MATH4512 – Fundamentals of Mathematical Finance

Homework Three

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1. Suppose there are n mutually uncorrelated assets. The rate of return on asset i has variance σ_i^2. The expected rates of return are unspecified at this point. The dollar amount of asset i in the market is X_i. We let $T = \sum_{i=1}^{n} X_i$ and then set $x_i = X_i/T$, for $i = 1, 2, \cdots, n$. We take the market portfolio in normalized form to be $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$. Find an expression for the beta β_j in terms of the x_i's and σ_i's.

2. In Simpleland there are only two risky stocks, A and B, whose details are listed in the following table.

	Number of shares outstanding	Price per share	Expected rate of return	Standard deviation of return
Stock A	100	\$1.50	15%	15%
Stock B	150	\$2.00	12%	9%

   Furthermore, the correlation coefficient between the returns of stocks A and B is $\rho_{AB} = \frac{1}{3}$.

   There is also a risk-free asset, and Simpleland satisfies the CAPM exactly.

   (a) What is the expected rate of return of the market portfolio?
   (b) What is the standard deviation of the market portfolio?
   (c) What is the beta of stock A?
   (d) What is the risk-free rate in Simpleland?

3. Some risky assets are not easily marketable. For example, human capital, an individual's future lifetime income, cannot be sold. Some assets such as one's house may not be sold for psychological reasons or inertia, so it may be considered non-marketable. When some assets are not marketable, the CAPM can be reformulated. Show that the equilibrium rate of return on asset i is given by

$$\overline{R}_i = r + \beta_i^*(\overline{R}_m - r)$$

   where

$$\beta_i^* = \frac{\mathrm{cov}(R_i, R_m) + (V_N/V_m)\mathrm{cov}(R_i, R_N)}{\sigma_m^2 + (V_N/V_m)\mathrm{cov}(R_m, R_N)},$$

   V_N = value of all non-marketable assets, V_m = value of marketable assets, and R_N = one-period rate of return on non-marketable assets.

   Hint: Let w_j be the weight of the j^{th} risky asset within the universe of marketable assets and let $w_N = V_N/V_m$. The objective function to be minimized is

$$\mathrm{var}\left(\sum_{j=1}^{J} w_j R_j + w_N R_N\right).$$

   Here, $w_1, \cdots, w_J$ are the control variables while w_N is not a control variable. The model has close resemblance to the asset-liability model.

4. Let r_j denote the equilibrium rate of return of risky asset j as deduced from CAPM and S be its equilibrium price. Let P_0 be the market price of asset j and $\widetilde{P}_e$ be the random return of the asset. Let r'_j be the rate of return as deduced from the market price of the asset. Let r_m denote the equilibrium rate of return of the market portfolio. Show that

$$E[r'_j] - r_f = (1 + r_f)\left(\frac{S}{P_0} - 1\right) + \frac{\operatorname{cov}(\widetilde{P}_e/P_0, r_m)}{\sigma_m^2}(\mu_m - r_f),$$

where $\sigma_m^2 = \operatorname{var}(r_m)$ and $\mu_m = E[r_m]$, r_f is the risk free interest rate.

Hint: First, find the relation between $E[r'_j]$ and $E[r_j]$. Note that they are the same if and only if $S = P_0$.

5. Take a subset of N risky assets from the financial market and assume their beta values to be $\boldsymbol{\beta}_m = (\beta_{1m} \quad \beta_{2m} \quad \beta_{Nm})^T$. We would like to construct the market proxy $\widehat{m}$ from these N risky assets such that the beta of $\widehat{m}$ is one and $\boldsymbol{\beta}_{\widehat{m}} = \boldsymbol{\beta}_m$, that is, $\beta_{jm} = \beta_{j\widehat{m}}$ for all j. Consider the following minimization problem

$$\min_{\substack{\boldsymbol{w} \\ \boldsymbol{\beta}_m^T \boldsymbol{w} = 1}} \boldsymbol{w}^T \Omega \boldsymbol{w}$$

where Ω is the covariance matrix of the random returns of the N assets. The constraint indicates that the beta of the market proxy $\widehat{m}$ is one. The first order conditions give

$$\Omega \boldsymbol{w} - \lambda \boldsymbol{\beta}_m = 0 \quad \text{and} \quad \boldsymbol{\beta}_m^T \boldsymbol{w} = 1.$$

Show that

$$\lambda = \frac{1}{\boldsymbol{\beta}_m^T \Omega^{-1} \boldsymbol{\beta}_m} \quad \text{and} \quad \boldsymbol{w}^* = \frac{\Omega^{-1}\boldsymbol{\beta}_m}{\boldsymbol{\beta}_m^T \Omega^{-1} \boldsymbol{\beta}_m}.$$

We set the market proxy to be $\boldsymbol{w}^*$. Check whether $\boldsymbol{\beta}_{\widehat{m}} = \boldsymbol{\beta}_m$, where $\boldsymbol{\beta}_{\widehat{m}} = (\beta_{1\widehat{m}} \cdots \beta_{N\widehat{m}})^T$, and recall $\beta_{j\widehat{m}} = \dfrac{\operatorname{cov}(r_j, r_{\widehat{m}})}{\sigma_{\widehat{m}}^2}$.

