MATH 4321 - Game Theory

Final Exam Solution, 2018

1. (a) We calculate the first-order partial derivatives of the payoff function for player i:

$$u_i(s_i, s_{-i}) = (1-c)s_i + s_1 + \cdots + s_{i-1} + s_{i+1} + \cdots + s_n$$
$$\Longrightarrow \frac{\partial u_i}{\partial s_i}(s_i, s_{-i}) = 1 - c, \quad s_i \in [0,5].$$

   (i) When $c < 1$, $\frac{\partial u_i}{\partial s_i}(s_i, s_{-i}) > 0$. Then in order to maximize the payoff function $u_i(s_i, s_{-i})$, player i should choose $s_i^* = 5$. Since the players are symmetric, we induce that all the players will choose the strategy $s_i^* = 5$ (for all i). Then the Nash equilibrium is all the players spending 5 hours on cleaning the apartment.

   (ii) When $c > 1$, $\frac{\partial u_i}{\partial s_i}(s_i, s_{-i}) < 0$. Then in order to maximize the payoff function $u_i(s_i, s_{-i})$, player i should choose $s_i^* = 0$. Since the players are symmetric, we induce that all the players will choose the strategy $s_i^* = 0$ (for all i). Then the Nash equilibrium is all the players not spending any time on cleaning the apartment.

   (b) When $n = 5$ and $c = 2$, we have the payoff function for player i is:

$$u_i(s_i, s_{-i}) = -2s_i + \sum_{j=1}^{5} s_j, \quad s_i \in [0,5].$$

   According to the discussion in (a), all the players will not spend any time on cleaning the apartment under Nash equilibrium, corresponding to payoff 0 to each player. It is obviously not a Pareto efficient outcome. For example, we can have $s_i = 1$ for all i and the payoff to each player i is $u_i(1,1,1,1,1) = 3 > 0$. Then the outcome is better off for all the players in this case.

2. (a) The payoff for firm i is given by

$$u_i(q_i, q_{-i}) = (P(Q) - c)q_i = (a - \sum_{j=1}^{n} q_i - c)q_i.$$

   We calculate the first-order partial derivatives of this payoff function and get

$$\frac{\partial u_i}{\partial q_i}(q_i, q_{-i}) = -q_i + a - \sum_{j=1}^{n} q_i - c.$$

   By setting $\frac{\partial u_i}{\partial q_i}(q_i, q_{-i}) = 0$, we get

$$q_i = \frac{a - c - q_1 - \cdots - q_{i-1} - q_{i+1} - \cdots - q_n}{2}$$

   for each firm i. Observing that all the n firms are symmetric, we can see that the players will adopt the same strategy and the final solution for the equations will be q^* satisfying

$$q^* = \frac{a - c - (n-1)q^*}{2},$$

   which yields $q^* = \frac{a-c}{1+n}$. Then each firm will choose to produce quantity $q^* = \frac{a-c}{1+n}$ under Nash equilibrium.

(b) First consider the total quantity in the Nash equilibrium as a function of n:

$$Q^* = nq^* = \frac{n(a-c)}{n+1}$$

and the resulting limit price is

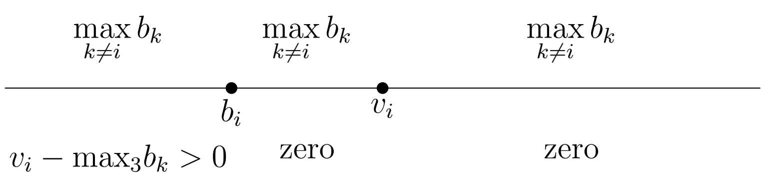
$$\lim_{n\to\infty} P(Q^*) = \lim_{n\to\infty}\left(a - \frac{n(a-c)}{n+1}\right) = c.$$

This means that as the number of firms grow, the Nash equilibrium price will also fall and will approach the marginal costs of the firms as the number of firms grows to infinity. Those familiar with a standard economics class know that in perfect competition price will equal marginal costs, which is what happens here when n approaches infinity.

