MATH 4321 - Game Theory

Final Exam Solution, 2019

1. (a) Candidate i's expected payoff is given by

$$f_i(s_i, s_{-i}) = \begin{cases} \frac{1}{2}v, & s_1 = s_2 = 0 \\ \frac{s_i}{s_1+s_2}v - s_i = \left(\frac{v}{s_1+s_2} - 1\right)s_i, & \text{otherwise} \end{cases},$$

specifically,

$$f_1(s_1, s_2) = \begin{cases} \frac{1}{2}v, & s_1 = s_2 = 0 \\ \left(\frac{v}{s_1+s_2} - 1\right)s_1, & \text{otherwise} \end{cases}$$

and

$$f_2(s_1, s_2) = \begin{cases} \frac{1}{2}v, & s_1 = s_2 = 0 \\ \left(\frac{v}{s_1+s_2} - 1\right)s_2, & \text{otherwise} \end{cases}.$$

- For a given $s_2 = 0$,

$$f_1(s_1) = \begin{cases} \frac{1}{2}v, & s_1 = 0 \\ v - s_1, & s_1 > 0 \end{cases},$$

which does not have maximum value. Therefore, the best-response function for player 1 given $s_2 = 0$ is $s_1(0) = 0^+$. Intuitively, player 1 spends slightly larger than zero and still wins. The similar fact also holds for player 2.
Alternatively: Or you can say the best-response functions for both players do not exist when the other player's spending level is zero.

- For a given $s_2 > 0$, taking the first-order derivative and setting it to zero, we get the best-response function of player 1:

$$\frac{\partial f_1}{\partial s_1} = \frac{vs_2}{(s_1+s_2)^2} - 1 = 0 \implies s_1(s_2) = \sqrt{vs_2} - s_2.$$

To verify, we calculate the second-order derivative:

$$\frac{\partial^2 f_1}{\partial s_1^2} = -\frac{2vs_2}{(s_1+s_2)^3} < 0.$$

Similarly,

$$s_2(s_1) = \sqrt{vs_1} - s_1, \quad s_1 > 0.$$

(b)
- When either $s_1 = 0$ or $s_2 = 0$, according to (a), the best-response function for at least one of the players does not exist. We thus cannot find any Nash equilibrium in this case.
- When $s_1 s_2 > 0$, the unique Nash equilibrium is given by the intersection of the two best-response functions:

$$\begin{cases} s_1 = \sqrt{vs_2} - s_2 \\ s_2 = \sqrt{vs_1} - s_1 \end{cases} \implies s_1^* = s_2^* = \frac{v}{4}.$$

(c) In this case, the expected payoff of player 2 changes to

$$f_2(s_2) = \begin{cases} \frac{1}{2}kv, & s_1 = s_2 = 0 \\ \left(\frac{kv}{s_1+s_2} - 1\right) s_2, & \text{otherwise} \end{cases}.$$

Her best-response function changes to

$$s_2(s_1) = \sqrt{kvs_1} - s_1, \quad s_1 > 0.$$

Together with $s_1(s_2) = \sqrt{vs_2} - s_2$, $s_2 > 0$, we find $s_1^{*\prime} = \frac{k}{(k+1)^2}v < \frac{v}{4}$ and $s_2^{*\prime} = \left(\frac{k}{k+1}\right)^2 v > \frac{v}{4}$. Therefore, the Nash equilibrium spending level decreases for player 1 but increases for player 2.

