MATH4512 – Fundamentals of Mathematical Finance

Topic Four — Utility optimization and stochastic dominance for investment decisions

4.1 Optimal long-term investment criterion – log utility criterion

4.2 Axiomatic approach to the construction of utility functions

4.3 Maximum expected utility criterion

4.4 Characterization of utility functions

4.5 Stochastic dominance

4.1 Optimal long-term investment strategy – log utility

Suppose there is an investment opportunity that the investor will either double her investment or return nothing. The probability of the favorable outcome is p. Suppose the investor has an initial capital of X_0, and she can repeat this investment many times. How much should she invest at each time in order to maximize the long-term growth of capital?

Statement of the problem

Let α be the proportion of capital invested during each play. The investor would like to find the optimal value of α which maximizes the long-term growth. The possible proportional changes are given by

$$\begin{cases} 1+\alpha & \text{if outcome is favorable} \\ 1-\alpha & \text{if outcome is unfavorable} \end{cases}, \qquad 0 \leq \alpha \leq 1.$$

General formulation

Let X_k represent the capital after the k^{th} trial, then

$$X_k = R_k X_{k-1}$$

where R_k is the random return variable.

We assume that all R_k's have *identical probability distribution* and they are *mutually independent*. The capital at the end of n trials is

$$X_n = R_n R_{n-1} \cdots R_2 R_1 X_0.$$

Taking logarithm on both sides

$$\ln X_n = \ln X_0 + \sum_{k=1}^{n} \ln R_k$$

or

$$\ln\left(\frac{X_n}{X_0}\right)^{1/n} = \frac{1}{n}\sum_{k=1}^{n} \ln R_k.$$

Since the random variables $\ln R_k$ are independent and have identical probability distribution, by the law of large numbers, the sample average tends to the true mean. We have

$$\frac{1}{n}\sum_{k=1}^{n} \ln R_k \longrightarrow E[\ln R_1], \text{ as } \ n \to \infty.$$

Remark

Since the expected value of $\ln R_k$ is independent of k, so we simply consider $E[\ln R_1]$. Suppose we write $m = E[\ln R_1]$, we have

$$\left(\frac{X_n}{X_0}\right)^{1/n} \longrightarrow e^m \quad \text{or} \quad X_n \longrightarrow X_0 e^{mn}.$$

For asymptotically large n, the capital grows exponentially with n at a rate m. Here, e^m is the growth factor for each investment period.

Log utility of single-period investment model

$$m + \ln X_0 = E[\ln R_1] + \ln X_0 = E[\ln R_1 X_0] = E[\ln X_1].$$

If we define the log utility form: $U(x) = \ln x$, then the problem of maximizing the growth rate m in the long-term investment strategy is equivalent to maximizing the expected utility $E[U(X_1)]$ of single-period terminal wealth.

Essentially, we transform the optimal long-term investment growth problem into a single-period model. The single-period maximization of log utility of terminal wealth guarantees the maximum growth of wealth in the long run.

Back to the investment strategy problem, how to find the optimal value of α such that the growth factor e^m, or equivalently, m is maximized:

$$m = E[\ln R_1] = p \ln(1+\alpha) + (1-p)\ln(1-\alpha).$$

The decision variable is α. Setting $\dfrac{dm}{d\alpha} = 0$, we obtain

$$p(1-\alpha) - (1-p)(1+\alpha) = 0$$

giving $\alpha = 2p - 1$.

Suppose we require $\alpha \geq 0$, then the existence of the above solution implicitly requires $p \geq 0.5$.

What happen when $p < 0.5$? The value for α for optimal growth is given by $\alpha = 0$ since m is a decreasing function of α when $\alpha \geq 0$.

Lesson learnt If the game is unfavorable to the player, then he should stay away from the game.

Example (volatility pumping)

Stock: In each period, its value either doubles or reduces by half.

riskless asset: just retain its value.

How to use these two instruments in combination to achieve growth?

$$\text{Return vector } \boldsymbol{R} = \begin{cases} \left(\frac{1}{2} \quad 1\right) & \text{if stock price goes down} \\ (2 \quad 1) & \text{if stock price goes up} \end{cases}.$$

Strategy of 50-50 portfolio

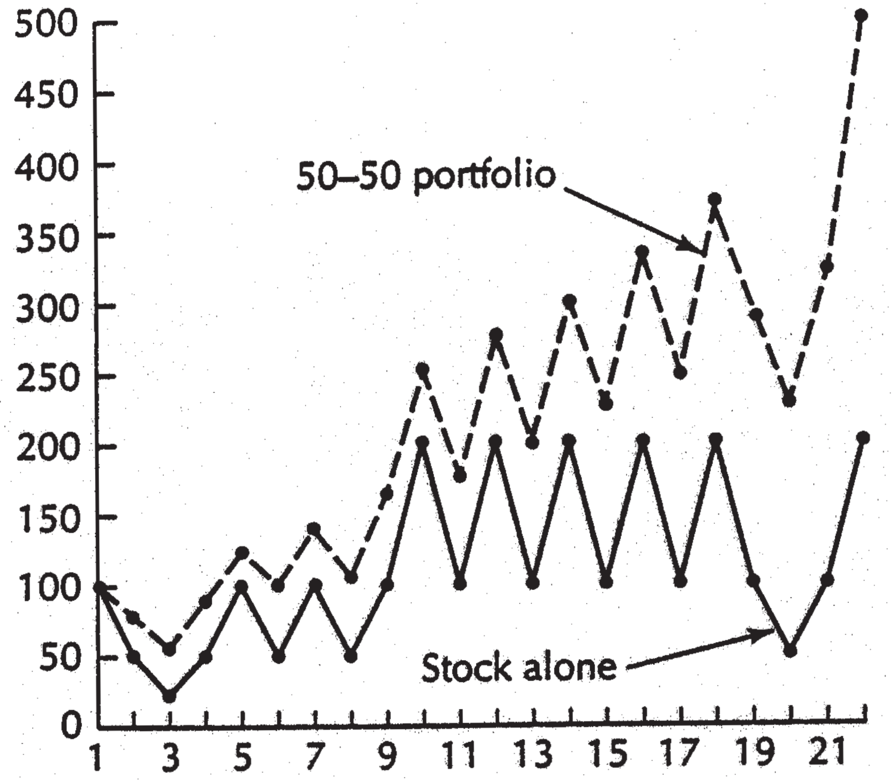
Invest *one half* of the capital in each asset for every period. Do the rebalancing at the beginning of each period so that one half of the capital is invested in each asset.

The expected growth rate

$$m = \underset{\substack{\uparrow \\ \text{prob of doubling}}}{\frac{1}{2}} \ln\left(\frac{1}{2}+1\right) + \underset{\substack{\uparrow \\ \text{prob of halving}}}{\frac{1}{2}} \ln\left(\frac{1}{2}+\frac{1}{4}\right) \approx 0.059.$$

We obtain $e^m \approx 1.0607$, so the gain on the portfolio is about 6% per period.

Remark This strategy follows the dictum of "buy low and sell high" via the process of rebalancing.



Combination of 50-50 **portfolio of risky stock and riskless asset gives an enhanced growth.**

Example (equal weight portfolio strategy)

Both risky assets either double or halve in value over each period with probability 1/2; and the price moves over successive periods are independent. Suppose we invest one half of the capital in each asset, and rebalance at the end of each period. The expected growth rate of the portfolio is found to be

$$m = \frac{1}{4}\ln 2 + \frac{1}{2}\ln\frac{5}{4} + \frac{1}{4}\ln\frac{1}{2} = \frac{1}{2}\ln\frac{5}{4} = 0.1116,$$

so that $e^m = \sqrt{\frac{5}{4}} = 1.118$. This gives an 11.8% growth rate for each period.

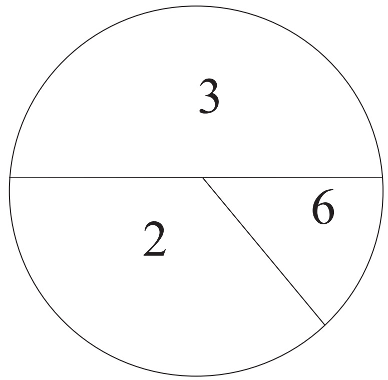
Remark Advantage of the index tracking fund, say, Dow Jones Industrial Average. The index automatically

(i) exercises stock splitting,

(ii) get rids of the weaker performers periodically.

Investment wheel

The number shown in a sector is the payoff for one-dollar investment on that sector.



1. Top sector: paying 3 to 1, though the area is 1/2 of the whole wheel (favorable odds).

2. Lower left sector: paying only 2 to 1 for an area of 1/3 of wheel (unfavorable odds).

3. Lower right sector: paying 6 to 1 for an area of 1/6 of the wheel (even odds).

Aggressive strategy

Invest all money in the top sector. This produces the highest single-period expected return. This is too risky for long-term investment! Why? The investor *goes broke* half of the time and cannot continue with later spins.

Fixed proportion strategy

Prescribe wealth proportions to each sector; apportion current wealth among the sectors as bets at each spin.

$$(\alpha_1, \alpha_2, \alpha_3) \text{ where } \alpha_i \geq 0 \quad \text{and} \quad \alpha_1 + \alpha_2 + \alpha_3 \leq 1.$$

α_1: top sector

α_2: lower left sector

α_3: lower right sector

If "top" occurs, $R(\omega_1) = 1 + 2\alpha_1 - \alpha_2 - \alpha_3$.

If "bottom left" occurs, $R(\omega_2) = 1 - \alpha_1 + \alpha_2 - \alpha_3$.

If "bottom right" occurs, $R(\omega_3) = 1 - \alpha_1 - \alpha_2 + 5\alpha_3$.

