MATH 4512 — Fundamentals of Mathematical Finance

Solution to Homework One

Course instructor: *Prof. Y.K. Kwok*

---

1. Recall that

$$D = \frac{1}{B}\sum_{i=1}^{n}\frac{c_i}{(1+y)^i}\frac{i}{m}$$

(cash flow c_i occurs at time $\dfrac{i}{m}$ years), where

$$B = \sum_{i=1}^{n} c_i(1+y)^{-i}.$$

Taking the derivative of B with respect to y, we have

$$\frac{dB}{dy} = -\frac{1}{1+y}\sum_{i=1}^{n} ic_i(1+y)^{-i} = -\frac{mDB}{1+y}$$

so that

$$D = -\frac{1+y}{m}\frac{1}{B}\frac{dB}{dy}.$$

Comparing to a similar formula (see p.5 in Topic One), where

$$D = -\frac{1+\lambda}{B}\frac{dB}{d\lambda}, \qquad \lambda = my.$$

In this problem, the growth factor over an extra period is $1+y$ instead of $1+\lambda$. Also, we recall

$$B = B_T\frac{c}{y}[1-\frac{1}{(1+y)^n}] + \frac{B_T}{(1+y)^n},$$

where c is the coupon rate per period and y is the yield per period (see p.2 in Topic One), so

$$\frac{d}{dy}\ln\frac{B}{B_T} = \frac{1}{B}\frac{dB}{dy} = -\frac{1}{y} + \frac{cn(1+y)^{-1}+1+y(1+y)^{-1}(-n)}{c[(1+y)^n-1]+y}.$$

Combining these relations, we have

$$D = -\frac{1+y}{m}\frac{1}{B}\frac{dB}{dy} = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n-1]+my}.$$

Note that $n = mT$. For fixed value of m, we take $T\to\infty$, which is equivalent to take $n\to\infty$. We then have

$$\lim_{T\to\infty} D = \frac{1}{m}+\frac{1}{\lambda} - \lim_{n\to\infty}\frac{1+y+n(c-y)}{mc[(1+y)^n-1]+my}.$$

By virtue of L'Hospital's rule, we obtain

$$\lim_{T\to\infty} D = \frac{1}{m}+\frac{1}{\lambda} - \frac{c-y}{mc\lim\limits_{n\to\infty}\ln(1+y)(1+y)^n} = \frac{1}{m}+\frac{1}{\lambda}.$$

2. The term $\frac{1}{B(t,T)}$ represents the "deterministic" return received by an investor holding a zero-coupon bond to maturity. The right-hand side is the *expected* return from time t to T generated by rolling over a \$1 investment in one-period maturity bonds, each of which has a yield equal to the future spot rate r_t, assuming that the investor cannot quit the annual rolling over strategy in the period $[t, T]$. The relationship represents an equilibrium condition, in which the expected returns for equal holding periods are themselves equal. On one hand, one may argue that in an environment of economic equilibrium, the returns on zero-coupon bonds of similar maturity cannot be significantly different since investors would not hold the bonds with the lower return. On the other hand, the subjective expectation of an individual investor determines the expected return for the rolling over strategy. She would choose among the two strategies based on the one with higher expected return. For example, under the current low interest rate environment, suppose the investor expects future hikes in interest rates, she would prefer the rolling over strategy to the long-term bond investment strategy.

3. We write
$$B_t = \frac{c}{i} + \frac{1}{(1+i)^T}\left(B_T - \frac{c}{i}\right)$$
so that
$$B_{t+1} = \frac{c}{i} + \frac{1}{(1+i)^{T-1}}\left(B_T - \frac{c}{i}\right).$$
We then have
$$B_{t+1} - B_t = \left(B_T - \frac{c}{i}\right)\frac{i}{(1+i)^T} = iB_t - c.$$
Rearranging the terms, we obtain
$$\frac{\Delta B}{B} = \frac{B_{t+1} - B_t}{B_t} = i - \frac{c}{B_t}.$$
In the continuous time limit, we deduce that
$$\frac{1}{B(t)}\frac{dB}{dt} = i(t) - \frac{c(t)}{B(t)}.$$
Note that $i(t)$ and $c(t)$ in the differential equation are visualized as cash flow rates so that $i(t)dt$ and $c(t)dt$ are dollar amounts collected over $(t, t+dt)$. Given that the governing equation for $B(t)$ is
$$\frac{dB(t)}{dt} = i(t)B(t) - c(t), \quad t < T,$$
with $B(T) = B_T$, the closed form solution is seen to be
$$B(t) = e^{-\int_t^T i(s)\ ds}\left[B_T + \int_t^T c(u)e^{\int_u^T i(s)\ ds}du\right], \quad t < T.$$
For the coupon amount $c(u)du$ received within the differential time interval $(u, u+du)$, it grows by the growth factor $e^{\int_u^T i(s)\ ds}$ by time T. Together with the par payment B_T received at time T, we apply the discount factor $e^{-\int_t^T i(s)\ ds}$ to obtain the present bond value at time T.

