MATH 4512 – Fundamentals of Mathematical Finance

Solution to Homework Three

Course Instructor: Prof. Y.K. Kwok

---

1. The market portfolio consists of n uncorrelated assets with weight vector $(x_1 \cdots x_n)^T$. Since the assets are uncorrelated, we obtain

$$\sigma_M^2 = \mathrm{cov}(x_1 r_1 + \cdots + x_n r_n, x_1 r_1 + \cdots + x_n r_n) = \sum_{j=1}^n x_j^2 \sigma_j^2.$$

and

$$\sigma_{iM} = \mathrm{cov}(x_i r_i, x_1 r_1 + \cdots + x_n r_n) = x_i \sigma_i^2.$$

We then have

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \frac{x_i \sigma_i^2}{\sum_{j=1}^n x_j^2 \sigma_j^2}.$$

2. The market consists of \$150 in shares of A and \$300 in shares of B. Hence, the market rate of return is

$$r_M = \left(\frac{150}{450}\right) r_A + \left(\frac{300}{450}\right) r_B = \frac{1}{3} r_A + \frac{2}{3} r_B.$$

   (a) $\overline{r}_M = \dfrac{1}{3} \times 0.15 + \dfrac{2}{3} \times 0.12 = 0.13;$

   (b) $\sigma_M = \left[\dfrac{1}{9}(0.15)^2 + \dfrac{4}{9 \times 3}(0.15)(0.09) + \dfrac{4}{9}(0.09)^2\right]^{\frac{1}{2}} = 0.09;$

   (c) $\sigma_{AM} = \dfrac{1}{3}\sigma_A^2 + \dfrac{2}{3}\rho_{AB}\sigma_A\sigma_B = \dfrac{1}{3}(0.15)^2 + \dfrac{2}{9}(0.15)(0.09) = 0.0105; \beta_A = \dfrac{\sigma_{AM}}{\sigma_M^2} = 1.2963.$

   (d) Since Simpleland satisfies the CAPM exactly, stocks A and B lie on the security market line. Specifically,

$$\overline{r}_A - r_f = \beta_A(\overline{r}_M - r_f).$$

   Hence, the risk-free rate is given by

$$r_f = \frac{\overline{r}_A - \beta_A \overline{r}_M}{1 - \beta_A} = 0.0625.$$

3. Assuming that there are J risky assets and one riskfree asset in the set of marketable assets, with total dollar value V_m. Define the weight vector $\boldsymbol{w} = (w_1 \cdots w_J)^T$, where w_j represents the weight of the j^{th} risky asset within the universe of marketable assets. That is, the dollar value V_j of the j^{th} risky asset is $w_j V_m$. Let w_0 denote the weight of the riskfree asset. Using V_m as the numeraire, the weight of the non-marketable asset, denoted by w_N, is equal to V_N/V_m. Note that $\sum_{j=0}^J w_j = 1$ and define the covariance matrix Ω such that the $(i,j)^{\text{th}}$ entry of $\Omega = \sigma_{ij} = \mathrm{cov}(R_i, R_j)$, where R_i is the rate of return of the i^{th} risky asset. Let $\overline{R}_m$ denote the expected rate of return of the portfolio of the marketable

assets, including the riskfree asset. Defining $\overline{\boldsymbol{R}} = (\overline{R}_1 \cdots \overline{R}_J)^T$ and $\boldsymbol{w} = (w_1 \cdots w_J)^T$. We then have

$$\overline{R}_m - r = \sum_{j=1}^{J} w_j(\overline{R}_j - r) = \boldsymbol{w}^T(\overline{\boldsymbol{R}} - r\mathbf{1}),$$

since the expected rate of return of the portfolio of marketable assets above the riskfree rate r is the weighted average of all expected rate of the return above r of the risky assets. The control variables in the optimal portfolio selection problem are $x_1, x_2, \cdots, x_J$. The objective function to be minimized is

$$\text{var}\left(\sum_{j=1}^{J} w_j R_j + w_N R_N\right)$$

which is proportional to the variance of portfolio's return. Note that the riskfree asset does not contribute to the portfolio risk. As a remark, the actual weight $\widehat{w}_j$ of the j^{th} risky asset in the overall portfolio is