3. Let $\max_3 b_k$ denote the third largest bid among all the bidders. We discuss different cases for the bid b_i of bidder i.

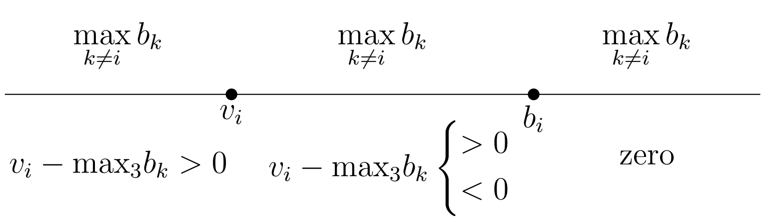
- When $b_i < v_i$:
  The payoff to player i should be

$$u_i(b_i, b_{-i}) = \begin{cases} v_i - \max_3 b_k > 0 & \text{if } \max_{k\neq i} b_k < b_i \\ 0 & \text{if } \max_{k\neq i} b_k > b_i \end{cases}$$



- When $b_i > v_i$:
  The payoff to player i should be

$$u_i(b_i, b_{-i}) = \begin{cases} v_i - \max_3 b_k > 0 & \text{if } \max_{k\neq i} b_k < v_i \\ v_i - \max_3 b_k > 0 & \text{if } v_i < \max_{k\neq i} b_k < b_i \ \&\ \max_3 b_k < v_i \\ v_i - \max_3 b_k < 0 & \text{if } v_i < \max_{k\neq i} b_k < b_i \ \&\ v_i < \max_3 b_k < \max_{k\neq i} b_k \\ 0 & \text{if } \max_{k\neq i} b_k > b_i \end{cases}$$



- When $b_i = v_i$:
  The payoff to player i should be

$$u_i(b_i, b_{-i}) = \begin{cases} v_i - \max_3 b_k > 0 & \text{if } \max_{k\neq i} b_k < b_i = v_i \\ 0 & \text{if } \max_{k\neq i} b_k > b_i = v_i \end{cases}$$

$\max_{k \neq i} b_k$ $\qquad\qquad$ $\max_{k \neq i} b_k$

$b_i = v_i$

$v_i - \max_3 b_k > 0$ $\qquad\qquad$ zero

Under this circumstance, we can see that strategy $b_i = v_i$ still weakly dominates $b_i < v_i$ but does not dominate $b_i > v_i$ since player i will get higher payoff under $b_i > v_i$ when $v_i < \max_{k \neq i} b_k < b_i$ and $\max_3 b_k < v_i$ ($v_i - \max_3 b_k > 0$ under $b_i > v_i$ and 0 under $b_i = v_i$).

4. (a)
   - When $x > y$, the Row player shoots first. If the Row player kills the Column player with probability $P_1(x)$ in the first place, the payoff to the Column player is -1. On the other hand, if the Row player misses the target with probability $1 - P_1(x)$, the Column player will certainly win with payoff 1 since the he can wait until distance 0 to shoot. Then the expected payoff to the Column player under this case is given by $P_1(x)(-1) + (1 - P_1(x))(1) = 1 - 2P_1(x)$.
   - When $x < y$, the Column player shoots first. It is symmetric to the first case and the expected payoff to the Column player will depend on $P_2(y)$ and is represented by $P_2(y)(1) + (1 - P_2(y))(-1) = 2P_2(y) - 1$.
   - When $x = y$, the two players shoot simultaneously. If the Row player hits the target but the Column player does not, the payoff to the Column player should be -1 and wise versa. Then the expected payoff payoff to the Column player is represented by $P_1(x)(1 - P_2(x))(-1) + (1 - P_1(x))P_2(x)(1) = P_2(x) - P_1(x)$.