4. Recall $r_H(i) = \left[\frac{B(i)}{B_0}\right]^{\frac{1}{H}}(1+i) - 1$. When $i = i_0$, $B(i)$ is simply the bond value at $t = 0$ since the initial interest rate is i_0, so
$$r_H(i_0) = \left[\frac{B_0}{B_0}\right]^{\frac{1}{H}}(1+i_0) - 1 = i_0.$$
Hence, $r_H(i_0) = i_0$ for any H; so all curves $r_H(i)$ pass through (i_0, i_0).

5. It is necessary to examine the Taylor expansion of $\frac{dB}{B}$ up to the third order, where

$$\frac{\Delta B}{B} \approx \frac{1}{B}\frac{dB}{di}di + \frac{1}{2}\frac{1}{B}\frac{d^2B}{di^2}(di)^2 + \frac{1}{6}\frac{1}{B}\frac{d^3B}{di^3}(di)^3.$$

Note that the third order derivative is always negative since

$$\frac{d^3B}{di^3} = -\sum_{t=1}^{T} t(t+1)(t+2)c_t(1+i)^{-t-3} < 0.$$

When $di > 0$, the value i is to the right of the tangency point, we have

$$\frac{1}{6}\frac{1}{B}\frac{d^3B}{di^3}(di)^3 < 0.$$

Therefore, the actual bond value is below the quadratic approximation to the right of the tangency point. On the other hand, since convexity of the bond value is greater than zero, so the actual bond value lies above the linear approximation.

Similarly, when $di < 0$, both $\frac{d^2B}{di^2}(di)^2$ and $\frac{d^3B}{di^3}(di)^3$ are positive. Therefore, the actual bond value lies above the quadratic approximation and linear approximation curves to the left of the tangency point.

6. (a) Bond A: maturity is 15 years and coupon rate is 10%

Time of payment t (year)	t(t+1)	Cash flows in nominal value	Discount rate	Cash flows in present value	Share of cash flows in present value in bond price	Weighted time of payment	t(t+1) times share of discounted cash flows
1	2	100	0.8929	89.2857	0.1034	0.1034	0.2067
2	6	100	0.7972	79.7194	0.0923	0.1846	0.5537
3	12	100	0.7118	71.1780	0.0824	0.2472	0.9888
4	20	100	0.6355	63.5518	0.0736	0.2943	1.4715
5	30	100	0.5674	56.7427	0.0657	0.3285	1.9707
6	42	100	0.5066	50.6631	0.0587	0.3519	2.4634
7	56	100	0.4523	45.2349	0.0524	0.3666	2.9326
8	72	100	0.4039	40.3883	0.0468	0.3741	3.3665
9	90	100	0.3606	36.0610	0.0417	0.3757	3.7573
10	110	100	0.3220	32.1973	0.0373	0.3727	4.1002
11	132	100	0.2875	28.7476	0.0333	0.3661	4.3931
12	156	100	0.2567	25.6675	0.0297	0.3566	4.6356
13	182	100	0.2292	22.9174	0.0265	0.3449	4.8287
14	210	100	0.2046	20.4620	0.0237	0.3316	4.9746
15	240	1100	0.1827	200.9659	0.2327	3.4899	55.8379
				863.7827		**7.8880**	**76.9145**

Calculation results for Bond A: duration is 7.8880 and convexity is 76.9145.