The expected value of the log return is given by

$$m = \frac{1}{2}\ln(1+2\alpha_1-\alpha_2-\alpha_3)+\frac{1}{3}\ln(1-\alpha_1+\alpha_2-\alpha_3)+\frac{1}{6}\ln(1-\alpha_1-\alpha_2+5\alpha_3).$$

To maximize m, we compute $\dfrac{\partial m}{\partial \alpha_i}, i = 1, 2, 3$, and set them be zero:

$$\frac{2}{2(1+2\alpha_1-\alpha_2-\alpha_3)} - \frac{1}{3(1-\alpha_1+\alpha_2-\alpha_3)} - \frac{1}{6(1-\alpha_1-\alpha_2+5\alpha_3)} = 0$$

$$\frac{-1}{2(1+2\alpha_1-\alpha_2-\alpha_3)} + \frac{1}{3(1-\alpha_1+\alpha_2-\alpha_3)} - \frac{1}{6(1-\alpha_1-\alpha_2+5\alpha_3)} = 0$$

$$\frac{-1}{2(1+2\alpha_1-\alpha_2-\alpha_3)} - \frac{1}{3(1-\alpha_1+\alpha_2-\alpha_3)} + \frac{5}{6(1-\alpha_1-\alpha_2+5\alpha_3)} = 0.$$

There is a whole family of optimal solutions, and it can be shown that they all give the same value for m.

(i) $\alpha_1 = 1/2, \alpha_2 = 1/3, \alpha_3 = 1/6$

One should invest in every sector of the wheel, and the bet proportions are equal to the probabilities of occurrence. Now,

$$m = \frac{1}{2}\ln\frac{3}{2} + \frac{1}{3}\ln\frac{2}{3} + \frac{1}{6}\ln 1 = \frac{1}{6}\ln\frac{3}{2}$$

so $e^m \approx 1.06991$ (a growth rate of about 7%).

Remark: Betting on the unfavorable sector is like buying insurance.

(ii) $\alpha_1 = 5/18, \alpha_2 = 0$ and $\alpha_3 = 1/18$.

Nothing is invested in the unfavorable sector. Note that $\alpha_1 + \alpha_2 + \alpha_3 < 1$ in this case. The corresponding value of m is also equal to $\frac{1}{6}\ln\frac{3}{2}$.

Log utility and growth function

Let $\boldsymbol{w}_i = (w_{i1} \cdots w_{in})^T$ be the weight vector of holding n risky securities at the i^{th} period, where weight is defined in terms of wealth. Write the random return vector at the i^{th} period as $\boldsymbol{R}_i = (R_{i1} \cdots R_{in})^T$. Here, R_{ij} is the random return of holding the j^{th} security after the i^{th} play.

Write S_m as the total return of the portfolio after m periods:

$$S_m = \prod_{i=1}^{m} \boldsymbol{w}_i \cdot \boldsymbol{R}_i.$$

Define $B = \{\boldsymbol{w} \in \mathbb{R}^n : \mathbf{1} \cdot \boldsymbol{w} \leq 1 \text{ and } \boldsymbol{w} \geq \mathbf{0}\}$, where $\mathbf{1} = (1 \cdots 1)^T$. This represents a trading strategy that does not allow short selling. When the successive plays are identical, we may drop the dependence on i by assuming that the gambler follows the same strategy for all plays.

Single-period growth function

Based on the *log-utility* criterion, we define the *growth function* by

$$W(\boldsymbol{w}; F) = E[\ln(\boldsymbol{w} \cdot \boldsymbol{R})],$$

where $F(\boldsymbol{R})$ is the distribution function of the stochastic return vector $\boldsymbol{R}$. The growth function is seen to be a function of the *trading strategy* $\boldsymbol{w}$ together with dependence on F. The optimal growth function is defined by

$$W^*(F) = \max_{\boldsymbol{w} \in B} W(\boldsymbol{w}; F).$$

Remark

To achieve the maximization of the long-term growth, we maximize $E[\ln(\boldsymbol{w} \cdot \boldsymbol{R})]$ instead of $E[\boldsymbol{w} \cdot \boldsymbol{R}]$. The maximization of $E[\boldsymbol{w} \cdot \boldsymbol{R}]$ is the optimal strategy for single play of the game.

Betting wheel revisited

Let the payoff upon the occurrence of the i^{th} event (denoted by ω_i, which corresponds to the pointer landing on the i^{th} sector) be $(0 \cdots a_i \cdot 0)^T$ with probability p_i. That is, $\boldsymbol{R}(\omega_i) = (0 \cdots a_i \cdot 0)^T$. Take the earlier example, the random return vector is given by

$$\begin{aligned}
\boldsymbol{R}(\omega_1) &= (3 \quad 0 \quad 0)^T \\
\boldsymbol{R}(\omega_2) &= (0 \quad 2 \quad 0)^T \\
\boldsymbol{R}(\omega_3) &= (0 \quad 0 \quad 6)^T.
\end{aligned}$$

$\omega_1 =$ top sector, $\omega_2 =$ bottom left sector, $\omega_3 =$ bottom right sector.

For this betting wheel game, the gambler betting on the i^{th} sector (equivalent to investment on security i) is paid a_i if the pointer lands on the i^{th} sector and loses the whole bet if otherwise.

Suppose the gambler chooses the weights $\boldsymbol{w} = (w_1 \cdots w_n)^T$ as the betting strategy with $\sum_{i=1}^{n} w_i = 1$, then

$$\begin{aligned} W(\boldsymbol{w}; F) &= \sum_{i=1}^{n} p_i \ln(\boldsymbol{w} \cdot \boldsymbol{R}(\omega_i)) = \sum_{i=1}^{n} p_i \ln w_i a_i \\ &= \sum_{i=1}^{n} p_i \ln \frac{w_i}{p_i} + \sum_{i=1}^{n} p_i \ln p_i + \sum_{i=1}^{n} p_i \ln a_i, \end{aligned}$$

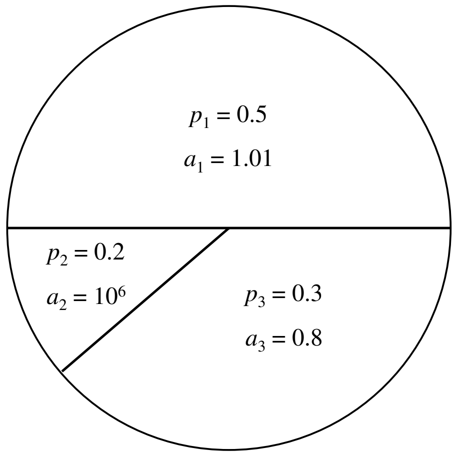
where the last two terms are known quantities. Using the inequality: $\ln x \leq x - 1$ for $x > 0$, with equality holds when $x = 1$, we have

$$\sum_{i=1}^{n} p_i \ln \frac{w_i}{p_i} \leq \sum_{i=1}^{n} p_i \left(\frac{w_i}{p_i} - 1 \right) = \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} p_i = 0.$$

The upper bound of $\sum_{i=1}^{n} p_i \ln \frac{w_i}{p_i}$ is zero, and this maximum value is achieved when we choose $w_i = p_i$ for all i. It also occurs that $\sum_{i=1}^{n} w_i = 1$. Therefore, an optimal betting strategy within B is $w_i^* = p_i$, for all i; and $W(\boldsymbol{w}^*; F) = \sum_{i=1}^{n} p_i \ln p_i a_i$.

Remarks

1. Consider the following example



   Though the return of the second sector is highly favorable, we still apportion only $w_2 = 0.2$ to this sector, given that our goal is to achieve the long-term growth. However, if we would like to maximize the one-period return, we should place all bets in the second sector.

2. An optimal long-term strategy as characterized by w_i depends on p_i but not a_i. The growth function $W(\boldsymbol{w}; F)$ surely depends on a_i.

4.2 Axiomatic approach to the construction of utility functions

How do we rank the following 4 investment choices?

Investment A		Investment B		Investment C		Investment D	
x	$p(x)$	x	$p(x)$	x	$p(x)$	x	$p(x)$
4	1	5	1	-5	$1/4$	-10	$1/5$
				0	$1/2$	10	$1/5$
				40	$1/4$	20	$2/5$
						30	$1/5$

When there is no risk, we choose the investment with the highest rate of return. — *Maximum Return Criterion*.

e.g. Investment B dominates Investment A, but this criterion fails to compare Investment B with Investment C.

Identify the investment with the highest expected return by comparing

$$E_C(x) = \frac{1}{4}(-5) + \frac{1}{2}(0) + \frac{1}{4}(40) = 8.75$$
$$E_D(x) = \frac{1}{5}(-10) + \frac{1}{5}(10) + \frac{2}{5}(20) + \frac{1}{5}(30) = 14.$$

According to the maximum expected return criterion, D is preferred over C. However, some investors may prefer C on the ground that it has a smaller downside loss of -5 and a higher upside gain of 40.