2. (a) Suppose $\Gamma > \sum_{j=1}^N q_j$. Take the first-order derivative and set it to zero:

$$\frac{\partial u_i}{\partial q_i} = \Gamma - 2q_i - \sum_{j\neq i}^N q_j - c_i = 0 \implies q_i(q_{-i}) = \frac{\Gamma - \sum_{j\neq i}^N q_j - c_i}{2}, \quad i = 1, 2, \cdots, N.$$

To verify the critical point is a maximum, we check the second-order derivative is negative:

$$\frac{\partial^2 u_i}{\partial q_i^2} = -2 < 0.$$

Using the relation

$$2q_i + \sum_{j\neq i}^N q_j = \Gamma - c_i, \quad i = 1, 2, \cdots, N,$$

we get the following equation

$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} \Gamma - c_1 \\ \Gamma - c_2 \\ \vdots \\ \Gamma - c_N \end{pmatrix},$$

which yields

$$\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} = \frac{1}{N+1} \begin{pmatrix} N & -1 & -1 & \cdots & -1 \\ -1 & N & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & N \end{pmatrix} \begin{pmatrix} \Gamma - c_1 \\ \Gamma - c_2 \\ \vdots \\ \Gamma - c_N \end{pmatrix} = \begin{pmatrix} \frac{\Gamma - Nc_1 + \sum_{j\neq 1}^N c_j}{N+1} \\ \frac{\Gamma - Nc_2 + \sum_{j\neq 2}^N c_j}{N+1} \\ \vdots \\ \frac{\Gamma - Nc_N + \sum_{j\neq N}^N c_j}{N+1} \end{pmatrix}.$$

Therefore, the optimal quantity produced by firm i is given by

$$q_i^* = \frac{\Gamma - Nc_i + \sum_{j\neq i}^N c_j}{N+1}, \quad i = 1, 2, \cdots, N.$$

We argue that the optimal strategy profile (q_i^*, q_{-i}^*) we get forms a Nash equilibrium. For any firm i, the profit it gets under the strategy profile (q_i^*, q_{-i}^*) is given by

$$u_i(q_i^*, q_{-i}^*) = \left(\frac{\Gamma - Nc_i + \sum_{j\neq i}^N c_j}{N+1}\right)^2.$$

Suppose player i deviates from the profile to a quantity $q_i' \neq q_i^*$ and others do not change, her profit function changes to

$$\begin{aligned}
u_i(q_i', q_{-i}^*) &= q_i' \left(\frac{2\Gamma - 2Nc_i + 2\sum_{j\neq i}^{N} c_j}{N+1} - q_i' \right) \\
&= -\left(q_i' - \frac{\Gamma - Nc_i + \sum_{j\neq i}^{N} c_j}{N+1} \right)^2 + \left(\frac{\Gamma - Nc_i + \sum_{j\neq i}^{N} c_j}{N+1} \right)^2 \\
&< \left(\frac{\Gamma - Nc_i + \sum_{j\neq i}^{N} c_j}{N+1} \right)^2 = u_i(q_i^*, q_{-i}^*).
\end{aligned}$$

Then player i will not deviate from the profile (q_i^*, q_{-i}^*), otherwise she will be worse off. Therefore, we conclude that (q_i^*, q_{-i}^*) is a Nash equilibrium.

(b) Assuming $c_1 = c_2 = \cdots = c_N = c$, we have

$$q^* = \frac{\Gamma - c}{N+1}.$$

When $N \to \infty$, the optimal quantity $q^* \to 0$ for each firm.

3. (a) According to the charity auction rule, the expected payment for each bidder is equal to her bidding amount, namely,

$$\beta(v) = D(v) = vF^{N-1}(v) - \int_{v_{min}}^{v} F^{N-1}(u)\,\mathrm{d}u.$$

Since $F(v)$ is uniform over $[v_{min}, v_{max}]$, we have

$$\begin{aligned}
\beta(v) &= v\left(\frac{v - v_{min}}{v_{max} - v_{min}}\right)^{N-1} - \int_{v_{min}}^{v} \left(\frac{u - v_{min}}{v_{max} - v_{min}}\right)^{N-1} \mathrm{d}u \\
&= v\left(\frac{v - v_{min}}{v_{max} - v_{min}}\right)^{N-1} - \frac{1}{N}\left(\frac{v - v_{min}}{v_{max} - v_{min}}\right)^{N} (v_{max} - v_{min}) \\
&= \left(\frac{v - v_{min}}{v_{max} - v_{min}}\right)^{N-1} \left(v - \frac{v - v_{min}}{N}\right), \quad v \in [v_{min}, v_{max}]
\end{aligned}$$