Bond B: maturity is 11 years and coupon rate is 5%

Time of payment t (year)	t(t+1)	Cash flows in nominal value	Discount rate	Cash flows in present value	Share of cash flows in present value in bond price	Weighted time of payment	t(t+1) times share of discounted cash flows
1	2	50	0.8929	44.6429	0.0764	0.0764	0.1528
2	6	50	0.7972	39.8597	0.0682	0.1364	0.4093
3	12	50	0.7118	35.5890	0.0609	0.1827	0.7308
4	20	50	0.6355	31.7759	0.0544	0.2175	1.0875
5	30	50	0.5674	28.3713	0.0486	0.2428	1.4565
6	42	50	0.5066	25.3316	0.0433	0.2601	1.8207
7	56	50	0.4523	22.6175	0.0387	0.2709	2.1675
8	72	50	0.4039	20.1942	0.0346	0.2765	2.4882
9	90	50	0.3606	18.0305	0.0309	0.2777	2.7770
10	110	50	0.3220	16.0987	0.0275	0.2755	3.0304
11	132	1050	0.2875	301.8499	0.5165	5.6820	68.1842
				584.3611		**7.8985**	**67.2073**

Calculation results for Bond B: duration is 7.8985 and convexity is 67.2073.

(b) Bond A has rate of return of 12.06% at horizon $H = D = 7.8880$ if interest rate jumps to 10% or 14%. Bond B has rate of return of 12.03% at horizon $H = D = 7.8985$ if interest rate jumps to 10% or 14%. These sample calculations show that the rates of return almost stay at the same level of 12% at horizon that equals duration.

Rates of return of Bond A when YTM changes

Horizon	**Scenario (YTM)**		
	Current	10%	14%
1	12.00%	27.35%	-0.45%
2	12.00%	18.36%	6.53%
3	12.00%	15.50%	8.97%
4	12.00%	14.10%	10.20%
5	12.00%	13.27%	10.95%
6	12.00%	12.72%	11.45%
7	12.00%	12.33%	11.81%
7.888	12.00%	12.06%	12.06%
8	12.00%	12.03%	12.09%
9	12.00%	11.80%	12.30%
10	12.00%	11.62%	12.47%

Rates of return of Bond B when YTM changes

Horizon	**Scenario (YTM)**		
	Current	10%	14%
1	12.00%	27.11%	-0.65%
2	12.00%	18.25%	6.42%
3	12.00%	15.43%	8.89%
4	12.00%	14.05%	10.15%
5	12.00%	13.23%	10.91%
6	12.00%	12.68%	11.42%
7	12.00%	12.30%	11.78%
7.8985	12.00%	12.03%	12.03%
8	12.00%	12.01%	12.06%
9	12.00%	11.78%	12.27%
10	12.00%	11.60%	12.44%

(c) I would choose bond A. These two bonds have pretty much the same duration, but bond A has a higher value of convexity. As a result, bond A has a higher rate of return compared to that of bond B, no matter interest rates increase or decrease, and for any choice of horizon.

7. Let t be the tax rate, x_i be the number of units of bond i bought, c_i be the coupon of bond i, p_i be the price of bond i, $i = 1, 2$. The par value of each bond is 100.

   To create a zero coupon bond, we require that the after-tax coupons match. This gives

$$100[x_1(1-t)c_1 + x_2(1-t)c_2] = 0,$$

   which reduces to an equation independent of t:

$$x_1c_1 + x_2c_2 = 0.$$

   Next, we require that the after-tax final cash flows match (mutually consistent). This gives another equation that relates the bond prices p_1, p_2 and p_0.

$$x_1[100 - (100 - p_1)t] + x_2[100 - (100 - p_2)t] = [100 - (100 - p_0)t].$$

   The price of the zero-coupon bond will be

$$p_0 = x_1p_1 + x_2p_2.$$

   The last relation for matching the par payments at maturity is given by

$$x_1 + x_2 = 1.$$

   Combining $x_1 + x_2 = 1$, $c_1x_1 + c_2x_2 = 0$, and $p_0 = x_1p_1 + x_2p_2$, we obtain

$$p_0 = \frac{c_2p_1 - c_1p_2}{c_2 - c_1} = \frac{0.07 \times 92.21 - 0.1 \times 75.84}{0.07 - 0.1} = 37.64.$$