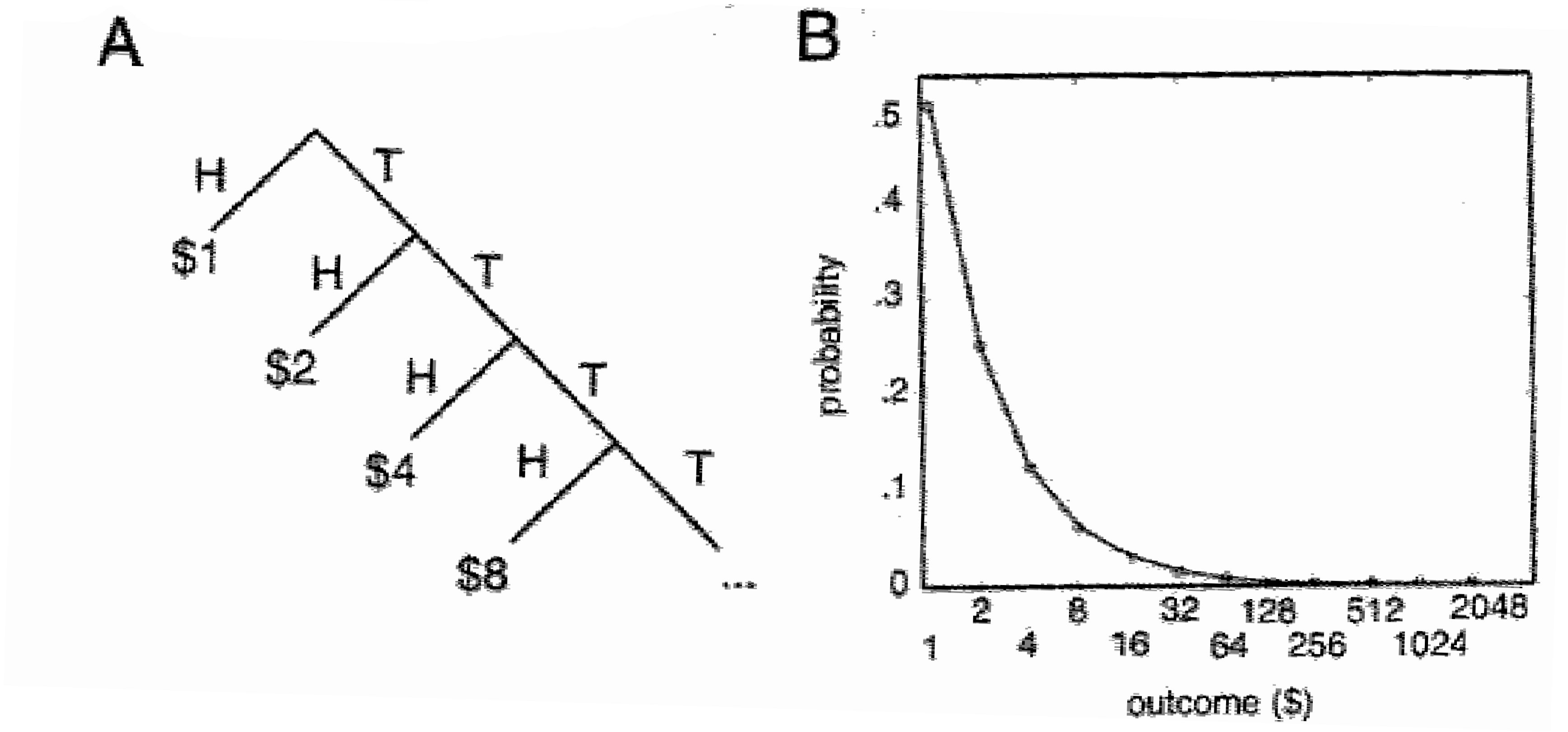
Expected value criterion is not sufficient. How to construct a mathematical function that is used to correct the expected value (with dependence only on probability) to account for the risk appetite of an individual investor into the decision procedure? The risk appetite changes with respect to the wealth level of the investor.

St Petersburg paradox (failure of Maximum Expected Return Criterion)

- Published by Bernuolli in the St Petersburg Academy Proceedings (1738)

Tossing of a fair coin until the first head shows up. The prize is 2^{k-1}, where k is the number of tosses until the first head shows up (the game is then ended). For example, suppose the head shows up in the first toss, the price is 1. This occurs with probability $\frac{1}{2}$ for a fair coin. There is a very small chance to receive a large sum of money, which occurs when k is large. There is no upper bound on the potential rewards from very low probability events.

Expected prize of the game $= \sum_{k=1}^{\infty} \frac{1}{2^k} 2^{k-1} = \infty$.



A. Outcome tree for the St. Petersburg gamble. The St. Petersburg gamble consists of a series of coin flips offering a 50% chance of \$1, a 25% chance of \$2, a 12.5% chance of \$4, and so on. The gamble may continue indefinitely.

B. The probability of each possible outcome decreases as a function of the outcome amount. The probability of a large reward is very low, but not zero.

- The decision criterion which takes only the expected value into account would recommend a course of action that no (real) rational person would be willing to take.

- Given the finite resources of the participants, people can only buy a lottery with a finite price. On the other hand, sellers would not produce a lottery whose potential loss were unacceptable. One simply cannot buy that which is not sold.

  If the total resources (or maximum jackpot) of the casino is W, then the expected value of the lottery is

$$E = \sum_{k=1}^{\infty} \frac{1}{2^k} \min(2^{k-1}, W).$$

  Suppose $2^{L-1} \le W < 2^L$, that is, $L = 1 + \text{floor}(\log_2 W)$. With the very low probability events neglected, we have

$$E = \sum_{k=1}^{L} \frac{1}{2^k} 2^{k-1} + \sum_{k=L+1}^{\infty} \frac{1}{2^k} W = \frac{L}{2} + \frac{W}{2^L}.$$

The following table shows the expected value E of the game with various potential backers and their bankroll W

Backer	*Bankroll*	*Expected value of lottery*
Friendly game	\$100	\$4.28
Millionaire	\$100,000,000	\$10.95
Billionaire	\$1,000,000,000	\$15.93
Bill Gates (2008)	\$58,000,000,000	\$18.84
U.S. GDP (2007)	\$13.8 trillion	\$22.79
World GDP (2007)	\$54.3 trillion	\$23.77
Googolaire	\$$10^{100}$	\$166.50

Notes: The estimated net worth of Bill Gates is from Forbes. The GDP data are as estimated for 2007 by the International Monetary Fund, where one trillion dollars equals \$$10^{12}$. A "googolaire" is a hypothetical person worth a googol dollars (\$$10^{100}$).

Is the expected payoff of the St. Petersburg gamble infinite?

Buffon (1777) had a child play the St. Petersburg game $2,048$ times.

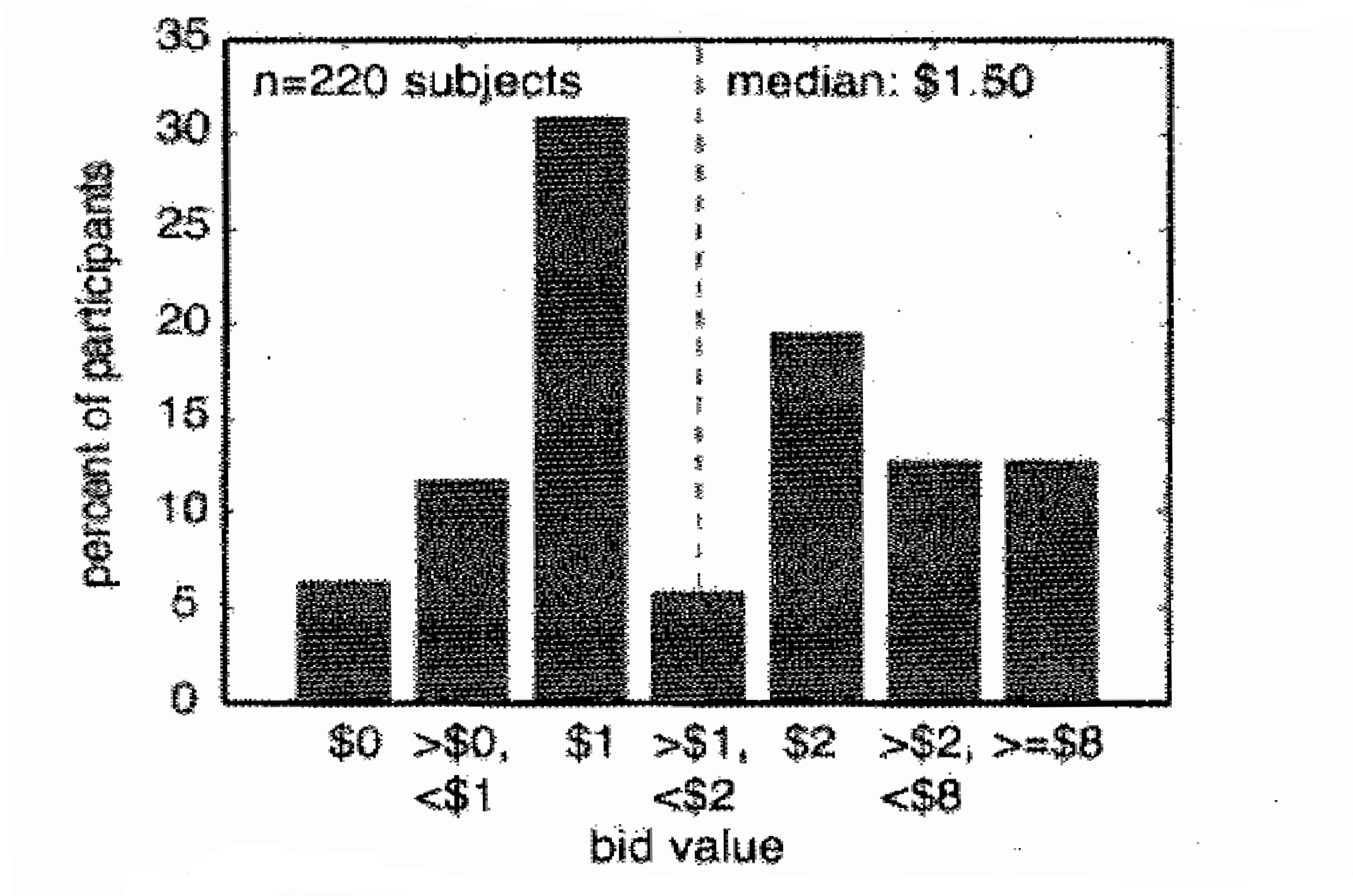
Tosses (k)	*Frequency*	Payoff (2^{k-1})
1	1061	1
2	494	2
3	232	4
4	137	8
5	56	16
6	29	32
7	25	64
8	8	128
9	6	256

Based on the above, Buffon concluded that the St. Petersburg game becomes fair with an entrance fee of approximately \$5.

- In more recent times, computers have made it possible to simulate coin flips more rapidly. Though estimated values are higher, the fundamental result does not change.

- Statistically, expected value is the central tendency of the distribution embodied in a risky game. For highly non-Gaussian distributions, the mean is not considered a valid estimator. Some researchers conclude that the true expected value of the St. Petersburg gamble is undefined, but not infinite.

