MATH 4321 - Game Theory

Mid-term Test Solution, 2018

1. (a) Observe that

$$v^- = \max_i \min_j a_{ij} \le \max_i a_{ij}.$$

The above inequality is independent of j (holds for $\forall\, j$), so it remains to be valid when we take $\min_j(\max_i a_{ij})$, which is precisely v^+. Hence $v^- \le v^+$.

   (b) If a pure strategy saddle point exists, we have

$$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j},$$

for all rows and columns ($\forall\, i, j$). Then we have

$$v^- = \min_j \max_i a_{ij} \le \max_i a_{ij^*} \le a_{i^*j^*} \le \min_j a_{i^*j} \le \max_i \min_j a_{ij} = v^+,$$

which indicates that $v^- \ge v^+$. Together with $v^- \le v^+$, we get $v^- = v^+$.

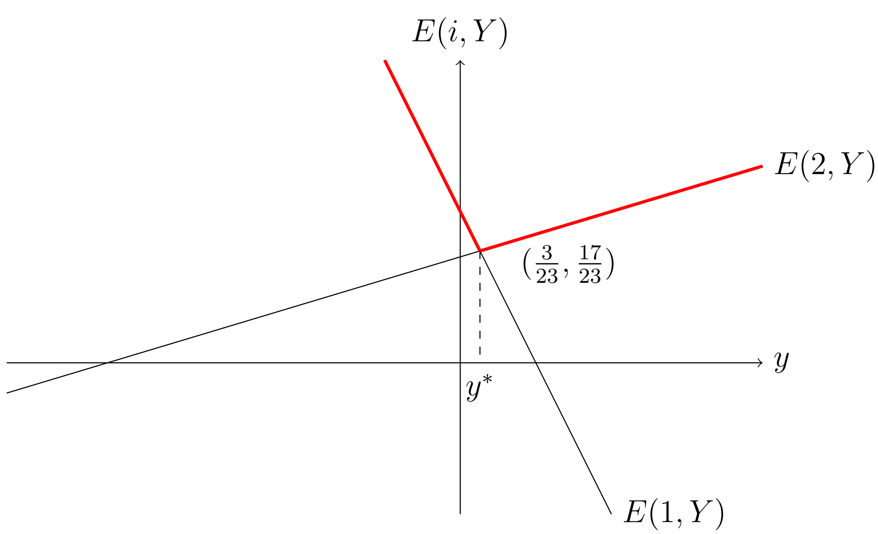
2. (a) Let Curly be the row player and the thief be the column player. We consider four strategy profiles:

   - **(Home, Home):** The thief will break into Curly's home for sure and Curly will lose his gold bar. -1 to Curly and $+1$ to the thief.
   - **(Home, Office) & (Office, Home):** The thief has chosen a wrong place without the gold bar and get caught. $+1$ to Curly and -1 to the thief.
   - **(Office, Office):** The thief has chosen the right place in office with the gold bar and has only 15% chance to break the safe and get the gold. The expected payoff to Curly should be $(+1) \times 85\% + (-1) \times 15\% = +0.7$ and that to the thief should be the inverse.

   (b) Suppose the strategy for Curly is $X = (x, 1-x)$ and the strategy for the thief is $Y = (y, 1-y)$. For the two pure strategies of the row player, we have two linear functions

$$E(1, Y) = -y + 1 - y = -2y + 1,$$
$$E(2, Y) = y + 0.7(1-y) = 0.3y + 0.7.$$

We plot the two lines:



The intersection point is $(\frac{3}{23}, \frac{17}{23})$, so $y^* = \frac{3}{23}$, $Y^* = (\frac{3}{23}, \frac{20}{23})$. The value of the game is found to be $v = \frac{17}{23}$.

Similarly, we get the two linear functions for the two pure strategies of the column player:

$$
\begin{aligned}
E(X,1) &= -x + 1 - x = -2x + 1, \\
E(X,2) &= x + 0.7(1-x) = 0.3x + 0.7.
\end{aligned}
$$

It is seen that the two functions $E(X,1)$ and $E(X,2)$ are actually the same as $E(1,Y)$ and $E(2,Y)$. Then we get the same result $x^* = \frac{3}{23}$ and $X^* = (\frac{3}{23}, \frac{20}{23})$.

3. (a) Let X^* be the probability vector X maximizing the expected payoff $E(X, Y^0)$. Then $\max_{X \in S_n} E(X, Y^0) = E(X^*, Y^0)$.