$$\widehat{w}_j = \frac{V_j}{V_{total}} = \frac{w_j V_m}{V_{total}},$$

where V_{total} is the total value of all assets, including the marketable risky assets, non-marketable risky assets and the marketable riskfree assets. Expanding the objective function, we obtain

$$\begin{aligned}
&\text{var}\left(\sum_{j=1}^{J} w_j R_j + w_N R_N\right) \\
&= \boldsymbol{w}^T \Omega \boldsymbol{w} + 2w_N \sum_{j=1}^{J} w_j \text{cov}(R_j, R_N) + w_N^2 \text{var}(R_N).
\end{aligned}$$

Since w_N is fixed, so it is not one of the control variables. We form the following Lagrangian:

$$L = \frac{\boldsymbol{w}^T \Omega \boldsymbol{w}}{2} + w_N \sum_{j=1}^{J} w_j \text{cov}(R_j, R_N) + \lambda\left[\overline{R}_m - r - \sum_{j=1}^{J} w_j(\overline{R}_j - r)\right].$$

The first order conditions are

$$\frac{\partial L}{\partial w_i} = \sum_{j=1}^{J} w_j \sigma_{ij} + w_N \text{cov}(R_i, R_N) - \lambda(\overline{R}_i - r) = 0, \quad i = 1, 2, \cdots, J, \tag{1a}$$

$$\overline{R}_m - r = \sum_{j=1}^{J} w_j(\overline{R}_j - r). \tag{1b}$$

The first J equations can be rewritten as

$$\begin{aligned}
\lambda(\overline{R}_i - r) &= \sum_{j=1}^{J} w_j \sigma_{ij} + w_N \text{cov}(R_i, R_N) \\
&= \text{cov}\left(\sum_{j=1}^{J} w_j R_j + w_N R_N, R_i\right) \\
&= \sum_{j=1}^{J} w_j \sigma_{ij} + w_N \text{cov}(R_i, R_N), \quad i = 1, 2, \cdots, J.
\end{aligned} \tag{2}$$

Combining Eqs. (1b) and (2), we obtain

$$\sum_{i=1}^{J} w_i \left(\sum_{j=1}^{J} \sigma_{ij} w_j + w_N \mathrm{cov}(R_i, R_N) \right) = \lambda \boldsymbol{w}^T (\overline{\boldsymbol{R}} - r\mathbf{1}) = \lambda(\overline{R}_m - r). \tag{3}$$

Eliminating λ in Eqs. (2) and (3), we obtain

$$\begin{aligned}\frac{\overline{R}_i - r}{\overline{R}_m - r} &= \frac{\mathrm{cov}\left(\sum_{j=1}^{J} w_j R_j + w_N R_N, R_i\right)}{\sum_{i=1}^{J} w_i \left[\sum_{j=1}^{J} \sigma_{ij} w_j + w_N \mathrm{cov}(R_i, R_N)\right]} \\ &= \frac{\mathrm{cov}(R_m, R_i) + \frac{V_N}{V_m}\mathrm{cov}(R_i, R_N)}{\sigma_m^2 + \frac{V_N}{V_m}\mathrm{cov}(R_i, R_N)},\end{aligned}$$

where

$$\begin{aligned}\sigma_m^2 &= \sum_{i=1}^{J}\sum_{j=1}^{J} w_i w_j \sigma_{ij} \\ \mathrm{cov}(R_i, R_m) &= \mathrm{cov}\left(R_i, \sum_{j=1}^{J} w_j R_j\right).\end{aligned}$$