- An alternative estimator of central tendency is median, which is robust to noise and favored for highly skewed distributions. The median of the distribution associated with the St. Petersburg gamble is between \$1 and \$2. Apparently, people estimate the value of the gamble using the median.



Histogram of bids offered for the standard St. Petersburg paradox. Although the expected value of the gamble is infinite, all bids were finite. The median bid was $1.50. The distribution was bimodal, with large modes at $1 and $2. Bids ranged from zero to $50,000.

Survey results on opinion polls

Category	Comment type	Example	#	%
1	Utility Curve	\$10 — "If my utility function were not concave I'd rather pay an infinite amount of money."	8	3.2
2	Finitude/trust	\$0 — "I would never expect to be offered this gamble, and if offered it would not play, suspecting a trick."	8	3.2
3	Risk/gambling aversion	\$1.5 — "A completely rational person would pay anything to play the game, because the expected utility is infinite… I'm risking less than that because I'm risk (or loss) adverse."	29	11.4
4	Performed simulation	\$2 — "Determined by flipping a coin in my pocket a few times - took 30 seconds … it's quicker than figuring the math."	13	5.12
5	Median / chance of given outcome	\$8 — "I would have a 12.5% chance of at least breaking even, and a 6.25% chance of at least doubling my money."	105	41.3
6	Other / irrelevant	"I'd be interested to see the results of this survey."	91	35.8

Preference relation and utility function

Building block – Pairwise comparison

Consider the set of alternatives B, how to determine which element in the choice set B that is preferred?

The individual first considers two arbitrary elements: $x_1, x_2 \in B$. He then picks the preferred element x_1 and discards the other. From the remaining elements, he picks the third one and compares with the winner. The process continues and the best choice among all alternatives is identified.

Choice set and preference relation

Let the choice set B be a *convex* subset of the n-dimensional Euclidean space. The component $x^{(i)}$ of the n-dimensional vector x may represent $x^{(i)}$ units of commodity i. By convex, we mean that if $x_1, x_2 \in B$, then $\alpha x_1 + (1-\alpha)x_2 \in B$ for any $\alpha \in [0,1]$.

- An individual is endowed with a *preference relation*, $\succeq$, for determining the preference between 2 elements.

- Given any elements x_1 and $x_2 \in B, x_1 \succeq x_2$ means either that x_1 is preferred to x_2 or that x_1 is indifferent to x_2.

Three axioms for $\succeq$

Reflexivity

For any $x_1 \in B, x_1 \succeq x_1$.

Comparability

For any $x_1, x_2 \in B$, either $x_1 \succeq x_2$ or $x_2 \succeq x_1$.

Transitivity

For $x_1, x_2, x_3 \in B$, given $x_1 \succeq x_2$ and $x_2 \succeq x_3$, then $x_1 \succeq x_3$.

Remarks

1. Without the comparability axiom, an individual could not determine an optimal choice. There would exist at least two elements of B between which the individual could not discriminate.
2. The transitivity axiom ensures that the choices are consistent.

Example 1 – Total quantity

Let $B = \{(x, y) : x \in [0, \infty)$ and $y \in [0, \infty)\}$ represent the set of alternatives. Let x represent ounces of orange soda and y represent ounces of grape soda. It is easily seen that B is a convex subset of $\mathbb{R}^2$.

Suppose the individual is concerned only with the total quantity of soda available, the more the better, then the individual is endowed with the following preference relation:

For $(x_1, y_1), (x_2, y_2) \in B$,

$$(x_1, y_1) \succeq (x_2, y_2) \text{ if and only if } x_1 + y_1 \geq x_2 + y_2.$$

Example 2 – Dictionary order

Let the choice set $B = \{(x, y) : x \in [0, \infty), y \in [0, \infty)\}$, the dictionary order $\succeq$ is defined as follows:

Suppose $(x_1, y_1) \in B$ and $(x_2, y_2) \in B$, then

$$(x_1, y_1) \succeq (x_2, y_2) \text{ if and only if}$$
$$[x_1 > x_2] \text{ or } [x_1 = x_2 \text{ and } y_1 \geq y_2].$$

It is easy to check that the dictionary order satisfies the three basic axioms of a preference relation.

Definition

Given $x, y \in B$ and a preference relation $\succeq$ satisfying the above three axioms.

1. x is indifferent to y, written as

$$x \sim y \quad \text{if and only if} \quad x \succeq y \text{ and } y \succeq x.$$

2. x is strictly preferred to y, written as

$$x \succ y \quad \text{if and only if} \quad x \succeq y \text{ but not } x \sim y.$$

Axiom 4 – Order Preserving

For any $x, y \in B$ where $x \succ y$ and $\alpha, \beta \in [0, 1]$,

$$[\alpha x + (1 - \alpha)y] \succ [\beta x + (1 - \beta)y] \quad \text{if and only if } \alpha > \beta.$$

Example 1 revisited – checking the Order Preserving Axiom

Recall the preference relation defined in Example 1, we take (x_1, y_1), $(x_2, y_2) \in B$ such that $(x_1, y_1) \succ (x_2, y_2)$ so that $x_1 + y_1 - x_2 - y_2 > 0$.

Take $\alpha, \beta \in [0, 1]$ such that $\alpha > \beta$, and observe

$$\alpha[(x_1 + y_1) - (x_2 + y_2)] > \beta[(x_1 + y_1) - (x_2 + y_2)].$$

Adding $x_2 + y_2$ to both sides, we obtain

$$\alpha(x_1 + y_1) + (1 - \alpha)(x_2 + y_2) > \beta(x_1 + y_1) + (1 - \beta)(x_2 + y_2).$$

Axiom 5 – Intermediate Value

For any $x, y, z \in B$, if $x \succ y \succ z$, then there exists a *unique* $\alpha \in (0, 1)$ such that

$$\alpha x + (1 - \alpha)z \sim y.$$

Remark

Given 3 alternatives with rankings of $x \succ y \succ z$, there exists a convex combination of x and z that is indifferent to y. Trade-offs between the alternatives exist.

Example 1 revisited – checking the Intermediate Value Axiom

Given $x_1 + y_1 > x_2 + y_2 > x_3 + y_3$, we choose

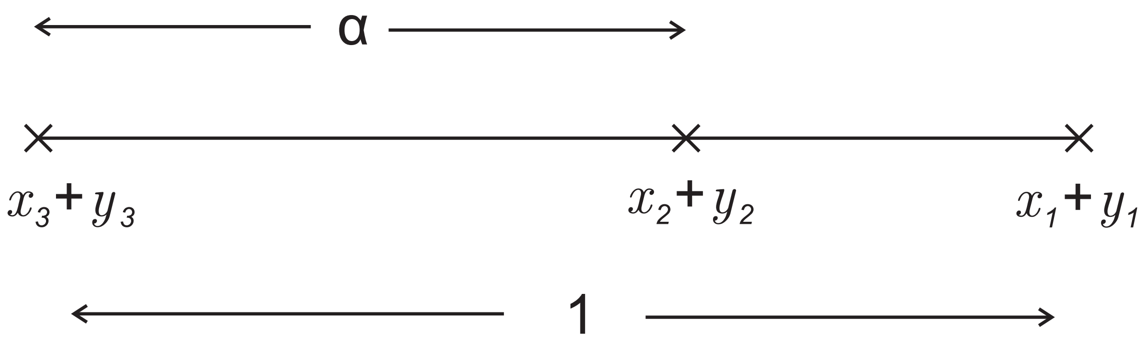
$$\alpha = \frac{(x_2 + y_2) - (x_3 + y_3)}{(x_1 + y_1) - (x_3 + y_3)}.$$

Rearranging gives

$$\alpha(x_1 + y_1) + (1 - \alpha)(x_3 + y_3) = x_2 + y_2$$

so that

$$[\alpha(x_1, y_1) + (1 - \alpha)(x_3, y_3)] \sim (x_2, y_2).$$



When $x_2 + y_2$ is getting closer to $x_3 + y_3$, α becomes smaller.

Dictionary order does not satisfy the intermediate value axiom

We quote a counter example. Suppose $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in B$ such that $(x_1, y_1) \succ (x_2, y_2) \succ (x_3, y_3)$ and $x_1 > x_2 = x_3$ and $y_2 > y_3$. For any $\alpha \in (0, 1)$, we consider the convex combination

$$\begin{aligned} & \alpha(x_1, y_1) + (1-\alpha)(x_3, y_3) \\ = \; & \alpha(x_1, y_1) + (1-\alpha)(x_2, y_3) \\ = \; & (\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_3). \end{aligned}$$

But for $\alpha > 0$, we have $\alpha x_1 + (1-\alpha)x_2 > x_2$ so

$$\alpha(x_1, y_1) + (1-\alpha)(x_3, y_3) \succ (x_2, y_2) \quad \text{for all} \quad \alpha \in (0, 1).$$

In other words, there does not exist $\alpha \in (0, 1)$ such that

$$\alpha x + (1-\alpha)z \sim y.$$

Axiom 6 – Boundedness

There exist $x^*, y^* \in B$ such that $x^* \succeq z \succeq y^*$ for all $z \in B$.

- This Axiom ensures the existence of a most preferred element $x^* \in B$ and a least preferred element $y^* \in B$.

Example 1 revisited – checking the Boundedness Axiom

Recall $B = \{(x, y) : x \in [0, \infty) \text{ and } y \in [0, \infty)\}$. Given any $(z_1, z_2) \in B$, we have

$$(z_1 + 1, z_2) \succeq (z_1, z_2) \text{ since } z_1 + z_2 + 1 > z_1 + z_2.$$

Therefore, a maximum does not exist.

Motivation for defining utility

Knowledge of the preference relation $\succeq$ effectively requires a complete listing of preferences over all pairs of elements from the choice set B. We define a *utility* function that assigns a numeric value to each element of the choice set such that a larger numeric value implies a higher preference.

- Firstly, we establish the theorem on the existence of utility function.

- Next, we show that the optimal criterion for ranking alternative investments is based on the ranking of the expected utility values of various investments.