(b) Bidder 1's expected payoff is expressed by

$$\Pi(x; v) = vF^{N-1}(x) - D(x) = vF^{N-1}(x) - \beta(x), \quad x = \beta^{-1}(b)$$

for the charity auction. Since $b^* = \beta(v)$, we have $x = v$ at $b = b^*$.

To maximize $\Pi(v)$ with respect to the bidding amount b, we take the first-order derivative and set it to zero:

$$\begin{aligned}
&\left.\frac{\mathrm{d}\Pi}{\mathrm{d}b}\right|_{b^*=\beta(v)} = v \left.\frac{\mathrm{d}F^{N-1}}{\mathrm{d}x}\right|_{x=v} \left.\frac{\mathrm{d}x}{\mathrm{d}b}\right|_{b^*=\beta(v)} - \left.\frac{\mathrm{d}\beta}{\mathrm{d}x}\right|_{x=v} \left.\frac{\mathrm{d}x}{\mathrm{d}b}\right|_{b^*=\beta(v)} = 0 \\
\iff & v\frac{\mathrm{d}F^{N-1}(v)}{\mathrm{d}v} = \frac{\mathrm{d}\beta(v)}{\mathrm{d}v} \iff \int v\,\mathrm{d}F^{N-1}(v) = \int \mathrm{d}\beta(v) \\
\iff & vF^{N-1}(v) - v_{min}F^{N-1}(v_{min}) - \int F^{N-1}(u)\,\mathrm{d}u = \beta(v) - \beta(v_{min})
\end{aligned}$$

Since $\Pi(x; v_{min})|_{x=v_{min}} = v_{min}F^{N-1}(v_{min}) - \beta(v_{min}) = 0$, we have

$$b^* = \beta(v) = vF^{N-1}(v) - \int_{v_{min}}^{v} F^{N-1}(u)\,\mathrm{d}u,$$

which maximizes bidder 1's expected payoff. Then bidder 1 will always get a lower payoff if she deviates to any bidding rule other than $b^* = \beta(v)$. Therefore, the bidding rule we calculated above is exactly a Nash equilibrium.

4. (a) (i) When $x > y$, player 1 shoots earlier than player 2, her expected payoff is given by

$$M(x, y) = (1)P_1(x) + (-1)[1 - P_1(x)] = 2P_1(x) - 1.$$

(ii) When $x = y$, the two players shoot simultaneously, the expected payoff for player 1 is

$$M(x, y) = (1)P_1(x)[1 - P_2(x)] + (-1)[1 - P_1(x)]P_2(x) = P_1(x) - P_2(x).$$

(iii) When $x < y$, player 2 shoots first, the expected payoff for player 1 is given by

$$M(x, y) = (-1)P_2(y) + (1)[1 - P_2(y)] = 1 - 2P_2(y).$$

Therefore, we have

$$M(x, y) = \begin{cases} 2P_1(x) - 1, & x > y \\ P_1(x) - P_2(x), & x = y \\ 1 - 2P_2(y), & x < y \end{cases}.$$

(b) **<u>Method I:</u>** For player 1, she chooses x to maximize $M(x, y)$. Since it is a zero-sum game, she takes into account $\min\limits_{y} M(x, y)$. When $x < y$, player 2 minimizes $M(x, y)$ by choosing $y = x^+$. Then player 1 considers the following maximin problem:

$$\max_x \min_y M(x, y) = \max_x \min[2P_1(x) - 1, P_1(x) - P_2(x), 1 - 2P_2(x)].$$

- When $x \le x^*$, we have $P_1(x) + P_2(x) \ge 1$. Then

$$2P_1(x) - 1 \ge P_1(x) - P_2(x) \ge 1 - 2P_2(x)$$

and

$$\max_x \min_y M(x, y) = \max_{x \le x^*}[1 - 2P_2(x)],$$

which equals $P_1(x) - P_2(x)$ when $x = x^*$.

