# MATH4512 – Fundamentals of Mathematical Finance

**Homework Four**

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1. In a betting game with $m$ possible outcomes, the gain from a unit bet on $i$ if outcome $j$ occurs is given by
$$g_{ij} = \begin{cases} d_i & \text{if } j = i \\ -1 & \text{if } j \neq i \end{cases}, \quad j = 1, 2, \cdots, m,$$
where $d_i > 0$, for all $i$. Assuming
$$\sum_{i=1}^{m} \frac{1}{1+d_i} < 1,$$
show that the betting strategy
$$\alpha_i = \frac{\frac{1}{1+d_i}}{1 - \sum_{i=1}^{m} \frac{1}{1+d_i}}, \quad i = 1, 2, \cdots, m,$$
always yields a gain of exactly 1.

*Hint*: As a numerical example, suppose a "stupid" casino sets the return of the game of throwing one dice with 6 faces to be 20 dollars for each dollar bet on a particular face (a fair game should return 6 dollars). In this case, $d_i = 19$ for all $i$ and
$$\sum_{i=1}^{6} \frac{1}{1+di} = \frac{6}{20} < 1.$$
A "clever" gambler will set
$$\alpha_i = \frac{\frac{1}{20}}{1 - \frac{6}{20}} = \frac{1}{14}$$
and places bets on all faces so that the gain in each game $= \frac{1}{14} \times 20 - \frac{1}{14} \times 6 = 1$, independent of the occurrence of any particular face of the dice.

2. The random return vector of three securities achieves the following values: $(4 \;\; 2 \;\; 3)$ and $(2 \;\; 4 \;\; 3)$ with equal probabilities. Show that the optimal strategy based on the logarithm utility criterion is not unique. Find two choices of optimal strategies.

3. Consider the class of power utility functions
$$U(x) = \frac{x^\gamma}{\gamma} \quad \text{for} \quad \gamma \leq 1.$$
This class includes the logarithm utility. (Hint: add $-\dfrac{1}{\gamma}$ to $U(x)$ and consider $\gamma \to 0^+$). The log-optimal strategy has been shown to exhibit the property that the maximization of $E[U(X_k)]$ with a fixed-proportions strategy only requires the maximization of the expected utility of single-period investment as given by $E[U(X_1)]$. Check whether such property can be extended to the power utility function.

4. This exercise is related to the *Dictionary Order*. Consider the choice set

$$B = \{(x, y) : x \in [0, \infty) \quad \text{and} \quad y \in [0, \infty)\}.$$

Consider the following preference relation:

$$\begin{array}{l} (x_1, y_1) \in B \quad \text{and} \quad (x_2, y_2) \in B \\ (x_1, y_1) \succeq (x_2, y_2) \text{ if and only if} \\ [x_1 > x_2] \quad \text{or} \quad [x_1 = x_2 \text{ and } y_1 \geq y_2]. \end{array}$$

Show that $\succeq$ satisfies the three axioms of Reflexivity, Comparability and Transitivity.

5. Recall the "Order Preserving" Axiom:

For any $x, y \in B$, where $x \succ y$ and $\alpha, \beta \in [0, 1]$,

$[\alpha x + (1-\alpha)y] \succ [\beta x + (1-\beta)y]$ if and only if $\alpha > \beta$. Show that the above Dictionary Order satisfies this Axiom.

6. Show that the function

$$U(x, y) = \ln(x + y)$$

cannot be an utility function representing the Dictionary Order.

*Hint*: A utility function $U : B \to R$ satisfies

(i) $x \succ y$ if and only if $U(x) > U(y)$.

(ii) $x \sim y$ if and only if $U(x) = U(y)$.

7. The HARA (for hyperbolic absolute risk aversion) class of utility functions is defined by

$$U(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^{\gamma}, \qquad b > 0.$$

The functions are defined for those values of $x$ where the term in parentheses is nonnegative. Show how the parameters $\gamma, a$ and $b$ can be chosen to obtain the following special cases (or an equivalent form).

(a) Linear or risk neutral: $U(x) = x$

(b) Quadratic: $U(x) = x - \frac{1}{2}cx^2$

(c) Exponential: $U(x) = e^{-ax}$ [Try $\gamma = -\infty$.]

(d) Power: $U(x) = cx^{\gamma}$

(e) Logarithmic: $U(x) = \ln x$ [Try $U(x) = (1-\gamma)^{1-\gamma}(x^{\gamma} - 1)/\gamma$.]

Show that the Arrow-Pratt risk aversion coefficient is of the form $1/(cx + d)$.

8. There is a useful approximation to the certainty equivalent that is easy to derive. A second-order expansion near $\overline{x} = E[x]$ gives

$$U(x) \approx U(\overline{x}) + U'(\overline{x})(x - \overline{x}) + \frac{1}{2}U''(\overline{x})(x - \overline{x})^2.$$

Hence, we deduce the following approximation:

$$E[U(x)] \approx U(\overline{x}) + \frac{1}{2}U''(\overline{x}) \text{ var}(x).$$

On the other hand, if we let $c$ denote the certainty equivalent and assume that it is close to $x$, we can use the first-order expansion

$$U(c) \approx U(\overline{x}) + U'(\overline{x})(c - \overline{x}).$$

Using these approximations, show that

$$c \approx \overline{x} + \frac{U''(\overline{x})}{2U'(\overline{x})} \operatorname{var}(x).$$

9. Consider the following investments:

| A | | B | | C | |
|---|---|---|---|---|---|
| probability | return (%) | probability | return (%) | probability | return (%) |
| 0.4 | 3 | 0.1 | 5 | 0.1 | 5 |
| 0.3 | 4 | 0.2 | 6 | 0.1 | 7 |
| 0.1 | 6 | 0.1 | 8 | 0.2 | 8 |
| 0.1 | 7 | 0.2 | 9 | 0.2 | 9 |
| 0.1 | 9 | 0.4 | 10 | 0.4 | 11 |

(a) What can be said about the desirability of the investments using first-order and second-order stochastic dominance?

(b) Using geometric mean return as a criterion, which investment is preferred?

10. Assume that the utility function $u(x)$ satisfies (i) $u'(x) > 0$, (ii) $u''(x) < 0$ and $u'''(x) > 0$. The distribution $F$ dominates the other distribution $G$ by the third order stochastic dominance if and only if

$$\int_C u(x)\, dF(x) \geq \int_C u(x)\, dG(x),$$

where $C$ is the set of all possible outcomes. Show that $F(x)$ dominates $G$ by the third order dominance if

(i) $\displaystyle\int_a^x \int_a^t [F(y) - G(y)]\; dydt \leq 0$ for all $x$, where $t$ lies between $a$ and $b$, and

(ii) $\displaystyle\int_a^b F(t)\, dt \leq \int_a^b G(t)\, dt.$

*Hint*: Consider the integration by parts of

$$\int_a^b u''(x) \int_a^x [F(y) - G(y)]\; dydx.$$