Theorem – Existence of Utility Function

Let B denote the set of payoffs from a finite number of choices, also being a convex subset of $\mathbf{R}^n$. Let $\succeq$ denote a preference relation on B. Suppose $\succeq$ satisfies the following axioms

(i) $\forall x \in B, x \succeq x$.

(ii) $\forall x, y \in B, x \succeq y$ or $y \succeq x$.

(iii) For any $x, y, z \in B$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

(iv) For any $x, y \in B, x \succeq y$ and $\alpha, \beta \in [0, 1]$,

$$\alpha x + (1 - \alpha)y \succeq \beta x + (1 - \beta)y \quad \text{if and only if} \quad \alpha > \beta.$$

(v) For any $x, y, z \in B$, suppose $x \succ y \succ z$, then there exists a unique $\alpha \in (0, 1)$ such that $\alpha x + (1 - \alpha)z \sim y$.

(vi) There exist $x^*, y^* \in B$ such that $\forall z \in B, x^* \succeq z \succeq y^*$.

Then there exists a utility function $U : B \to \mathbf{R}$ such that

(a) $x \succ y$ if and only if $U(x) > U(y)$.

(b) $x \sim y$ if and only if $U(x) = U(y)$.

To show the existence of $U : B \to \mathbf{R}$, we write down one such function and show that it satisfies the stated conditions.

Based on Axiom 6, we choose $x^*, y^* \in B$ such that

$$x^* \succeq z \succeq y^* \quad \text{for all} \quad z \in B.$$

Without loss of generality, let $x^* \succ y^*$. [Otherwise, $x^* \sim z \sim y^*$ for all $z \in B$. In this case, $U(z) = 0$ for all $z \in B$, which is a trivial utility function that satisfies condition (b).]

Consider an arbitrary $z \in B$. There are 3 possibilities:

1. $\quad z \sim x^*$; $\quad$ 2. $\quad x^* \succ z \succ y^*$; $\quad$ 3. $\quad z \sim y^*$.

We define U by giving its value under all 3 cases:

1. $U(z) = 1$. The most preferred element has utility value of one.

2. By Axiom 5, there exists a unique $\alpha \in (0, 1)$ such that
$$[\alpha x^* + (1 - \alpha)y^*] \sim z.$$
   Define $U(z) = \alpha$.

3. $U(z) = 0$. The least preferred element has utility value of zero.

Such U satisfies properties (a) and (b).

The Boundness Axiom gives the lower and upper bound of U. The Intermediate Value Axiom gives the utility value α. Finally, the Order Preserving Axiom gives the ranking of the alternatives based on their utility values.

Proof of property (a)

Necessity

Suppose $z_1, z_2 \in B$ are such that $z_1 \succ z_2$, we need to show

$$U(z_1) > U(z_2).$$

Consider the four possible cases.

1. $z_1 \sim x^* \succ z_2 \succ y^*$
2. $z_1 \sim x^* \succ z_2 \sim y^*$
3. $x^* \succ z_1 \succ z_2 \succ y^*$
4. $x^* \succ z_1 \succ z_2 \sim y^*$.

Case 1 By definition, $U(z_1) = 1$ and $U(z_2) = \alpha$, where $\alpha \in (0, 1)$ uniquely satisfies

$$\alpha x^* + (1 - \alpha)y^* \sim z_2.$$

Now, $U(z_1) = 1 > \alpha = U(z_2)$.

Case 2 By definition, $U(z_1) = 1 > 0 = U(z_2)$.

Case 3 By defintion, $U(z_i) = \alpha_i$, where $\alpha_i \in (0, 1)$ uniquely satisfies

$$\alpha_i x^* + (1 - \alpha_i) y^* \sim z_i,$$

so that

$$z_1 \sim [\alpha_1 x^* + (1 - \alpha_1) y^*] \text{ and } [\alpha_2 x^* + (1 - \alpha_2) y^*] \sim z_2.$$

We claim $U(z_1) = \alpha_1 > \alpha_2 = U(z_2)$. Assume not, then $\alpha_1 \leq \alpha_2$. By Axiom 4,

$$[\alpha_2 x^* + (1 - \alpha_2) y^*] \succeq [\alpha_1 x^* + (1 - \alpha_1) y^*].$$

This is a contradiction. Hence, $\alpha_1 > \alpha_2$ is true and

$$U(z_1) = \alpha_1 > U(z_2) = \alpha_2.$$

Case 4 By definition, $U(z_1) = \alpha_1$, where $\alpha_1 \in (0, 1)$ uniquely satisfies

$$\alpha_1 x^* + (1 - \alpha_1) y^* \sim y_1 \quad \text{and} \quad U(z_2) = 0.$$

We have $U(z_1) = \alpha_1 > 0 = U(z_2)$.

Sufficiency

Suppose, given $z_1, z_2 \in B$, that $U(z_1) > U(z_2)$, we would like to show $z_1 \succ z_2$. Consider the following 4 cases

1. $U(z_1) = 1$ and $U(z_2) = \alpha_2$, where $\alpha_2 \in (0,1)$ uniquely satisfies

$$[\alpha_2 x^* + (1-\alpha_2)y^*] \sim z_2.$$

2. $U(z_1) = 1$, where $z_1 \sim x^*$ and $U(z_2) = 0$, where $z_2 \sim y^*$.

3. $U(z_i) = \alpha_i$, where $\alpha_i \in (0,1)$ uniquely satisfies

$$[\alpha_i x^* + (1-\alpha_i)y^*] \sim z_i.$$

4. $U(z_1) = \alpha_1$ and $U(z_2) = 0$, where $z_2 \sim y^*$.

Case 1 $z_1 \sim x^* \sim [1 \cdot x^* + 0 \cdot y^*]$ and $z_2 \sim [\alpha_2 x^* + (1 - \alpha_2) y^*]$.

By Axiom 4, $1 > \alpha_2$ so that $z_1 \succ z_2$.

Case 2 $z_1 \sim x^* \succ y^* \sim z_2$.

Case 3 $z_1 \sim [\alpha_1 x^* + (1 - \alpha_1) y^*]$

$z_2 \sim [\alpha_2 x^* + (1 - \alpha_2) y^*]$

Since $\alpha_1 = U(z_1) > U(z_2) = \alpha_2$, by Axiom 4, $z_1 \succ z_2$.

Case 4 $z_1 \sim [\alpha_1 x^* + (1 - \alpha_1) y^*]$ and

$z_2 \sim y^* \sim [0 x^* + (1 - 0) y^*]$.

By Axiom 4 and since $\alpha_1 > 0, z_1 \succ z_2$.

Proof of Property (b)

Necessity

Suppose $z_1 \sim z_2$ but $U(z_1) \neq U(z_2)$, then

$$U(z_1) > U(z_2) \quad \text{or} \quad U(z_2) > U(z_1).$$

By property (a), this implies $z_1 \succ z_2$ or $z_2 \succ z_1$, a contradiction. Hence,

$$U(z_1) = U(z_2).$$

Sufficiency

Suppose $U(z_1) = U(z_2)$, but $z_1 \succ z_2$ or $z_1 \prec z_2$. By property (a), this implies $U(z_1) > U(z_2)$ or $U(z_2) > U(z_1)$, a contradiction. Hence, $z_1 \sim z_2$.

4.3 Maximum expected utility criterion

How to make a choice between the following two lotteries:

$$\begin{aligned} L_1 &= \{p_1, A_1; p_2, A_2; \cdots; p_n, A_n\} \\ L_2 &= \{q_1, A_1; q_2, A_2; \cdots; q_n, A_n\}? \end{aligned}$$

The outcomes are $A_1, \cdots, A_n$; p_i and q_i are the probabilities of occurrence of A_i in L_1 and L_2, respectively. These outcomes are mutually exclusive and only one outcome can be realized under each lottery. We are not limited to lotteries with the same set of outcomes. Suppose outcome A_i will not occur in Lottery L_1, we can simply set $p_i = 0$.

Comparability

When faced with two monetary outcomes A_i and A_j, the investor must say $A_i \succ A_j, A_j \succ A_i$ or $A_i \sim A_j$.

Continuity

If $A_3 \succeq A_2$ and $A_2 \succeq A_1$, then there exists unique $U(A_2)$ $[0 \leq U(A_2) \leq 1]$ such that

$$L = \{[1 - U(A_2)], A_1; U(A_2), A_3\} \sim A_2.$$

For a given set of outcomes A_1, A_2 and A_3, these probabilities are a function of A_2, hence the notation $U(A_2)$.

Why is it called continuity axiom? When $U(A_2) = 1$, we obtain $L = A_3 \succeq A_2$; when $U(A_2) = 0$, we obtain $L = A_1 \preceq A_2$. If we increase $U(A_2)$ continuously from 0 to 1, we hit a value $U(A_2)$ such that $L \sim A_2$.

Remark
Though $U(A_2)$ is a probability value, we will see that it is also the investor's utility function.

Interchangeability

Given $L_1 = \{p_1, A_1; p_2, A_2; p_3, A_3\}$ and $A_2 \sim A = \{q, A_1; (1-q), A_3\}$, the investor is indifferent between L_1 and $L_2 = \{p_1, A_1; p_2, A; p_3, A_3\}$. Note that L_2 has monetary values A_1 and A_3 and a lottery A as prizes.

Transitivity

Given $L_1 \succ L_2$ and $L_2 \succ L_3$, then $L_1 \succ L_3$.

Also, if $L_1 \sim L_2$ and $L_2 \sim L_3$, then $L_1 \sim L_3$.

Decomposability

A complex lottery has lotteries as prizes. A simple lottery has monetary values A_1, A_2, as prizes.

Consider a complex lottery $L^* = \{1 - q, L_1; q, L_2\}$, where

$$L_1 = \{p_1, A_1; (1 - p_1), A_2\} \quad \text{and} \quad L_2 = \{p_2, A_1; (1 - p_2), A_2\},$$

L^* can be decomposed into a simple lottery $L = \{p^*, A_1; (1-p^*), A_2\}$, with A_1 and A_2 as prizes where $p^* = (1 - q)p_1 + qp_2$.