6. Assume that the following two-index model describes the rate of return of asset i as

$$R_i = b_{1i} I_1 + b_{i2} I_2 + e_i.$$

Assume that the following three portfolios are observed.

Portfolio	Expected Return	b_{i1}	b_{i2}
A	12.0	1	0.5
B	13.4	3	0.2
C	12.0	3	-0.5

According to the APT, the expected rate of return $\overline{R}_i$ is given by

$$\overline{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}.$$

Determine λ_0, λ_1, and λ_2.

7. Someone who believes that the collection of all stocks satisfies a single-factor model with the market portfolio serving as the factor gives you information on three stocks which make up a portfolio. (See Table.) In addition, you know that the market portfolio has an expected rate of return of 12% and a standard deviation of 18%. The riskfree rate is 5%.

(a) What is the portfolio's expected rate of return?

(b) Assuming the factor model is accurate, what is the standard deviation of this rate of return?

Simple Portfolio

Stock	Beta	Standard deviation of random error term	Weight in portfolio
A	1.10	7.0%	20%
B	0.80	2.3%	50%
C	1.00	1.0%	30%

8. Two stocks are believed to satisfy the two-factor model

$$\begin{aligned} r_1 &= a_1 + 2f_1 + f_2 \\ r_2 &= a_2 + 3f_1 + 4f_2. \end{aligned}$$

In addition, there is a riskfree asset with a rate of return of 10%. It is known that $\overline{r}_1 = 15\%, \overline{r}_2 = 20\%$. What are the values of λ_0, λ_1 and λ_2 for this model?

9. David and Sue are portfolio managers for a defined benefit pension fund that has \$20 billion dollars invested in U.S. stocks. A large portion of these funds are passively invested to match the returns of the S & P 500 index. Recently, top administrators of the pension plan have asked David and Sue to develop an active investment strategy that will initially be used on a small portion of the \$20 billion. If the active strategy they develop is successful, it will be used on a larger percentage of the \$20 billion. David and Sue have decided to use the concept of arbitrage pricing theory (APT) in developing their active investment strategy.

(a) They decide their initial APT model will include the following three common factors:
N = percentage change in consumer non-durable good purchases
D = percentage change in consumer durable good purchases
I = percentage change in consumer price inflation
Following the standard way of symbolically expressing the APT, they write the following equation for stock i:

$$\widetilde{R}_{it} = a_{0t} + b_{iN}(\widetilde{N}_t) + b_{iD}(\widetilde{D}_t) + b_{iI}(\widetilde{I}_t) + e_{it}.$$

Define what the following terms mean: $a_{0t}, b_{iN}, \widetilde{N}_t$.

(b) After considerable statistical analysis, they develop the following factor estimates for stock 1:

$$b_{1N} = 1.0; b_{1D} = 1.5; b_{1I} = -0.5.$$

They believe a_{01} should be the one-year risk-free rate at 4%. In addition, they believe the security markets expect N, D, and I to be the following during the next year.

Factor	Expected value
N	2.0%
D	3.0%
I	1.5%

If these assessments are correct, what expected return does the market expect on stock 1 during the next year?

(c) David and Sue agree with the expected value of N and I. But they believe the percentage growth of durable consumer purchases (common factor D) during the next year will be 5.0%. How should they use this opinion in developing their active management strategy?

(d) What is the role of the e_{it} term in the equation?

(e) What types of difficulties do you see in using the APT to develop an active management strategy?

10. An investor purchased one share of Goldman Sachs (GS) at \$100 at $T = 0$. She again purchased one more share of GS at \$120 at $T = 1$. She received total dividend of \$2 at $T = 1$ and a dividend of \$3 per share at $T = 2$. The investor sold the shares at $T = 2$ for a total consideration of \$260 (or \$130 per share). What are the money weighted and time weighted returns?

11. There are two funds A and B. Fund A has a sample mean of 0.13 and fund B has a sample mean of 0.18, with the riskier fund B having double the beta at 2.0 as fund A. The respective standard deviations are 15% and 19%. The mean return for the market index is 0.12 with a standard deviation of 8%, while the riskfree rate is 0.08.