- When $x \ge x^*$, we have $P_1(x) + P_2(x) \le 1$. Then

$$2P_1(x) - 1 \le P_1(x) - P_2(x) \le 1 - 2P_2(x)$$

and

$$\max_x \min_y M(x, y) = \max_{x \ge x^*}[2P_1(x) - 1],$$

which equals $P_1(x) - P_2(x)$ when $x = x^*$.

- When $x = x^*$, we have $P_1(x) + P_2(x) = 1$. Then

$$2P_1(x) - 1 = P_1(x) - P_2(x) = 1 - 2P_2(x)$$

  and and

$$\max_x \min_y M(x, y) = P_1(x^*) - P_2(x^*).$$

In conclustion, player 1's expected payoff $M(x, y)$ is maximized at $x = x^*$ taking into account $\min_y M(x, y)$. Similarly, player 2 will also choose the distance y^* satisfying $P_1(y^*)+P_2(y^*) = 1$ to maximize her own payoff given that player 1 is trying to minimize it.

Method II: We argue that the strategy profile (x^*, y^*) forms a Nash equilibrium, under which player 1's payoff is given by

$$M(x^*, y^*) = P_1(x^*) - P_2(x^*) = P_1(y^*) - P_2(y^*)$$

since $x^* = y^*$.

- When $x \le x^* = y^*$, $M(x, y^*) = 1 - 2P_2(y^*) = P_1(y^*) - P_2(y^*) = M(x^*, y^*)$.
- When $x > x^* = y^*$, $M(x, y^*) = 2P_1(x) - 1 < P_1(x^*) - P_2(x^*) = M(x^*, y^*)$.
- When $y \le y^* = x^*$, $M(x^*, y) = 2P_1(x^*) - 1 = P_1(x^*) - P_2(x^*) = M(x^*, y^*)$.
- When $y > y^* = x^*$, $M(x^*, y) = 1 - 2P_2(y) > P_1(y^*) - P_2(y^*) = M(x^*, y^*)$.

In conclusion, $M(x, y^*) \le M(x^*, y^*) \le M(x^*, y)$ for all x and y, which implies that (x^*, y^*) is a saddle point for the zero-sum game and therefore a Nash equilibrium.

5. (a) A non-permanent member can be marginal in the following condition:

   - The "big five" approve and there are other 3 non-permanent countries approving.

   There are C_3^9 such coalitions. Therefore, the probability for a non-permanent member to make a difference is given by

$$\pi(p) = C_3^9 \cdot p^8(1 - p)^6.$$

   (b) Under the assumption of homogeneity and uniform distribution of the probability p, the power index is equal to the Shapley-Shubik index, so the Shapley-Shubik index for a non-permanent member is given by

$$\phi_s = \int_0^1 \pi(p)f(p)\,\mathrm{d}p = \int_0^1 C_3^9 \cdot p^8(1 - p)^6\,\mathrm{d}p = C_3^9 \cdot \frac{8! \cdot 6!}{15!}.$$

   (c) Let $p_1, \cdots, p_5$ be the voting probability for the "big five" and $q_6, \cdots, q_{15}$ be the voting probability for the non-permanent members. The probability that a non-permanent member (say player 6) can make a difference is given by

$$\begin{aligned}\pi(p_1, \cdots, p_5, q_7, \cdots, q_{15}) = &\prod_{j=1}^{5} p_j \cdot q_7 q_8 q_9 (1 - q_{10}) \cdots (1 - q_{15}) + \cdots \\ &+ \prod_{j=1}^{5} p_j \cdot (1 - q_7) \cdots (1 - q_{12}) q_{13} q_{14} q_{15},\end{aligned}$$

   where there are totally C_3^9 terms, choosing 3 non-permanent members from the remaining 9 members with probability q_j and the other 6 members with probability $1 - q_j$.