   (b) For silent duel game, even if the player who shoots first misses the target, the other player will also stick to his own strategy since he will not know if the opponent has shot or not. Then the player will not be guaranteed to lose given that he misses the target in the first place. Based on this fact, the payoff function for the Column player can be modified to (Suppose the Row player chooses to shoot at distance x and Column player chooses to shoot at distance y)

$$S(x,y) = \begin{cases} P_1(x)(-1) + (1 - P_1(x))P_2(y)(1) = P_2(y) - P_1(x) - P_1(x)P_2(y) & \text{if } x > y \\ P_1(x)(1 - P_2(x))(-1) + (1 - P_1(x))P_2(x)(1) = P_2(x) - P_1(x) & \text{if } x = y \\ P_2(y)(1) + (1 - P_2(y))P_1(x)(-1) = P_2(y) - P_1(x) + P_1(x)P_2(y) & \text{if } x < y \end{cases}.$$

5. (a) (i) For the game $[18; 6, 6, 6, 6, 2, 2]$, a winning coalition must contain at least 3 voters with 6 votes. There are already 18 votes from the three voters with 6 votes, so a winning coalition will always win the game with and without voters with 2 votes. Therefore, the two voters with 2 votes are dummy in this game.

   (ii) For the voters with 2 votes, he cannot be pivot or marginal in any coalition since he is the dummy player. Then the Shapley-Shubik indices for them should be $\phi_5 = \phi_6 = 0$ and the Banzhaf indices for them should also be $\beta_5 = \beta_6 = 0$.
   Voters with 6 voters then have the same Shapley-Shubik and Banzhaf indices calculated by

$$\phi_1 = \phi_2 = \phi_3 = \phi_4 = \frac{1-0}{4} = \frac{1}{4},$$
$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{1}{4}.$$

   (b) Proof by contradiction. Suppose there are n dummy players (represented by player 1 to player n, respectively) and the collection of dummies $D = \cup_{i=1}^{n}\{i\}$ can turn a losing coalition S in to a winning coalition $S \cup D$. Then given that $S \cup D$ is a winning coalition and player 1 is dummy, we can deduce by the definition of dummy that coalition $S \cup D \setminus \{1\}$ also wins the game. Furthermore, player 2 is also a dummy, so coalition $S \cup D \setminus \{1, 2\}$ also wins the game. So on and so forth, we have coalition $S \cup D \setminus \{1, \cdots, n\} = S$ also wins the game since all players from player 1 to player n are dummies. Then we get a contradiction since the initial coalition S loses the game. In conclusion, a collection of dummies can never turn a losing coalition into a winning coalition.

(c) (i) We fist set the weight of voters in majority group as 2 and then let the weight of the voters in minority group be x. According to the rule, we have

$$6 < q, \quad 4 + x = q, \quad \text{and} \quad 2x < q,$$

so $4 + x > 6$ and $2x < 4 + x$ giving $2 < x < 4$. We take $x = 3$, then $q = 7$. The game can be represented by

$$[7; 3, 3, 2, 2, 2].$$

(ii) For Voter 3 in the majority group, she is marginal in 5 coalitions:

$$\{1, 2, 3\}, \ \{1, 3, 4\}, \ \{1, 3, 5\}, \ \{2, 3, 4\}, \ \{2, 3, 5\}.$$

The conditional probability that player 3 makes a difference should be

$$\begin{aligned}\pi_3(p_1, p_2, p_4, p_5) = & p_1 p_2 (1 - p_4)(1 - p_5) + p_1(1 - p_2) p_4 (1 - p_5) \\ & + p_1 (1 - p_2)(1 - p_4) p_5 + (1 - p_1) p_2 p_4 (1 - p_5) \\ & + (1 - p_1) p_2 (1 - p_4) p_5.\end{aligned}$$

Given that the 3 voters in the majority group is with homogeneous voting probability p and the 2 voters in the minority group are independent with voting probabilities, q_1 and q_2, we can modify the above equation to

$$\pi_3(q_1, q_2, p) = q_1 q_2 (1 - p)^2 + 2 q_1 (1 - q_2) p (1 - p) + 2(1 - q_1) q_2 p (1 - p).$$