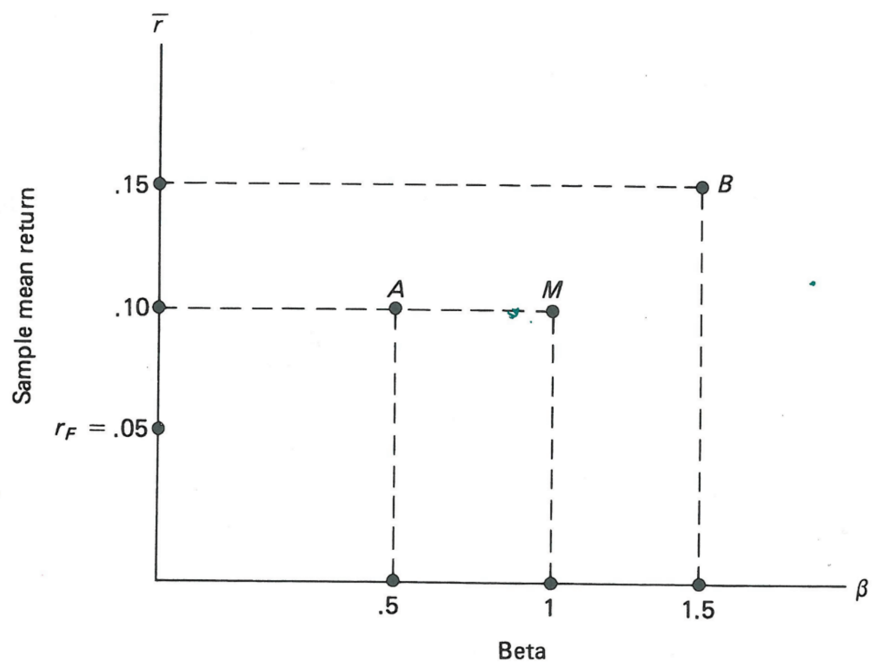
(a) Compute the Jensen Alpha for each fund. What does it indicate to you?

(b) Compute the Treynor Ratio for the funds and the market. Interpret the results.

(c) Compute the Sharpe ratio for the fund and the market.

(d) How would you comment on the performance of the two funds?

12. Suppose the returns and corresponding beta values for two assets (A and B) are as indicated on the following graph:

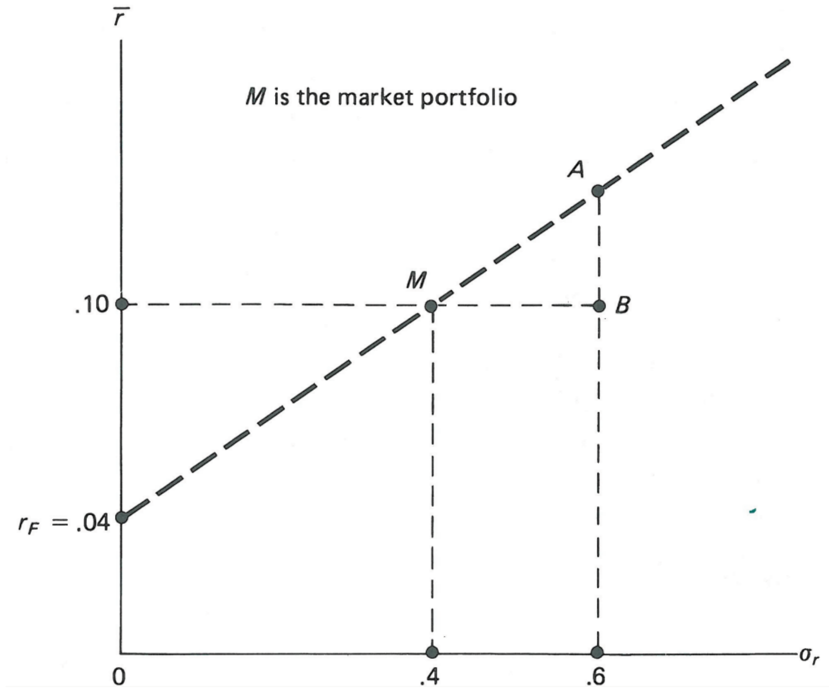


(a) Compute the Treynor Ratio for A and B. Interpret the results.

(b) Compute the Jensen Alpha for A and B. Interpret the results.

(c) Suppose one manager had selected a portfolio represented by A and another manager had selected a portfolio represented by B. Would you feel confident in evaluating the manager's relative performance with the Treynor or Jensen results?

13. Consider the following graph:



   (a) Compute the Sharpe Ratio for A and B. Interpret the results.

   (b) Suppose there is no borrowing of riskfree asset. How would your interpretation of A's performance be affected?

14. Two managers both are awarded a Jensen Alpha of 3 percent on their performance.

   (a) Does this indicate a good performance or a poor performance?

   (b) What shortfalls would you point out in this index?

15. Suppose you plot the performance of two funds in (β, μ) diagram in relation to your estimated security market line, and you find that they line up perfectly equidistant above the line. At a later date you discover your estimate of the security market line had too large a slope because you used the wrong riskfree rate; that in fact, the true SML had a very mild incline. What does this affect your evaluations of these two funds?

16. What are the advantages and disadvantages of the Jensen Alpha, Treynor Ratio, and Sharpe Ratio? Discuss the scenarios where these three measures work well, respectively.