4. Consider

$$\begin{aligned}E[r_j'] = E\left[\frac{\widetilde{P}_e - P_0}{P_0}\right] &= E\left[\frac{\widetilde{P}_e}{S}\right]\frac{S}{P_0} - 1 \\ &= (E[r_j] + 1)\frac{S}{P_0} - 1 \\ &= [1 + r_f + \beta_{jm}(\mu_m - r_f)]\frac{S}{P_0} - 1\end{aligned}$$

so that

$$\begin{aligned}E[r_j'] - r_f &= (r_f + 1)\left(\frac{S}{P_0} - 1\right) + \frac{\mathrm{cov}(r_j, r_m)}{\sigma_m^2}(\mu_m - r_f)\frac{S}{P_0} \\ &= (r_f + 1)\left(\frac{S}{P_0} - 1\right) + \frac{\mathrm{cov}(\widetilde{P}_e/P_0, r_m)}{\sigma_m^2}(\mu_m - r_f) \\ &= \alpha_j + \beta_{jm}'(\mu_m - r_f)\end{aligned}$$

where

$$\alpha_j = (1 + r_f)\left(\frac{S}{P_0} - 1\right) \quad \text{and} \quad \beta_{jm}' = \mathrm{cov}(\widetilde{P}_e/P_0, r_m)/\sigma_m^2.$$

Here, β_{jm}' is the beta as deduced from the market price.

5. Form the Lagrangian: $L = \dfrac{1}{2}\boldsymbol{w}^T\Omega\boldsymbol{w} + \lambda(1 - \boldsymbol{\beta}_m^T\boldsymbol{w})$. The first order conditions are

$$\Omega\boldsymbol{w} - \lambda\boldsymbol{\beta}_m = 0 \quad \text{and} \quad \boldsymbol{\beta}_m^T\boldsymbol{w} = 1.$$

We then obtain

$$\lambda = \frac{1}{\boldsymbol{\beta}_m^T\Omega^{-1}\boldsymbol{\beta}_m} \quad \text{and} \quad \boldsymbol{w}^* = \frac{\Omega^{-1}\boldsymbol{\beta}_m}{\boldsymbol{\beta}_m^T\Omega^{-1}\boldsymbol{\beta}_m}.$$

Consider

$$\beta_{j\widehat{m}} = \frac{\operatorname{cov}(r_j, r_{\widehat{m}})}{\sigma_{\widehat{m}}^2} = \frac{e_j^T \Omega \boldsymbol{w}^*}{\boldsymbol{w}^{*T} \Omega \boldsymbol{w}^*}, \text{ where } \boldsymbol{e}_j = (0 \cdots 1 \cdots 0)^T,$$

so that

$$\boldsymbol{\beta}_{\widehat{m}} = \frac{\Omega \boldsymbol{w}^*}{\boldsymbol{w}^{*T} \Omega \boldsymbol{w}^*} = \frac{\Omega(\lambda \Omega^{-1} \beta_m)}{\lambda^2 (\Omega^{-1} \boldsymbol{\beta}_m)^T \Omega (\Omega^{-1} \boldsymbol{\beta}_m)} = \frac{\boldsymbol{\beta}_m}{\lambda \boldsymbol{\beta}_m^T \Omega^{-1} \boldsymbol{\beta}_m} = \boldsymbol{\beta}_m.$$

6. According to the APT, the expected rate of return $\overline{R}_i$ of the i^{th} asset is given by

$$\overline{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}.$$

We obtain

$$\begin{cases} 12 - \lambda_0 = \lambda_1 + 0.5\lambda_2 \\ 13.4 - \lambda_0 = 3\lambda_1 + 0.2\lambda_2 \\ 12 - \lambda_0 = 3\lambda_1 - 0.5\lambda_2 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 0.5 & 1 \\ 3 & 0.2 & 1 \\ 3 & -0.5 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} 12 \\ 13.4 \\ 12 \end{pmatrix}.$$

Solving the algebraic system, we have $\lambda_0 = 10, \lambda = 1, \lambda_2 = 2$. Hence, $\overline{R}_i = 10 + b_{i1} + 2b_{i2}$.