Then the probability that Voter 3 will make a difference is calculated by

$$\begin{aligned} & \int_0^1 \int_0^1 \int_0^1 \pi_3(q_1, q_2, p) f_1(q_1) f_2(q_2) f(p) \, \mathrm{d}q_1 \, \mathrm{d}q_2 \, \mathrm{d}p \\ = & \int_0^1 \int_0^1 \int_0^1 [q_1 q_2 (1 - p)^2 + 2 q_1 (1 - q_2) p (1 - p) + \\ & \quad 2(1 - q_1) q_2 p (1 - p)] f_1(q_1) f_2(q_2) f(p) \, \mathrm{d}q_1 \, \mathrm{d}q_2 \, \mathrm{d}p \\ = & \int_0^1 q_1 f_1(q_1) \, \mathrm{d}q_1 \int_0^1 q_2 f_2(q_2) \, \mathrm{d}q_2 \int_0^1 (1 - p)^2 f(p) \, \mathrm{d}p + \\ & 2 \int_0^1 q_1 f_1(q_1) \, \mathrm{d}q_1 \int_0^1 (1 - q_2) f_2(q_2) \, \mathrm{d}q_2 \int_0^1 p(1 - p) f(p) \, \mathrm{d}p + \\ & 2 \int_0^1 (1 - q_1) f_1(q_1) \, \mathrm{d}q_1 \int_0^1 q_2 f_2(q_2) \, \mathrm{d}q_2 \int_0^1 p(1 - p) f(p) \, \mathrm{d}p. \end{aligned}$$

6. (a) Proof by contradiction. Without loss of generality, we only consider the case where we suppose there is another feasible point $(u', v') \in S$ such that $u' > \overline{u}$ and $v' \geq \overline{v}$. Since $g(u, v) = (u - u^*)(v - v^*)$, we must have

$$g(u', v') = (u' - u^*)(v' - v^*) > (\overline{u} - u^*)(\overline{v} - v^*) = g(\overline{u}, \overline{v}).$$

It is a contradiction to the fact that $(\overline{u}, \overline{v})$ maximizes the function $g(u, v)$. Therefore, $(\overline{u}, \overline{v})$ is Pareto-optimal.

(b) (i) We find v^+ and v^- for game matrices A and B^T separately:

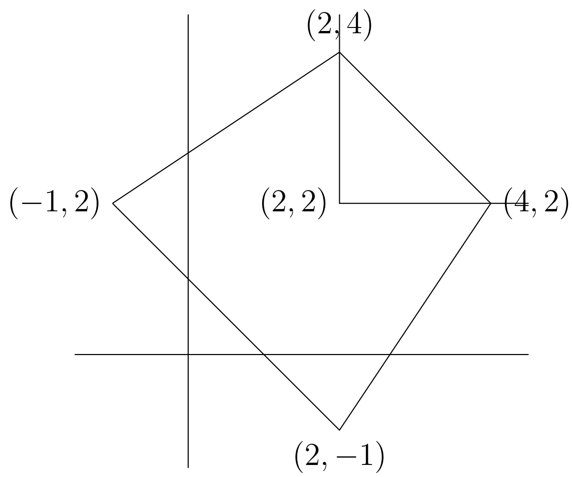
$$A: \begin{array}{ccccl} 4 & 2 & 2 & & \\ -1 & 2 & -1 & & \\ 4 & 2 & v^- & = \max\min = 2 \\ & & v^+ & = \min\max = 2 \end{array} \qquad B^T: \begin{array}{ccccl} 2 & 2 & 2 & & \\ -1 & 4 & -1 & & \\ 2 & 4 & v^- & = \max\min = 2 \\ & & v^+ & = \min\max = 2 \end{array}$$

We find that the upper value and lower value for A and B^T are the same. Therefore, $value(A) = value(B^T) = 2$.

