MATH 4512 — Fundamentals of Mathematical Finance

Homework One

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1. Let c be the coupon rate per period and y be the yield per period. There are m periods per year (say, $m = 4$ for quarterly coupon payments), and let n be the number of periods remaining until maturity. Show that the duration D is given by

$$D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my}.$$

Here, the yield per year λ equals my. Show that, as $T \to \infty$, we obtain

$$D \to \frac{1}{m} + \frac{1}{\lambda}.$$

Remark

The above analytic results are revealed in the following numerical example. Consider the duration calculated for various bonds as shown in the following table, where $\lambda = 0.05$ and $m = 2$. We obtain $D \to \frac{1}{2} + \frac{1}{0.05} = 20.5$.

Duration of a Bond Yielding 5% as Function of Maturity and Coupon Rate

	Coupon rate			
Years to maturity	1%	2%	5%	10%
1	0.997	0.995	0.988	0.977
2	1.984	1.969	1.928	1.868
5	4.875	4.763	4.485	4.156
10	9.416	8.950	7.989	7.107
25	20.164	17.715	14.536	12.754
50	26.666	22.284	18.765	17.384
100	22.572	21.200	20.363	20.067
∞	20.500	20.500	20.500	20.500

The table shows that duration does not increase appreciably with maturity. In fact, with fixed yield, duration increases only to a finite limit as maturity is increased.

2. The return-to-maturity expectations hypothesis states that the return generated by holding a bond for term t to T will equal the expected return generated by continually rolling over a bond whose term is a period evenly divisible into $T - t$. Explain why the above relationship can be expressed formally as

$$\frac{1}{B(t,T)} = E_t[(1+r_t)(1+\tilde{r}_{t+1})\cdots(1+\tilde{r}_{T-1})],$$

where $B(t,T)$ is the time-t price of a discount bond maturing at T and r_t is the one-period spot rate at time t. The operator E_t indicates that expectation is taken based on the information available at the current time t.

Remark

Suppose the investor starts with one dollar at time t and invests in a discount bond maturing one year later the "deterministic" return is

$$1 + r_t = \frac{1}{B(t, t+1)},$$

where $B(t, t+1)$ is known at time t. At time $t+1$, the investor uses the proceed $\dfrac{1}{B(t, t+1)}$ to invest in a discount bond maturing one year later. The bond price is $\tilde{B}(t+1, t+2)$, which is not known at time t. The return over $[t+1, t+2]$ is

$$1 + \tilde{r}_{t+1} = \frac{1}{\tilde{B}(t+1, t+2)},$$

where "tilde" quantities represent stochastic quantities. At time $t+2$, the investor again invests in $\dfrac{1}{B(t, t+1)\tilde{B}(t+1, t+2)}$ units of discount bond maturing one year later. After $T - t$ years, the random return at time T is

$$(1 + r_t)(1 + \tilde{r}_{t+1}) \cdots (1 + \tilde{r}_{T-1}) = \frac{1}{B(t, t+1)\tilde{B}(t+1, t+2) \cdots \tilde{B}(T-1, T)}.$$

This strategy is like investing in a money market account with annual rolling over.

3. Let B_t be the time-t value of the bond maturity T years later and paying annual coupon amount c. Let i be the interest rate, assuming to be constant throughout the life of the bond. Recall that