   - We first show that $E(X^*, Y^0) \geq E(i, Y^0)$, $\forall\, 1 \leq i \leq n$.
     Since X^* maximizes the value of $E(X, Y^0)$, then $E(X^*, Y^0) \geq E(X, Y^0)$ should hold for all mixed strategies X. Since pure strategy i for player I is also a special case of mixed strategies ($x_i = 1$ and $x_j = 0$ for all $j \neq i$), we also have $E(X^*, Y^0) \geq E(i, Y^0)$, $\forall\, 1 \leq i \leq n$.
   - We then should that the equality $E(X^*, Y^0) = E(i, Y^0)$ can be established for some row i.
     We prove by contradiction. Suppose the equality does not hold for all the row i. Then we have strict inequality $E(X^*, Y^0) > E(i, Y^0)$ for all i. On the other hand, the maximized expected payoff is calculated by

     $$E(X^*, Y^0) = \max_{X \in S_n} E(X, Y^0) = \max_{X \in S_n} \sum_{i=1}^{n} x_i E(i, Y^0) < E(X^*, Y^0),$$

     where $0 \leq x_i \leq 1$. Then we get a contradiction. Therefore, we conclude that equality $E(X^*, Y^0) = E(i, Y^0)$ must hold for some row i.

   Combined the above two results, we have $\max_{X \in S_n} E(X, Y^0) = \max_{1 \leq i \leq n} E(i, Y^0)$.

   (b) From (a) we have got $\max_{1 \leq i \leq n} E(i, Y^0) = E(X^*, Y^0)$ and

   $$E(X^*, Y^0) = \max_{X \in S_n} \sum_{i=1}^{n} x_i E(i, Y^0), \quad 0 \leq x_i \leq 1.$$

   Therefore, if we have $E(j, Y^0) < \max_{1 \leq i \leq n} E(i, Y^0)$ for some j, we must have $x_j = 0$. Otherwise, $\max_{X \in S_n} \sum_{i=1}^{n} x_i E(i, Y^0) < E(X^*, Y^0)$ and we have a contradiction.

   According to the above argument, from $E(i_1^*, Y^0) = E(i_2^*, Y^0) > E(i, Y^0)$, we guarantee that $x_i = 0$ ($i \neq i_1^*$, i_2^*) and $x_{i_1^*}$, $x_{i_2^*} \geq 0$. One of the corresponding best response mixed strategies can be

   $$X^* = \left\{(x_1, x_2, \cdots, x_n) \middle| x_{i_1^*} = \alpha,\, x_{i_2^*} = 1 - \alpha,\, x_i = 0, \forall\, i \neq i_1^*,\, i_2^* \right\}, \quad 0 \leq \alpha \leq 1.$$

   (c) Given that $Y^0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we want to find the best response strategy $X^* = (x_1, x_2, x_3)$ for the investor. The expected payoff of the investor for each of her pure strategy is calculated by

$$E(1,Y^0) = (15+10+4)\times\frac{1}{3} = \frac{29}{3},$$
$$E(2,Y^0) = (9+9+12)\times\frac{1}{3} = 10,$$
$$E(3,Y^0) = (10+10+10)\times\frac{1}{3} = 10.$$

It is seen that $E(1,Y^0) < E(2,Y^0) = E(3,Y^0) = 10$. According to the discussion in (b), we have $x_1 = 0$ and

$$X^* = (0,\alpha,1-\alpha), \quad 0\le\alpha\le 1.$$

In conclusion, the investor should not allocate any budget into stock and should allocate the $ 1000 into bond and deposit by any porportion.

4. (a) The two players are equal at shooting skills and have the same strategy sets $S_1 = S_2 = \{10, 6, 2\}$. Therefore, the two players only switch their roles under two strategy profiles (i,j) and (j,i) $(i,j = 10, 6, \text{ or } 2)$. The probability for one player to kill the other under strategy profile (i,j) is equal to the probability for him be killed under strategy profile (j,i). It further implies that the expected payoff for Burr at entry (i,j) is just the inverse of that at entry (j,i), so $A = -A^T$.

   For entry (i,i) (diagonal entries), the two players decide to shoot at the same position. Since they have equal skills for shooting, the probability to kill the other must be equal to the probability to be killed, which implies the expected payoff for both players must be zero.