8. Let P denote the principal left in the pool and r denote the annualized rate of return.

$$\text{Year 1} \quad P = (20)(1.085) + \frac{-10(20)(-0.005)}{1.085} - 12.5(0.07) = 21.75$$
$$r = \frac{21.75 - 20}{20 - 12.5} = 23.33\%;$$
$$\text{Year 2} \quad P = (21.75)(1.08) + \frac{-10(21.75)(-0.005)}{1.08} - 12.5(0.065) = 23.68$$
$$r = \frac{23.69 - 21.75}{21.75 - 12.5} = 20.86\%;$$
$$\text{Year 3} \quad P = (23.68)(1.075) + \frac{-10(23.68)(-0.005)}{1.075} - 12.5(0.06) = 25.81$$
$$r = \frac{25.81 - 23.68}{23.68 - 12.5} = 19.02\%;$$
$$\text{Year 4} \quad P = (25.81)(1.07) + \frac{-10(25.81)(-0.005)}{1.07} - 12.5(0.055) = 28.14$$
$$r = \frac{28.14 - 25.81}{25.81 - 12.5} = 17.51\%;$$
$$\text{Year 5} \quad P = (28.14)(1.065) - \frac{10(28.14)(0.02)}{1.065} - 12.5(0.05) = 24.06$$
$$r = \frac{24.06 - 28.14}{28.14 - 12.5} = -26.09\%;$$
$$\text{Year 6} \quad P = (24.06)(1.085) - \frac{10(24.06)(0.02)}{1.085} - 12.5(0.07) = 20.80$$
$$r = \frac{20.80 - 24.06}{24.06 - 12.5} = -28.20\%.$$

$$
\begin{aligned}
& \text{Net gain after 6 years} \\
= \; & \text{principal left in the pool at the end of Year 6} - \text{initial investment} - \text{borrowed amount} \\
= \; & 20.80 - 12.5 - 7.5 = 0.80.
\end{aligned}
$$

If invested in bank of amount 7.5 billion (without borrowing to gain leverage), then the net gain $= 7.5(1.06)(1.055)(1.05)(1.045)(1.04)(1.06) - 7.5 = 2.64$.

Note the significant mark-to-market losses in the bond portfolio when the interest rates increased by 2% in two consecutive years (Years 5 and 6).

9. The decisions at times after the initial time do not depend on d. At time 1, the upper and lower node values are $x_2 = 14 + 14d$ and $x_1 = 7 + 7d$, respectively. Then, the initial value is

$$x_0 = \max[14d(1+d), 7(1+d+d^2)].$$

The choice depends on d. The critical value of d is

$$d^* = \frac{\sqrt{5}-1}{2} \approx 0.618.$$

- For $d < d^*$, we choose x_2.
- For $r = 33\%$, we have $d = 0.75$ and for $r = 25\%$ we have $d = 0.8$, so solution is the same for both.

10. (a) Since we mine forever, we have $K_K = K_{K+1} =$ constant. We let this constant be K. So $K = \dfrac{(g - dK)^2}{2000} + dK$ implies $K = 220$ every period. Thus, the initial value of the mine, $V_0 = Kx_0 = 220x_0 = \$11$ million.

(b) The amount of gold remaining in the mine in period n, $x_n = x_{n-1} - z_{n-1}$ where z_n equals the amount mined in period n. Recall the relation:

$$z_j = \frac{(g - dK_{j+1})}{1000} x_j, \; j = 0, 1, 2, \ldots$$

According to the Table in the lecture note, we have

$$
\begin{aligned}
z_0 &= \frac{400 - dK_1}{1000} x_0 = \frac{400 - \frac{211.45}{1.1}}{1000} \times 50000 = 10389, \;\; x_1 = x_0 - z_0 = 39611; \\
z_1 &= \frac{400 - \frac{208.17}{1.1}}{1000} \times 50000 = 10538, \;\; x_2 = x_1 - z_1 = 39611 - 10538 = 29073; \text{etc.}
\end{aligned}
$$

Through successive iteration, we finally obtain $x_{10} = 2393$.
Thus, by part (a), the value of the mine in period 10 is found to be $220x_{10} = \$526,460$ (at that time).

(c) The optimal extraction rate in each period $= \dfrac{g - dK}{1000} = 20\%$, so after 10 years, $x_{10} = 0.8^{10} \times 50,000 = 5369$ ounces of gold remains with a value of $\$1,181,116$ (at that time). Note that the miner is more aggressive to get a larger amount of gold from the mine when the lease of the mine has finite number of years.

11. (a) Set up a trinomial lattice with arcs:
"up" = no pumping
"middle" = normal pumping
"down" = enhanced pumping
The reserve values can be entered on each node. At the final time, the maximum reserve is 100,000 and the minimum is 26,214 barrels.

(b) Work backward to find present value = $366,740. The optimal strategy is: enhanced pumping for the first two years, followed by normal pumping in the last year.