The decomposability property can be extended to the generalized case. Suppose

$$L^* = \{p_1, L_1; p_2, L_2; \ldots; p_n, L_n\}$$

and

$$L_i = \{1 - q_i, A_1; q_i, A_2\}, \; i = 1, 2, \ldots, n,$$

then

$$L^* = \{\sum_{i=1}^{n} p_i(1 - q_i), A_1; \sum_{i=1}^{n} p_i q_i, A_2\}.$$

Monotonicity

(a) For monetary outcomes, $A_2 > A_1 \Longleftrightarrow A_2 \succ A_1$.

(b) For lotteries

(i) Let $L_1 = \{p, A_1; (1-p), A_2\}$ and $L_2 = \{p, A_1; (1-p), A_3\}$, $0 < p < 1$. We have $A_3 > A_2$ if and only if $A_3 \succ A_2$ and $L_2 \succ L_1$. Under the same probability of occurrence, we compare monetary outcomes.

(ii) Let $L_1 = \{p, A_1; (1-p), A_2\}$ and $L_2 = \{q, A_1; (1-q), A_2\}$, also $A_2 > A_1$ (hence $A_2 \succ A_1$). We have $p < q \Longleftrightarrow L_1 \succ L_2$. Under the same set of monetary outcomes, we compare the respective probability of occurrence.

Theorem

The optimal criterion for ranking alternative investments is the *expected utility* of the various investments, where

$$L_1 \succ L_2 \Leftrightarrow \sum p_i U(A_i) > \sum q_i U(A_i).$$

Proof

How to make a choice between L_1 and L_2

$$\begin{aligned} L_1 &= \{p_1, A_1; p_2, A_2; \cdots; p_n, A_n\} \\ L_2 &= \{q_1, A_1; q_2, A_2; \cdots; q_n, A_n\} \end{aligned}$$

where A_i are distinct monetary outcomes arranged according to $A_1 < A_2 < \cdots < A_n$?

1. By *comparability axiom*, we can compare A_i. Further, by monotonicity axiom, we determine that

$$A_1 < A_2 < \cdots < A_n \text{ implies } A_1 \prec A_2 \prec \cdots \prec A_n.$$

2. By *continuity axiom*, for every A_i, there exists $U(A_i)$ such that $A_i \sim A_i^*$. Define the lottery $A_i^* = \{[1 - U(A_i)], A_1; U(A_i), A_n\}$ where $0 \le U(A_i) \le 1$.

   For $A_1, U(A_1) = 0$, hence $A_1^* \sim A_1$; for $A_n, U(A_n) = 1$. For other $A_i, 0 < U(A_i) < 1$. By the monotonicity and transitivity axioms, $U(A_i)$ increases from zero to one as A_i increases from A_1 to A_n.

3. Substituting A_i by A_i^* in L_1 successively and by the *interchangeability axiom*, we have

$$L_1 \sim \widetilde{L}_1 = \{p_1, A_1^*; p_2, A_2^*; \cdots; p_n, A_n^*\}.$$

   The lotteries $A_1^*, A_2^*, \ldots, A_n^*$ are dependent only on the monetary outcomes A_1 and A_n.

4. Note that $\tilde{L}_1$ is a complex lottery, which consists only the two monetary outcomes A_1 and A_n. By the *decomposability axiom*, we decompose the complex lottery $\widetilde{L}_1$ as a simple lottery that is in terms of A_1 and A_n, where

$$L_1 \sim \widetilde{L}_1 \sim L_1^* = \{\Sigma p_i[1 - U(A_i)], A_1; \Sigma p_i U(A_i), A_n\}.$$

Similarly, we decompose $\widetilde{L}_2$ as a simple lottery, where

$$L_2 \sim L_2^* = \{\Sigma q_i[1 - U(A_i)], A_1; \Sigma q_i U(A_i), A_n\}.$$

5. By the *monotonicity axiom*, $L_1^* \succ L_2^*$ if and only if

$$\Sigma p_i U(A_i) > \Sigma q_i U(A_i).$$

This is precisely the *expected utility criterion*. By transitivity, we obtain $L_1 \sim L_1^* > L_2^* \sim L_2$ if and only if $\sum p_i U(A_i) > \sum q_i U(A_i)$.

Remarks

Recall $A_i \sim A_i^* = \{[1-U(A_i)], A_1; U(A_i), A_n\}$, such a function $U(A_i)$ always exists, though not all investors would agree on the specific value of $U(A_i)$.

- By the monotonicity axiom, utility is increasing. Suppose

$$\begin{aligned} A_i \sim A_i^* &= \{[1-U(A_i)], A_1; U(A_i), A_n\} \\ \succ A_j \sim A_j^* &= \{[1-U(A_j)], A_1; U(A_j), A_n\}, \end{aligned}$$

  then $U(A_i) > U(A_j)$.

- A utility function is determined up to a positive linear transformation, so its value is not limited to the range $[0, 1]$. "Determined" means that the ranking of the projects by the Maximum Expected Utility Criterion does not change.

- The absolute difference or ratio of the utilities of two investment choices gives no indication of the degree of preference of one over the other since utility values can be expanded or suppressed by a linear transformation.

4.4 Characterization of utility functions

1. More is being preferred to less: $u'(w) > 0$

2. Investors' taste for risk
   We define certainty equivalent c of a gamble with random outcome X by

$$u(c) = E[u(X)].$$

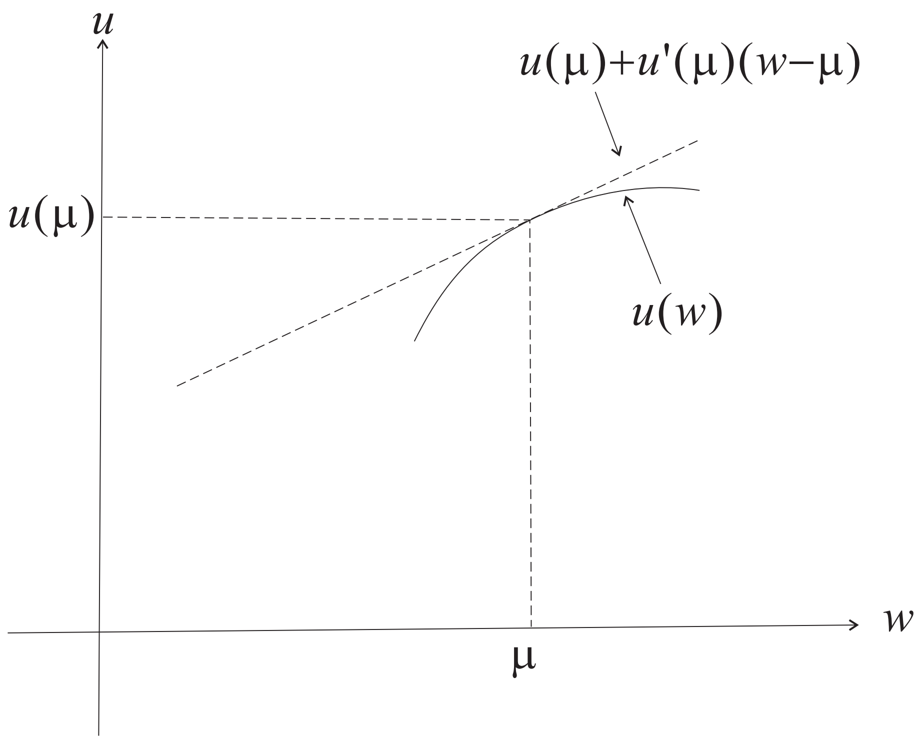
   – averse to risk (certainty equivalent $<$ expected value)
     The certainty equivalent may be visualized as the price of the game. The investor visualizes the price to be less than its expected value.
   – neutral toward risk (indifferent to a fair gamble)
   – seek risk (certainty equivalent $>$ expected value)

3. Investors' preference changes with a change in wealth. Percentage of wealth invested in risky asset changes as wealth changes.

Jensen's inequality

Suppose $u''(w) \leq 0$ and X is a random variable, then

$$u(E[X]) \geq E[u(X)].$$



Write $E[X] = \mu$; since $u(w)$ is concave, we have

$$u(w) \leq u(\mu) + u'(\mu)(w - \mu) \quad \text{for all values of } w.$$

Replace w by X and take the expectation on each side

$$E[u(X)] \leq u(\mu) = u(E[X]).$$

Interpretation

$E[u(X)]$ represents the expected utility of the gamble associated with X. The investor prefers a sure wealth of μ that is set to be equal to the expected value $E[X]$ rather than playing the game, if $u''(w) \leq 0$. This indicates *risk aversion*.

Recall that the certainty equivalent c is given by

$$u(c) = E[u(X)] \leq u(\mu)$$

so that $c \leq \mu$ since u is an increasing function. For example, a risk averse gambler prefers to receive \$4 with certainty than to play a game with expected value \$5.

Alternative viewpoint on risk aversion – Insurance premium

Individual's total initial wealth is w, and the wealth is subject to random loss Y during the period, $0 \leq Y < w$.

Let π be the insurable premium payable at time 0 that fully reimburses the loss (neglecting the time value of money).

1. If the individual decides not to buy insurance, then the expected utility is $E[u(w - Y)]$. The expectation is based on investor's *own subjective assessment* of the loss.

2. If he buys the insurance, the utility at the end of the period is $u(w - \pi)$. Note that $w - \pi$ is the sure wealth.

The fair value of insurance premium π is determined by

$$u(w - \pi) = E[u(w - Y)].$$

Recall that if the individual is risk averse [$u''(w) \leq 0$], then from Jensen's inequality (change X to $w - Y$), we obtain

$$u(w - E[Y]) \geq E[u(w - Y)].$$

Therefore, we deduce that $\pi \geq E[Y]$.