7. (a) The beta of the portfolio is a weighted average of the individual betas:

$$\beta = 0.2 \times 1.1 + 0.5 \times 0.8 + 0.3 \times 1 = 0.92.$$

Hence, by applying the CAPM to the portfolio, we find

$$\overline{r}_P = 0.05 + 0.92(0.12 - 0.05) = 11.44\%.$$

(b) Using the single-factor model, we have

$$\begin{aligned} \sigma_e^2 &= \sum_{i=A}^{C} w_i^2 \sigma_{e_i}^2 = 0.2^2 \times 0.07^2 + 0.5^2 \times 0.023^2 + 0.3^2 \times 0.01^2 \\ &= 0.00033725; \\ \sigma^2 &= \beta^2 \sigma_M^2 + \sigma_e^2 = 0.92^2 \times 0.18^2 + 0.00033725 = 0.2776; \\ \sigma &= 16.7\%. \end{aligned}$$

8. By the APT, we have $\lambda_0 = r_f = 10\%$. Also, we deduce the following pair of equations for λ_1 and λ_2:

$$\begin{aligned} 0.15 &= 0.10 + 2\lambda_1 + \lambda_2 \\ 0.20 &= 0.10 + 3\lambda_1 + 4\lambda_2. \end{aligned}$$

This yields $\lambda_1 = 0.02$ and $\lambda_2 = 0.01$.

9. (a) a_{0t} is the constant rate of return expected for a security whose b_{ik} values are all zero; b_{iN} is the sensitivity of stock i to percentage changes in non-durable consumer purchases (the average value across all stocks is 1.0);
$\widetilde{N}_t$ is the uncertain percentage change in non-durable goods purchases in period t.

(b) $E(R_1) = 4.0\% + 1.0(2.0\%) + 1.5(3.0\%) - 0.5(1.5\%) = 9.75\%$.

(c) Given that they expect the common factor D to have a larger payoff than that expected by other investors, their portfolio should have a sensitivity to factor D larger than normal. Usually this means the portfolio will have a factor sensitivity in excess of 1.0.

(d) The term e_{it} is the stock specific return during period t. To minimize the impacts of stock specific returns on portfolio returns, the portfolio should be broadly diversified.

(e) Probably the most difficult aspect of using the APT as a tool for active management is the identification of the common factors. If this can be done, then there are difficulties of creating a model that will successfully predict the common factor outcomes. For example, how should David and Sue develop their opinion that factor D would be greater than what is expected by others?

10. MWB is calculated as an IRR by

$$0 = -100 + \frac{-120 + 2}{1 + r} + \frac{260 + 3 \times 2}{(1 + r)^2},$$

yielding $r = 13.6\%$. The HPR for each period is $(120 + 2)/100 - 1 = 22\%$ and $(260 + 3 \times 2)/240 - 1 = 10.83\%$. So, TWR is $\sqrt{1.22 \times 1.1083} - 1 = 16.28\%$.

11. (a) $\alpha_A = 1\%$ and $\alpha_B = 2\%$. Fund B's Jensen Alpha is twice that of fund A.

(b) $T_A = 5$ and $T_B = 5$ while $T_M = 4$. The Treynor Ratio shows that both funds performing at the same level which is higher than the market level.

(c) $S_A = 0.333$, $S_B = 0.526$ and $S_M = 0.5$. The Sharpe ratio places B ahead of A.

(d) Both funds appear to be able to identify undervalued securities since they have positive Jensen indices. Fund B's Jensen Alpha is larger, but when considering the investors' ability to lever fund A 1 percent excess return, they both look pretty much the same in this respect as indicated by their equal Treynor ratios. Fund B, however, clearly has the better management since it can capture the excess return while diversifying over many individual issues, as indicated by its superior Sharpe ratio. Fund B gets the nod!

12. (a) $T_A = (0.1 - 0.05)/0.5 = 0.1$ and $T_B = 0.067$. A has a greater excess return per systematic risk than B.

(b) $\alpha_A = 0.025$ and $\alpha_B = 0.025$. A and B have the same Jensen Alpha.

(c) Both portfolios have picked undervalued securities and generates the same amount of the Jensen Alpha. However, taken the ability to lever into account portfolio A is more attractive as indicated by its larger Treynor Ratio. Nevertheless, one cannot confidently tell which is better without knowing their specific risks taken which is essential in evaluating their ability in capturing the breadth of investment performance.