$$B_t = \frac{c}{i}\left[1 - \frac{1}{(1+i)^T}\right] + \frac{B_T}{(1+i)^T}.$$

Show that

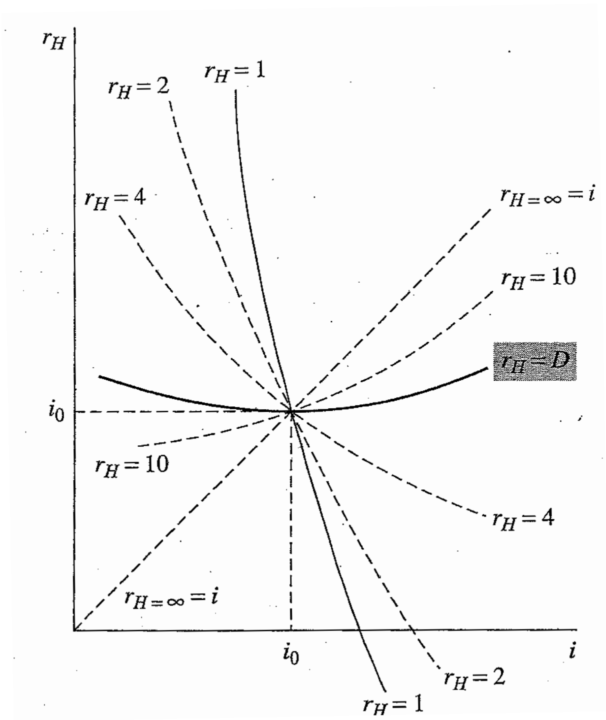
$$\frac{\Delta B_t}{B_t} = \frac{B_{t+1} - B_t}{B_t} = i - \frac{c}{B_t}.$$

In the continuous time limit, show that the governing equation for the bond value function $B(t)$ is given by

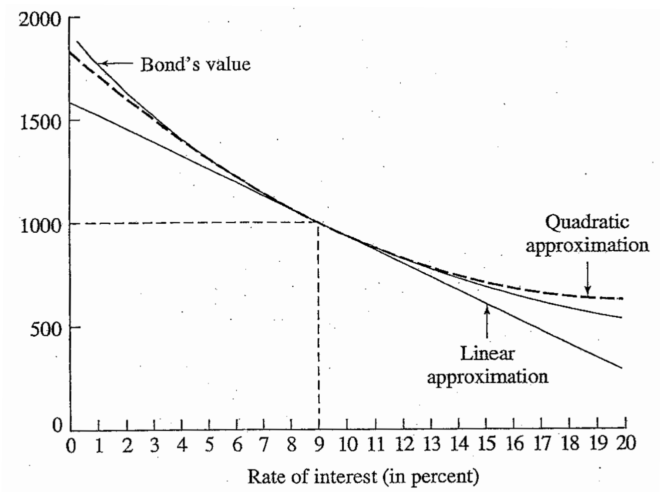
$$\frac{dB(t)}{dt} = i(t)B(t) - c(t), \quad t < T,$$

with the terminal condition: $B(T) = B_T$. There are two factors that affect the bond value $B(t)$, one is the instantaneous short rate $i(t)$ and the other is the coupon rate $c(t)$. The second term on the right hand side exhibits a negative sign in the coupon effect since the bond value decreases in value after paying a coupon amount $c(t)dt$ over $(t, t+dt)$. Deduce the closed form solution for $B(t)$.

4. Show that all curves $r_H = r_H(i)$ for various horizons $H(H = 1, 2, ..., \infty)$ go through the point (i_0, i_0). In other words, show that (i_0, i_0) is a fixed point for all curves $r_H(i)$.



5. Looking at the figure below, the quadratic approximation curve of the bond's value lies between the bond's value curve and the tangent line to the left of the tangency point and outside (above) these lines to the right of the tangency point. The fact that a quadratic (and hence "better") approximation behaves like this is not intuitive: we would tend to think that a "better approximation" would always lie between the exact value curve and its linear approximation. How can you explain this apparently non-intuitive result?



Linear and quadratic approximations of a bond's value.

6. Consider the following two bonds:

	Bond A	Bond B
Maturity	15 years	11 years
Coupon rate	10%	5%
Par value	\$1000	\$1000

(a) The current yield to maturity is taken to be 12%. Determine the convexity of each bond.

(b) Suppose you have a defensive strategy, and that you want to immunize the investor. What is each bond's rate of return at horizon $H = D$ if interest rates keep jumping from 12% to either 10% or 14%?

(c) By examining the rates of return of the two bonds under an increase or decrease of interest rates, and different choices of horizon, which bond would you choose?