(ii) The feasible set, taking into account the security point, is

$$S^* = \{(u, v) | v \le -u + 6, u \ge 2, v \ge 2\}.$$

We plot the feasible set as following:



The two players only negotiate in the top right region of the security point $(2, 2)$. From the figure, we can find that the Pareto-optimal boundary is given by the line segment on the top right:

$$u + v = 6, \quad 2 \le u \le 4, \quad 2 \le v \le 4.$$

We then set ip the nonlinear programming problem:

$$\begin{array}{ll} \text{Maximize} & g(u, v) = (u - 2)(v - 2) \\ \text{subject to} & (u, v) \in S^*. \end{array}$$

Given that the optimal point $(\overline{u}, \overline{v})$ occurs on the Pareto-optimal boundary $v = -u + 6$, $2 \le u \le v$, we then maximize the function

$$g(u, v) = f(u) = (u - 2)(-u + 6 - 2) = -u^2 + 6u - 8.$$

The first-order derivatives is given by

$$f'(u) = -2u + 6 = 0 \implies u = 3 \implies v = 3$$

which yields $g(3, 3) = 1$. We further check the second-order derivatives

$$f''(u)|_{u=3} = -2 < 0.$$

Then we guarantee that function $g(u, v)$ is maximized at point $(\overline{u}, \overline{v}) = (3, 3)$. The cooperation must be a linear combination of the strategies yielding the payoffs under pure strategies, so we solve

$$(3, 3) = \lambda(2, 4) + \lambda(4, 2)$$

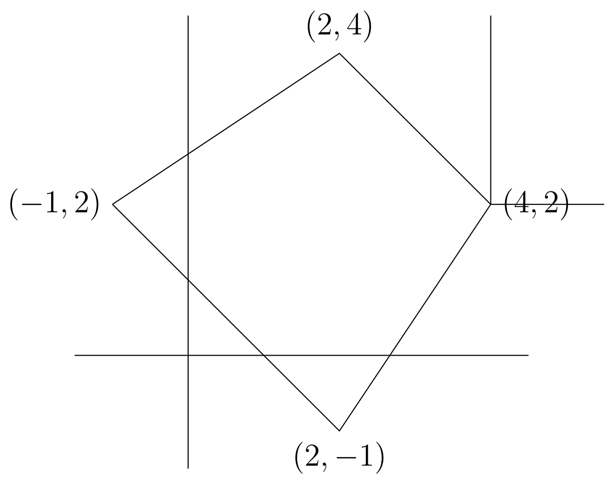
to get $\lambda = \frac{1}{2}$. This says that the two players must agree to play the pure strategy $(Row2, Col2)$ and $(Row1, Col1)$ with equal probability $\frac{1}{2}$.

(iii) We have threat strategies for the two players as $X_t = (1, 0)$ and $Y_t = (1, 0)$. Then the new security point of the game changes to $(u^t, v^t) = (4, 2)$. We solve the new nonlinear programming problem:

$$\begin{array}{ll} \text{Maximize} & g(u, v) = (u - 4)(v - 2) \\ \text{subject to} & (u, v) \in S', \end{array}$$

where

$$S' = \{(u, v) | v \leq -u + 6, u \geq 4, v \geq 2\}.$$



We can see the new feasible set in this case is one single point $(4, 2)$. We then get the bargaining outcome $(\overline{u}, \overline{v}) = (4, 2)$. For further check, we may use the formula:

$$\overline{u} = \frac{m_p u^t + v^t - b}{2m_p} = \frac{-1 \times 4 + 2 - 6}{2 \times (-1)} = 4,$$
$$\overline{v} = \frac{1}{2}(m_p u^t + v^t + b) = \frac{1}{2}(-1 \times 4 + 2 + 6) = 2.$$

We can also get the final outcome $(\overline{u}, \overline{v}) = (4, 2)$. Under the given threat strategies, the two players will finally play $(Row1, Col1)$ as a bargaining solution.