   (b) (2, 10):

$$\begin{aligned} x =&(1)P[\text{Hamilton misses at 10}]P[\text{Burr kills Hamilton at 2}] \\ &+(-1)P[\text{Burr is killed by Hamilton at 10}] \\ =&0.8\times 1 - 0.2 = 0.6. \end{aligned}$$

   (2,6):

$$\begin{aligned} y =&(1)P[\text{Hamilton misses at 6}]P[\text{Burr kills Hamilton at 2}] \\ &+(-1)P[\text{Burr is killed by Hamilton at 6}] \\ =&0.6\times 1 - 0.4 = 0.2. \end{aligned}$$

   (c) The game matrix is shown as following:

		Hamilton			
		10	6	2	
	10	0	−0.12	−0.6	−0.6
Burr	6	0.12	0	−0.2	−0.2
	2	0.6	0.2	0	0
		0.6	0.2	0	$v^- = \max\min = 0$
					$v^+ = \min\max = 0$

   We then find the upper and lower values in the above matrix. It is seen that $v^- = v^+ = 0$. Then there exists a pure strategies saddle point at $(2,2)$ and the value of the game is $v(A) = 0$.

5. (a) Suppose player 1 supports A and player 2 supports B.
    - If both vote, there is a tie. The payoffs for both are $1 - c$.
    - If player 1 votes while player 2 abstains, player 1 has payoff $2 - c$ while player 2 has zero payoff. Similar results are obtained if they swap their role.
    - If both abstain, there is a tie and no cost incurred, so the payoffs for both are 1.

    The bi-matrix game is depicted as follows:

I \ II	Vote	Abstain
Vote	$(1-c, 1-c)$	$(2-c, 0)$
Abstain	$(0, 2-c)$	$(1, 1)$

    Suppose Player I chooses the first row, then Player II will choose to vote since $1-c > 0$. Suppose Player I chooses the second row, then Player II will also choose to vote since $2 - c > 1$.
    Suppose Player II chooses the first column, then Player I will choose to vote since $1 - c > 0$.
    Suppose Player II chooses the second column, then Player II will also choose to vote since $2 - c > 1$.
    Therefore, we find that "Vote" is a dominant strategy for both players and (Vote, Vote) is the unique pure strategy Nash equilibrium.

   (b) $k = m > 1$.
    Suppose everyone votes, then the candidates A and B tie. Each voter has a payoff of $1 - c$. Now, if one voter chooses not to vote while all the other remain "Vote", this voter has a payoff zero (since the candidate he supported earlier will lose). The payoff of this voter worsens under unilateral deviation. Hence, "everyone votes" is a Nash equilibrium.

   (c) $k < m$.
    Suppose everyone votes. Since $k < m$ and candidate A loses. A supporter of A will be better off from saving the cost of casting vote if he changes from "Vote" to "Abstain" since A remains losing. Hence, the action profile is not a Nash equilibrium.

6. (a)
    - "if" part (A strategy profile is a dominant-strategy equilibrium if it is a Nash equilibrium).
      False. consider the following game matrix as a counter-example:

	Col1	Col2
Row1	(5,5)	(1,1)
Row2	(3,4)	(4,3)

      Here, Strategy profile (Row1, Col1) is Pareto-dominant and is a Nash equilibrium. However, it is not a dominant-strategy equilibrium since the row player will choose Row2 if the column player chooses Col2. Then Row1 is not the dominant strategy for the row player.
    - "only if" part (If a strategy profile is a dominant-strategy equilibrium, then it is a Nash equilibrium).
      True. Proof by contradiction. Suppose a dominant-strategy equilibrium (s_i^*, s_{-i}^*) is not a Nash equilibrium, then $\exists\, s_i'$, such that

$$\pi_i(s_i^*, s_{-i}^*) < \pi_i(s_i', s_{-i}^*).$$

Then strategy profile (s_i^*, s_{-i}^*) is not a dominant-strategy. There is a contradiction. Therefore, a dominant-strategy equilibrium must be a Nash equilibrium.

(b) (i) Consider the following game matrix as an example:

	Col1	Col2
Row1	(5,4)	(5,3)
Row2	(3,5)	(1,1)

In this case, Row1 and Col1 are dominant strategies for the row player and column player, respectively. Then (Row1, Col1) is a dominant-strategy equilibrium. However, it does not Pareto-dominate all other strategy profiles because the payoff to column player at (Row1, Col1) is not the largest among all the possible payoffs of column player.

(ii) Consider the following game matrix as an example:

	Col1	Col2
Row1	(5,5)	(1,1)
Row2	(3,4)	(4,3)

As is illustrated in (a), the strategy profile (Row1, Col1) Pareto-dominates all other strategy profiles but is not a dominant-strategy equilibrium.