]**

2. Curly has two safes, one at home and one at the office. The safe at home can be broken by any thief. The safe at the office is hard to crack and a thief has only 15% chance of breaking it. Curly (Row player) has to decide where to place his gold bar (worth 1). On the other hand, if the thief (Column player) hits the wrong place, he gets caught (worth −1 to the thief and +1 to Curly). The two strategies of both players are "Home" and "Office".

   (a) We formulate this game as a two-person zero sum matrix game. Explain why the game matrix is given as follows: **[1]**

| | Home | Office |
|---|---|---|
| Home | $-1$ | 1 |
| Office | 1 | 0.7 |

   (b) Solve the game by finding the mixed strategies of the two players and value of the game by graphical method. **[4]**

3. A mixed strategy $X^*$ for player I in a two-player zero sum game is a best response strategy to the fixed strategy $Y^0$ for player II if it satisfies
$$E(X^*, Y^0) = \max_{X \in S_n} E(X, Y^0),$$
   where $S_n$ is the set of $n$-component probability vectors.

   (a) Show that
$$\max_{X \in S_n} E(X, Y^0) = \max_{1 \leq i \leq n} E(i, Y^0).$$ **[4]**

   *Hint* Argue why the left side is always greater than or equal to the right side, then show that equality can be established for some row $i$.

(b) Suppose there exist $i_1^*$ and $i_2^*$ such that

$$E(i_1^*, Y^0) = E(i_2^*, Y^0) > E(i, Y^0), \ i \neq i_1^*, i_2^*.$$

Find the corresponding best response mixed strategy. **[1]**

(c) In the investment strategies game with investments on stock, bond and deposit, suppose the payoff to the investor is listed as follows:

| | state of economy | | |
|---|---|---|---|
| | good | neutral | bad |
| stock | 15 | 10 | 4 |
| bond | 9 | 9 | 12 |
| deposit | 10 | 10 | 10 |

Suppose the investor's belief on the state of economy is $Y^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, that is, equal chance of good, neutral or bad. Suppose the investor has \$1,000 in his investment budget, how to allocate the budget into these three investment instruments? **[3]**

4. We model the duel between Hamilton (US Secretary of Treasury) and Burr (US Vice President) more than 2 centuries ago as a symmetric two-person zero sum game. Each player holds a pistol that has exactly one bullet. They face each other starting at 10 paces apart and walk toward each other, each deciding when to shoot. We assume that each player does not know whether the opponent has taken the shot. Each player's chance of hitting the opponent increases when they are closer. An opponent who is hit would be killed. The payoff to the winner is 1 and the loser is $-1$. The payoff is zero if both hit together or miss together.

In this problem, we consider a simplified version of this symmetric zero sum game, where the duelists can choose to fire at 10 paces, 6 paces, or 2 paces; namely, there are 3 strategies. We assume that both are equal at shooting skill. Suppose also that the probability of a shot killing the opponent is 0.2 at 10 paces, 0.4 at 6 paces and 1 at 2 paces.

(a) Let $A$ be the game matrix of this duel game. Explain why $A = -A^T$ and show that the diagonal entries in $A$ are zero. **[2]**

(b) The entries of the game matrix are shown below:

| | | Hamilton | | |
|---|---|---|---|---|
| | | 10 | 6 | 2 |
| | 10 | 0 | $-0.12$ | $-x$ |
| Burr | 6 | 0.12 | 0 | $-y$ |
| | 2 | $x$ | $y$ | 0 |

The entry in $(6, 10)$ is found to be 0.12, which is the expected payoff to Burr. This is computed by

$$\begin{aligned} & (1)P[\text{Hamilton misses at } 10]P[\text{Burr kills Hamilton at } 6] \\ + \ & (-1)P[\text{killed by Hamilton at } 10] \\ = \ & 0.8 \times 0.4 - 0.2 = 0.12 \end{aligned}$$

Use similar methods to find $x$ and $y$, which are the entries in $(2, 10)$ and $(2, 6)$. **[2]**

(c) Once the game matrix $A$ is found, check whether a saddle point in pure strategies exists. If yes, find the value of the game. **[3]**

5. Two candidates, A and B, compete in an election. Of the $n$ citizens, $k$ support candidate A and $m$ $(= n - k)$ support candidate B. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain. A citizen who abstains receives the payoff of 2 if the candidate she supports wins, 1 if this candidate ties for first place, and 0 if this candidate loses. A citizen who votes receives the payoffs $2 - c$, $1 - c$ and $-c$ in these three cases, where the cost $c$ satisfies $0 < c < 1$. The payoff to each player is summarized as

| | win | tie | lose |
|---|---|---|---|
| vote | $2-c$ | $1-c$ | $-c$ |
| abstain | 2 | 1 | 0 |

(a) When $k = m = 1$, show that "vote" is a dominant strategy of each player. Find all pure Nash equilibriums. **[2]**

(b) When $k = m > 1$, explain why "everyone votes" is a pure Nash equilibrium. **[3]**

(c) For $k \geq 1$, does "everyone votes" remain to be a pure Nash equilibrium when $k < m$? Give your explanation. **[3]**

6. (a) Prove or disprove. If yes, give the proof; if not, provide a counter example.
A strategy profile is a dominant-strategy equilibrium if and only if it is a Nash equilibrium. **[3]**

*Hint* You have to consider both "if" part and "only if" part. It can be "yes" on one part and "no" on the other part. Also, it can be both "yes" or both "no".

(b) Quote an example for each of the following cases:

(i) A strategy profile is a dominant-strategy equilibrium but it does not weakly Pareto-dominate all other strategy profiles. **[2]**

(ii) A strategy profile Pareto-dominates all other strategy profiles but it is not a dominant-strategy equilibrium. **[2]**

— *End* —