Suppose the higher moments of Y are negligible, it can be deduced that the maximum premium that a risk-averse individual with wealth w is willing to pay to avoid a possible loss of Y is approximately

$$\pi \approx \mu_Y + \frac{\sigma_Y^2}{2} R_A(w - \mu),$$

where $R_A(w) = -u''(w)/u'(w)$, $0 \leq Y < w$ and $\mu = E[Y] < w$. With higher $R_A(w)$, the individual is willing to pay a higher premium to avoid risk.

Proof

We start from the governing equation for π

$$u(w-\pi) = E[u(w-Y)].$$

We proceed to find an analytic approximation of π in powers of a small perturbation parameter. Write $Y = \mu + zV$, where V is a random variable of finite value and with zero mean. Here, z is a small perturbation parameter. This is based on the assumption that the deviation of Y from its mean value μ is small. We then have

$$u(w-\pi) = E[u(w-\mu-zV)]. \tag{1}$$

We are seeking the perturbation expansion of π in powers of z in the form

$$\pi = a + bz + cz^2 + \cdots$$

Similar to the determination of the coefficient in a Taylor series expansion, we differentiate the governing equation with respect to the parameter z at successive orders and set $z = 0$ in the resulting equation.

(i) Setting $z = 0, u(w-a) = E[u(w-\mu)] = u(w-\mu)$ so that

$$a = \mu.$$

(ii) Differentiating (1) with respect to z and setting $z = 0$, we obtain

$$-\pi'(0)u'(w-\pi) = E[-Vu'(w-\mu)]. \qquad (2)$$

Since $E[V] = 0$ and $\pi'(0) = b$, so $b = 0$.

(iii) Differentiating (1) twice with respect to z, we have

$$-\pi''(z)u'(w-\pi) + [\pi'(z)]^2 u''(w-\pi) = E[V^2 u''(w-\mu)].$$

Setting $z = 0$ and observing $\pi'(0) = 0$, we obtain

$$-\pi''(0)u'(w-\pi) \;=\; E[V^2 u''(w-\mu)].$$

Note that $\text{var}(V) = E[V^2]$ since $E[V] = 0$ and $\pi''(0) = 2c$, we obtain

$$c = -\frac{\text{var}(V)}{2}\frac{u''}{u'}\bigg|_{w-\mu}.$$

Absolute risk aversion coefficient

Define the absolute risk aversion coefficient: $R_A(w) = -\dfrac{u''(w)}{u'(w)}$, we have

$$\begin{aligned}\pi &\approx \mu + \frac{R_A(w-\mu)}{2} z^2 \text{var}(V) \\ &= \mu + \frac{\sigma_Y^2}{2} R_A(w-\mu).\end{aligned}$$

Here, $\pi - \mu \approx \dfrac{\sigma_Y^2}{2} R_A(w-\mu)$ is called the risk premium. The risk premium represents the extra amount that the insurer charges since uncertainty of the random loss is transferred from the buyer to the insurer. For low level of risks, $\pi - \mu$ is approximately proportional to the product of one half of the variance of the loss distribution and individual's absolute risk aversion coefficient.

Note that R_A is evaluated at $w - \mu = w - E[Y]$, which is the expected resulting wealth of the investor when faced with the random loss Y.

Relative risk aversion coefficient

The whole wealth w is invested into the game. Let Zw denote the outcome of the game, where Z is the random return. Write var$(Z) = \sigma_Z^2$. If the game is fair, then $E[Z] = 1$.

Choice A	Choice B
wZ	w_C (with certainty)

According to the expected utilities criterion, the investor is indifferent to these two positions if and only if

$$E[u(Zw)] = u(w_C).$$

Note that $w_C = w - (w - w_C)$, indicating the payment of $w - w_C$ for Choice B. The payment $w - w_C$ represents the certainty amount the investor would be willing to pay in order to avoid the risk of the game.

Let q be the fraction of wealth an investor is giving up in order to avoid the gamble; then $q = \frac{w - w_C}{w}$ or $w_C = w(1 - q)$. Let Z be the return per dollar invested so that for a fair gamble, $E[Z] = 1$. Write $\text{var}(Z) = \sigma_Z^2$.

Suppose we invest w dollars, the random amount at the end of the game would be wZ. Expand $u(wZ)$ around w:

$$u(wZ) = u(w) + u'(w)(wZ - w) + \frac{u''(w)}{2}(wZ - w)^2 + \cdots$$

so that the expected utility value of the terminal wealth is given by

$$E[u(wZ)] = u(w) + 0 + \frac{u''(w)}{2}w^2\sigma_Z^2 + \cdots$$

since $\sigma_Z^2 = E[(Z - 1)^2]$.

On the other hand, by Taylor series expansion, we obtain

$$u(w_C) = u(w(1-q)) = u(w) - qwu'(w) + \cdots .$$

Equating $u(w_C)$ with $E[u(wZ)]$ of their leading order terms, we obtain

$$\frac{u''(w)}{2} w^2 \sigma_Z^2 = -u'(w)qw$$

so that

$$q = -\frac{\sigma_Z^2}{2} w \frac{u''(w)}{u'(w)}.$$

Define $R_R(w)$ = coefficient of relative risk aversion $= -w\dfrac{u''(w)}{u'(w)}$, then $q = \dfrac{w - w_C}{w}$ = percentage of risk premium $= \dfrac{\sigma_Z^2}{2} R_R(w)$. Again, R_R is evaluated at the expected resulting wealth of the investor, which equals $E[Zw] = w$ since $E[Z] = 1$.

Types of utility functions

1. *Exponential utility*

$$
\begin{aligned}
u(x) &= 1 - e^{-ax}, x > 0 \\
u'(x) &= ae^{-ax} \\
u''(x) &= -a^2 e^{-ax} < 0 \quad \text{(risk aversion)}
\end{aligned}
$$

so that $R_A(x) = a$ for all wealth level x.

2. *Power utility*

$$
\begin{aligned}
u(x) &= \frac{x^\alpha - 1}{\alpha}, \quad \alpha \leq 1 \\
u'(x) &= x^{\alpha-1} \\
u''(x) &= (\alpha - 1)x^{\alpha-2}
\end{aligned}
$$

so that $R_A(x) = \dfrac{1-\alpha}{x}$ and $R_R(x) = 1 - \alpha$.

3. *Logarithmic utility* (corresponds to $\alpha \to 0$ in power utility)

$$\begin{aligned} u(x) &= a \ln x + b, \quad a > 0 \\ u'(x) &= a/x \\ u''(x) &= -a/x^2 \end{aligned}$$

so that $R_A(x) = \dfrac{1}{x}$ and $R_R(x) = 1$.

Observe that

$$\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha} = \lim_{\alpha \to 0} \frac{(\ln x) x^\alpha}{1} = \ln x.$$

Properties of the power utility functions: $U(x) = x^\alpha/\alpha, \alpha \leq 1$

(i) $\alpha > 0$, aggressive utility

Consider $\alpha = 1$, corresponding to $U(x) = x$. This is the *expected value criterion*.

Recall that the strategy that maximizes the expected value bets all capital on the most favorable sector – prone to early bankruptcy.

For $\alpha = 1/2$; consider two opportunities:

(a) capital will double with a probability of 0.9 or it will go to zero with probability 0.10,

(b) capital will increase by 25% with certainty.

Since $0.9 \times \sqrt{2} > \sqrt{1.25}$, so opportunity (a) is preferred to (b). However, opportunity (a) is certain to go bankrupt if the game is repeated many times.

(ii) $\alpha < 0$, conservative utility

For $\alpha = -1/2$, consider two opportunities

(a) capital quadruples in value with certainty

(b) with probability 0.5 capital remains constant and with probability 0.5 capital is multiplied by 10 million.

Since $-4^{-1/2} > -0.5 - 0.5(10,000,000)^{-1/2}$, opportunity (a) is preferred to (b).

Apparently, the best choice for α may be negative, but close to zero. This utility function is close to the logarithm function.

Absolute risk aversion

$$A(w) = -\frac{u''(w)}{u'(w)}$$

If $A(w)$ has the same sign for all values of w, then the investor has the same risk preference (risk averse, neutral or seeker) for all values of w (global).

Relative risk aversion

$$R(w) = -\frac{wu''(w)}{u'(w)}.$$

Note that utility functions are only unique up to a strictly positive affine transformation. The second derivative alone cannot be used to characterize the intensity of risk averse behavior. The risk aversion coefficients are invariant to a strictly positive affine transformation of individual utility function, say $\widetilde{u}(w) = au(w) + b, a > 0$. We observe $u''(w)/u'(w) = \widetilde{u}''(w)/\widetilde{u}'(w)$.

Changes in Absolute Risk Aversion with Wealth

Condition	Definition	Property of $A(w)$	Example
Increasing absolute risk aversion	As wealth increases, hold fewer dollars in risky assets	$A'(w) > 0$	w^{-Cw^2}
Constant absolute risk aversion	As wealth increases, hold same dollar amount in risky assets	$A'(w) = 0$	$-e^{-Cw}$
Decreasing absolute risk aversion	As wealth increases, hold more dollars in risky assets	$A'(w) < 0$	$\ln w$

Changes in Relative Risk Aversion with Wealth

Condition	Definition	Property of $R'(W)$	Examples of Utility Functions
Increasing relative risk aversion	Percentage invested in risky assets declines as wealth increases	$R'(w) > 0$	$w - bw^2$
Constant relative risk aversion	Percentage invested in risky assets is unchanged as wealth increases	$R'(w) = 0$	$\ln w$
Decreasing relative risk aversion	Percentage invested in risky assets increases as wealth increases	$R'(w) < 0$	$-e^{2w^{-1/2}}$

Quadratic utility and mean-variance criterion

The mean-variance criterion can be reconciled with the expected utility approach by either: (1) using a quadratic utility function, or (2) making the assumption that the random returns of the risky assets are normal random variables.

Quadratic utility

The quadratic utility function can be defined as $U(x) = ax - \frac{b}{2}x^2$, where $a > 0$ and $b > 0$. This utility function is really meaningful only in the range $x \leq a/b$, for it is in this range that the function is increasing. Note also that for $b > 0$ the function is strictly concave everywhere and thus exhibits risk aversion.