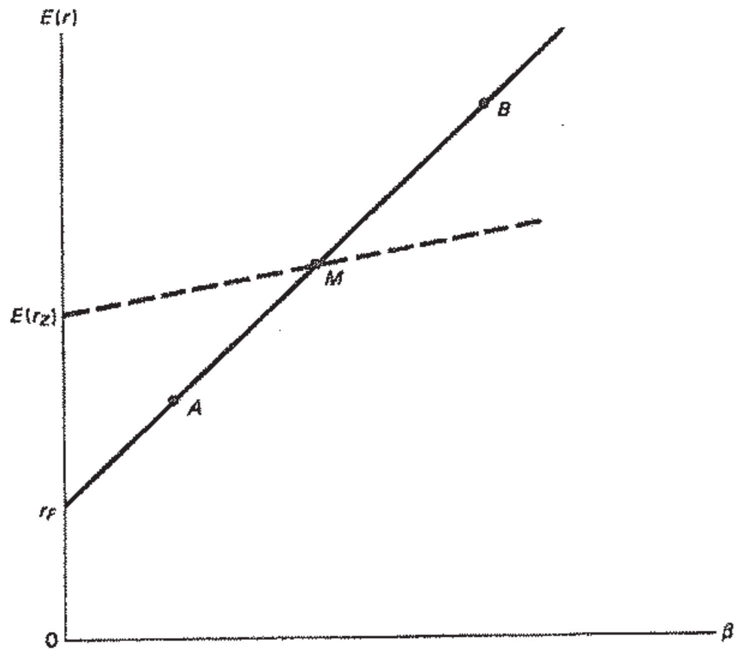
13. (a) $S_A = S_M = \frac{0.1-0.04}{0.4} = 0.15$, $S_B = 0.1$. Fund A is superior to fund B in terms of Sharpe ratio.

(b) If there is no borrowing of riskfree asset, then the CML may bent over to a hyperbola from the position of the market portfolio. As a result, A would be above the CML which indicates a superior performance relative to the efficient frontier.

14. (a) A sizable positive value of Jensen Alpha of 3% indicates superior performance.

    (b) A major problem with the Jensen Alpha is that it is only sensitive to depth, but not breadth. In effect, a "lucky" manager who invested all his funds in one stock and was thus not diversified against risk could have gotten a great return for one year on his investment and be ranked the same as a superior manager who diversified against risk and captured great returns on many securities in his portfolio.

15. Depicting this graphically, suppose you estimated the security market line as the solid line below, with both the low-beta portfolio A and the high-beta portfolio B positioned on the estimated security line. Note that the portfolio at A has performed below the true security market line, giving it a negative Jensen Alpha, while the portfolio B has performed better than the true security market line, thus giving it a positive Jensen Alpha. Your evaluations of the managers would also change accordingly, with the manager B having outperformed manager A.



16. (a) *Jensen Alpha*
    Advantages: It is an indicator adapted to market performance and insensitive to market scenarios. It is insensitive to fund risk.
    Disadvantages: It reflects only the depth of investment performance, but fails to incorporate the breadth. It can be used for performance evaluation for portfolios that are well diversified and have equal beta.

    (b) *Treynor Ratio*
    Advantages: It is equivalent to the beta-adjusted Jensen Alpha. It enables us to compare well-diversified portfolios with different betas.
    Disadvantages: The Treynor Index does not reflect breadth. It can be used to compare well-diversified portfolios with different betas. It can also be used to evaluate performance of well-diversified portfolios relative to the market, by comparing T_P with the model predicted beta-adjusted excess return $E[r_M] - r$.

    (c) *Sharp Ratio*
    Advantages: The Sharp Ratio is based on the total risk and it enables us to evaluate

the relative performance of portfolios that are not well-diversified and have different risk levels. It is sensitive to both depth and breadth, where the latter is captured by the diversified risk.

Disadvantages: Sharp Ratio may produce counterintuitive rankings. If the excess returns of two funds are equal and negative, the fund with higher standard deviation (risk) receives a higher rank. It can be used to rank portfolios with different levels of risk levels and portfolios that are not very well-diversified. It can also be used to evaluate portfolios that constitute an individual's total personal wealth.