Under the assumption of independence together with mean value of voting probability equals $\frac{1}{2}$, the absolute Banzhaf index for a non-permanent member (say player 6) is given by

$$\begin{aligned}\beta_6' &= \int_0^1 \pi(p_1, \cdots, p_5, q_7, \cdots, q_{15}) f_1(p_1) \cdots f_{15}(q_{15})\, \mathrm{d}p_1 \cdots \mathrm{d}q_{15} \\ &= \prod_{j=1}^{5} \int_0^1 p_j f_j(p_j)\, \mathrm{d}p_j \cdot \int_0^1 q_7 f_7(q_7)\, \mathrm{d}q_7 \int_0^1 q_8 f_8(q_8)\, \mathrm{d}q_8 \cdots \\ &= \frac{C_3^9}{2^{14}}.\end{aligned}$$

6. (a) Under the threat strategy (X_t, Y_t), the security point changes to $(u_0, v_0) = (X_t A Y_t^T, X_t B Y_t^T)$. Assuming an interior solution, the bargaining solution $(\overline{u}, \overline{v})$ must be on the Pareto-optimal boundary $v = m_p u + b$. Therefore, we can transform the objective function into

$$f(u) = (u - X_t A Y_t^T)(m_p u + b - X_t B Y_t^T).$$

To maximize it, we take the first-order derivative and set it to zero:

$$f'(u) = 2m_p u + X_t(-m_p A - B)Y_t^T + b = 0,$$

which yields

$$\overline{u} = \frac{X_t(-m_p A - B)Y_t^T + b}{-2m_p}, \quad m_p < 0.$$

Correspondingly,

$$\overline{v} = m_p \overline{u} + b = \frac{1}{2}[b - X_t(-m_p A - B)Y_t^T].$$

Both players aim to choose their optimal threat strategies (X_t and Y_t, respectively) to maximize their own payoffs ($\overline{u}$ and $\overline{v}$, respectively). From the above equations, we observe that player 1 can maximize the term $X_t(-m_p A - B)Y_t^T$ to maximize $\overline{u}$ and player 2 can minimize the same term $X_t(-m_p A - B)Y_t^T$ to maximize $\overline{v}$. Therefore, it changes to a zero-sum game with matrix $-m_p A - B$ where the row player (player 1) chooses X_t to maximize the entries while the column player (player 2) chooses Y_t to minimize the entries.

(b) (i) Nash bargaining solution.

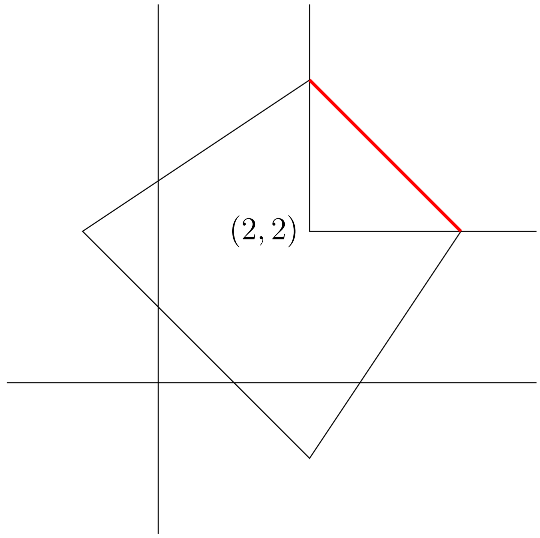
- *Find the security point.*
  The individual matrices are as follows:

$$A = \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}, \quad B^T = \begin{pmatrix} 2 & 2 \\ -1 & 4 \end{pmatrix}.$$

  It is easy to calculate that $value(A) = 2$, $value(B^T) = 2$, so the status quo security point for this game is at $(u^*, v^*) = (2, 2)$.
- *Find the feasible set and Pareto-optimal boundary.*
  The feasible set, taking into account the security point, is

$$S^* = \{(u, v) | v \leq -u + 6, 2 \leq u \leq 4, 2 \leq v \leq 4\}.$$

  The Pareto-optimal boundary is $v = -u + 6$, $2 \leq u \leq 4$.