7. An investor is considering the purchase of 10-year U.S. Treasury bonds and plans to hold them to maturity. Federal taxes on coupons must be paid during the year they are received, and tax must also be paid on the capital gain realized at maturity (defined as the difference between face value and original price). This investor's federal tax bracket rate is $r = 30\%$, as it is for most individuals. There are two bonds with par 100 that meet the investor's requirements. Bond 1 is a 10-year, 10% bond with a price (in decimal form) of $P_1 = 92.21$. Bond 2 is a 10-year, 7% bond with a price of $P_2 = 75.84$. Based on the price information contained in these two bonds, the investor would like to compute the theoretical price of a hypothetical 10-year zero-coupon bond that had no coupon payments and required tax payment only at maturity equal in amount to 30% of the realized capital gain (the par value minus the original price). This theoretical price should be such that the price of this bond and those of bonds 1 and 2 are mutually consistent (in mathematical term, they are equal in value) on an after-tax basis. Find this theoretical price, and show that it does not depend on the tax rate t. Assume all cash flows occur at the end of each year.

8. Orange County managed an investment pool into which several municipalities made short-term investments. A total of \$7.5 billion was invested in this pool, and this money was used to purchase securities. Using these securities as collateral, the pool borrowed \$12.5 billion from Wall Street brokerages, and these funds were used to purchase additional securities. The \$20 billion total was invested primarily in long-term fixed-income securities to obtain a higher yield than the short-term alternatives. Furthermore, as interest rates slowly declined, as they did in 1992-1994, an even greater return was obtained. Things fell apart in 1994, when interest rates rose sharply.

   Hypothetically, assume that initially the duration of the invested portfolio was 10 years, the short-term rate was 6%, the average coupon interest on the portfolio was 8.5% of face value, the cost of Wall Street money was 7%, and short-term interest rates were falling at $\frac{1}{2}\%$ per year.

   (a) What was the rate of return that the pool investors obtained during this early period? Does it compare favorably with 6% that these investors would have obtained by investing normally in short-term securities?

   (b) When interest rates had fallen two percentage points and began increasing at 2% per year, what rate of return was obtained by the pool?

   Additional assumptions made in the calculations:

   (a) Assume the bond portfolio is restructured annually to maintain a duration of 10 years.

   (b) Assume the value of money borrowed is maintained at \$12.5 billion every year.

(c) Assume Orange County makes interest on borrowed fund at the rate which prevailed at the beginning of the given year.

Hints

- In the first year, the coupon rate was 8.5%. With a duration of 10 years, the change in portfolio value due to change in interest rate is given by

$$-\frac{duration}{1+i} \cdot P \cdot change\ in\ i.$$

  The interest cost of borrowing of 12.5 billion per annum with cost of fund 7% is given by 12.5×(0.07).

- In the fifth year, the coupon rate became 6.5% while the change in interest rate is 2%. The cost of fund became 5%.

9. Find the solution to the fishing problem when the interest rate is 33%. Are the decisions different from those at the interest rate equals 25%? At what critical value of the discount factor does the solution change?

10. Consider the Complexico mine and assume a 10% constant interest rate; also assume the price of gold is constant at $400/oz.

    (a) Find the value of the mine (not a 10-year lease) if the current deposit is x_0. In particular, how much is the mine worth initially when $x_0 = 50,000$ ounces? [*Hint:* Consider the recursive equation for K_k as $k \to \infty$.]

    (b) For the 10-year lease considered in the text, how much gold remains in the mine at the end of the lease; and how much is the mine worth at that time?

    (c) If the mine were not leased, but instead operated optimally by an owner, what would the mine be worth after 10 years?

11. You have purchased a lease for the Little Bear Oil well. This well has initial reserves of 100 thousand barrels of oil. In any year you have three choices of how to operate the well: (a) you can *not* pump, in which case there is no operating cost and no change in oil reserves; (b) you can pump normally, in which case the operating cost is $50 thousand and you will pump out 20% of what the reserves were at the beginning of the year; or (c) you can use enhanced pumping using water pressure, in which case the operating cost is $120 thousand and you will pump out 36% of what the reserves were at the beginning of the year. The price of oil is $10 per barrel and the interest rate is 10%. Assume that both your operating costs and the oil revenues come at the beginning of the year (through advance sales). Your lease is for a period of 3 years.

    (a) Show how to set up a trinomial lattice to represent the possible states of the oil reserves.

    (b) What is the maximum present value of your profits, and what is the corresponding optimal pumping strategy?