- *Set up and solve the nonlinear programming problem.*
  The problem we then need to solve is

$$\begin{aligned}&\text{Maximize } g(u,v) = (u-2)(v-2)\\ &\text{subject to } (u,v) \in S^*.\end{aligned}$$

  If the optimal point $(\overline{u}, \overline{v})$ occurs on the Pareto-optimal boundary $v = -u+6$, $2 \leq u \leq 4$, then we maximize

$$g(u,v) = f(u) = (u-2)(-u+4).$$

  Take the first-order derivatives of function $f(u)$ and set it to zero:

$$f'(u) = -2u + 6 = 0 \implies \overline{u} = 3 \implies \overline{v} = 3,$$

  which yields $g(3,3) = 1$. Checking the second-order derivative is negative:

$$f''(u) = -2 < 0.$$

- *Find the strategies giving the negotiated solution.*
  The only points in the bimatrix that are of interest are the endpoints of the Pareto-optimal boundary, namely, $(2,4)$ and $(4,2)$. So the cooperation must be a linear combination of the strategies yielding these payoffs. Solve

$$(3,3) = \lambda(2,4) + (1-\lambda)(4,2)$$

  to get $\lambda = \frac{1}{2}$. This says that (I, II) must agree to play the pure strategies $(\mathrm{I}_1, \mathrm{II}_1)$ and $(\mathrm{I}_2, \mathrm{II}_2)$ half of the time, respectively.

(ii) Threat solution.

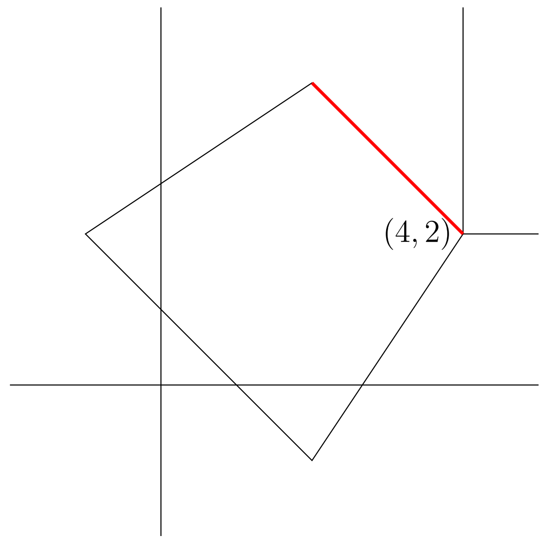
- *Identify the possible Pareto-optimal boundary.*

  The Pareto-optimal boundary is given by $v = -u + 6$ with $m_p = -1$ and $b = 6$, $2 \leq u \leq 4$.
- *Construct new matrix $-m_pA - B$ for a zero sum game.*
  We look for the value of the game with matrix $A - B$:

$$A - B = \begin{pmatrix} 2 & 3 \\ -3 & -2 \end{pmatrix}.$$

- *Find the optimal strategies X_t, Y_t for the zero sum game.*
  We find that $value(A-B) = 2$ and the optimal threat strategies are $X_t = Y_t = (1,0)$. Then we know that the security point is as follows:

$$u^t = 4 \quad \text{and} \quad v^t = 2.$$



- *Calculate solution $(\overline{u}, \overline{v})$ of the bargaining game.*

  This point is exactly the vertex of the feasible set. The two players have no choice but achieve the bargaining solution $(\overline{u}, \overline{v}) = (4, 2)$ with threat strategies $X_t = Y_t = (1,0)$.