Quadratic concave utility and mean-variance criterion

mean-variance analysis $\Leftrightarrow$ maximum expected utility criterion based on quadratic concave utility (risk averse)

Suppose that a portfolio has a random terminal wealth value of y. Using the expected utility criterion, we evaluate the portfolio using

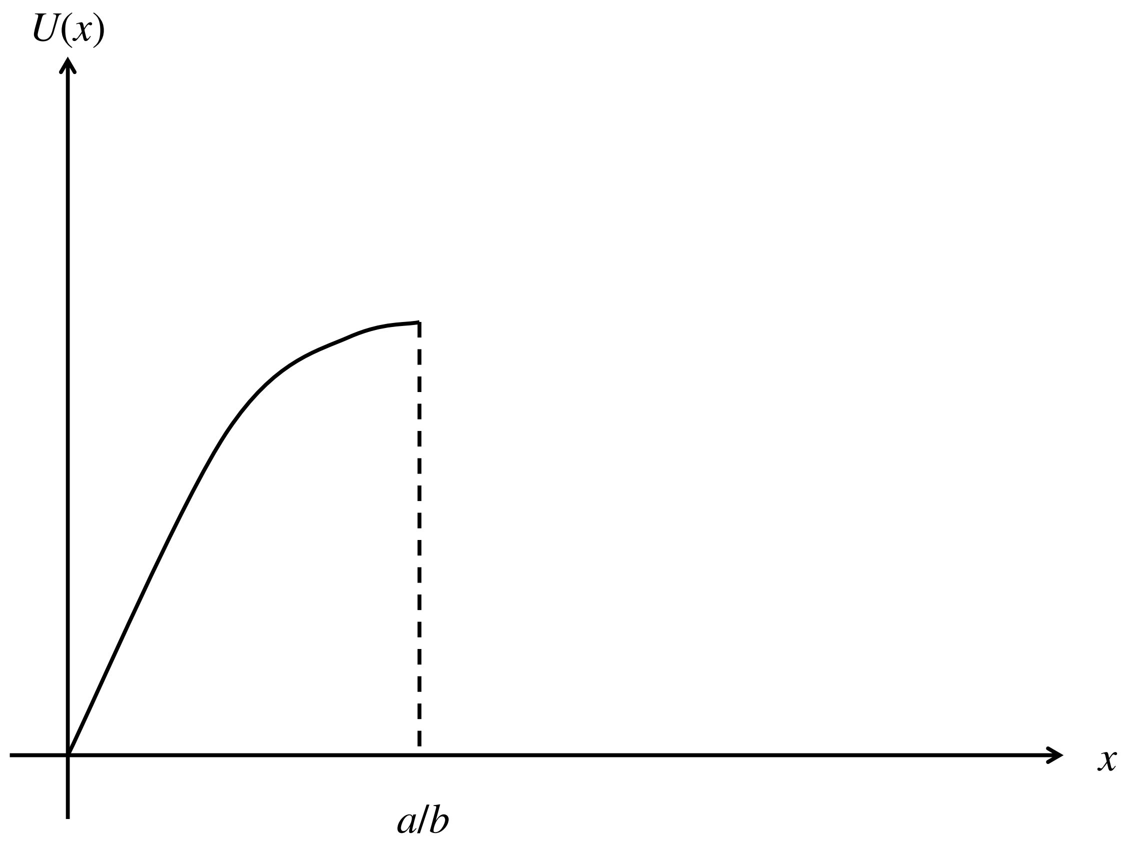
$$\begin{aligned} E[U(y)] &= E\left[ay - \frac{b}{2}y^2\right] \\ &= aE[y] - \frac{b}{2}E[y^2] \\ &= aE[y] - \frac{b}{2}(E[y])^2 - \frac{b}{2}\text{var}(y), \quad a > 0, b > 0. \end{aligned}$$

The expected utility value is seen to be dependent only on the mean and variance of the random wealth y. The optimal portfolio is the one that maximizes this value with respect to all feasible choices of the random wealth variable y.

- For a given value of $E[y]$, maximizing $E[U(y)] \Leftrightarrow$ minimizing var(y).

- For a given var(y), maximizing $E[U(y)] \Leftrightarrow$ maximizing $E[y]$.

  This is because $U(x) = ax - \frac{b}{2}x^2$ is an increasing function of x in the range $0 \leq x \leq a/b, a > 0$ and $b > 0$.



Normal returns and mean-variance criterion

When all returns of risky assets are normal random variables, the mean-variance criterion is also equivalent to the expected utility approach for any risk-averse utility function.

To deduce this, select an utility function U that is increasing and concave. Consider a random wealth variable y that is a normal random variable with mean value M and standard deviation σ. Since the probability distribution is completely defined by M and σ, it follows that the expected utility is a function of M and σ. Since U is increasing and risk averse, then

$$E[U(y)] = \int_{-\infty}^{\infty} U(y) \frac{1}{\sqrt{2\pi}\sigma}\ e^{-(y-M)^2/2\sigma^2}\, dy = f(M, \sigma),$$

with $\dfrac{\partial f}{\partial M} > 0$ and $\dfrac{\partial f}{\partial \sigma} < 0$. The sign of $\dfrac{\partial f}{\partial \sigma}$ is negative due to the risk averse property of U.

Portfolio selection procedure: minimizing portfolio variance

- Now suppose that the returns of all assets are normal random variables. Then the return of any linear combination of these assets is also a normal random variable. Hence any portfolio problem is therefore equivalent to the selection of combination of assets that maximizes the function $f(M_P, \sigma_P)$ with respect to all feasible combinations. Here, M_P and σ_P are the mean and standard deviation of portfolio's random return, respectively.

- For a risk-averse utility, this again implies that the variance should be minimized for any given value of the mean. This is because $f(M_P, \sigma_P)$ is a decreasing function of σ_P, a lower value of portfolio variance σ_P^2, the higher value of $E[U(y)]$. In other words, the solution must be mean-variance efficient.

- The portfolio selection problem is to find portfolio weights $\boldsymbol{w}^*$ such that $f(M_P, \sigma_P)$ is maximized with respect to all feasible combinations.

4.5 Stochastic dominance

- Once the investor's utility function, we have the full information on the investor's preference. Using the *maximum expected utility criterion*, we obtain a complete ordering of all the investments under consideration. What happens if we have only partial information on the choice of the utility funciton (say, prefer more to less and/or risk aversion)?

- In *the First Order Stochastic Dominance Rule*, we only consider the class of utility functions, call $\mathbf{U}_1$, such that $u' \geq 0$ (with strict inequality over some range). This is a very general assumption and it does not assume any specific utility function.

- Recall $E[u(x)] = \int u(x)\, dF(x) = U(F)$, where F is the probability distribution of the random variable x. We may consider expected utility value as a function of distribution on the underlying x.

Feasible set – set of all available investments under consideration.

Dominance under $\mathbf{U}_1$

Investment A dominates investment B under $\mathbf{U}_1$ if for all utility functions such that $u \in \mathbf{U}_1, E_A u(x) \geq E_B u(x)$; [equivalently, $U(F_A) \geq U(F_B)$, where F_A and F_B are the distribution function of choices A and B, respectively]; and for at least one utility function, there is a strict inequality.

- Dominance is transitive. If A dominates B and B dominates C, then A dominates C.

- Choices among investments amount to choices on probability distributions.

For any pair of distinct investments x and y, either one of the following cases holds:

(i) x dominates y for any choice of utility function in $\boldsymbol{U}_1$;

(ii) x dominates y under one utility function while y dominates x under another utility function;

(iii) y dominates x for any choice of utility function in $\boldsymbol{U}_1$.

Efficient set in $\mathbf{U}_1$ (collection of investments that are not being dominated)

An investment is included in the efficient set if there is no other investment that dominates it. Suppose investments A and B are efficient, then neither A nor B dominates the other. That is, there exists $u_1 \in \boldsymbol{U}_1$ such that $E_A u_1(x) > E_B u_1(x)$ while there exists another $u_2 \in \boldsymbol{U}_1$ such that $E_A u_2(x) < E_B u_2(x)$. Some prefer A and other prefer B (no dominance between A and B).

Inefficient set in $\mathbf{U}_1$ (being dominated)

Any investment that does not lie in the efficient set is included in the inefficient set. The inefficient set includes all inefficient investments. An inefficient investment is that there is at least one investment that dominates it. It is plausible that the inefficient set is null. For example, if there are only two investment choices and either one dominates the other, then the inefficient set is null.

- It is still possible that an inefficient investment is dominated by another inefficient investment, but that dominating investment is itself being dominated by an efficient investment. There is no need for an inefficient investment to be dominated by all efficient investments. One dominance is enough to relegate an investment to the inefficient set.

- An investment within the inefficient set cannot dominate an investment within the efficient set since if such dominance were to exist then the latter one would not be efficient.

Example

There are 5 investment choices: A, B, C, D and E.

inefficient set	efficient set
C, D, E	A, B

- $E_A u_1(x) > E_B u_1(x)$ while $E_A u_2(x) < E_B u_2(x)$.
- A dominates C and D, while B dominates E.

Since any investment must be either efficient or inefficient, the efficient and inefficient sets form a partition of the feasible set.

- The partition of the set of feasible choices into the efficient and inefficient sets depends on the choice of the class of utility functions. In general, the smaller the efficient set relative to the feasible set, the better for the decision maker.

- When we have only one utility function, we have complete ordering of all investment choices. The efficient set may likely contain one element (possibly more than one if we have investments whose expected utility values happen to tie with each other).

- Objective and subjective decisions
  The first stage provides the efficient set (objective decision) while the second state determines the optimal choice by maximizing the expected utility of an individual investor (subjective decision).

First order stochastic dominance

Two Investment Alternatives: Outcomes and Associated Probabilities

Investment A		Investment B	
Outcome	Probability	Outcome	Probability
12	1/3	11	1/3
10	1/3	9	1/3
8	1/3	7	1/3

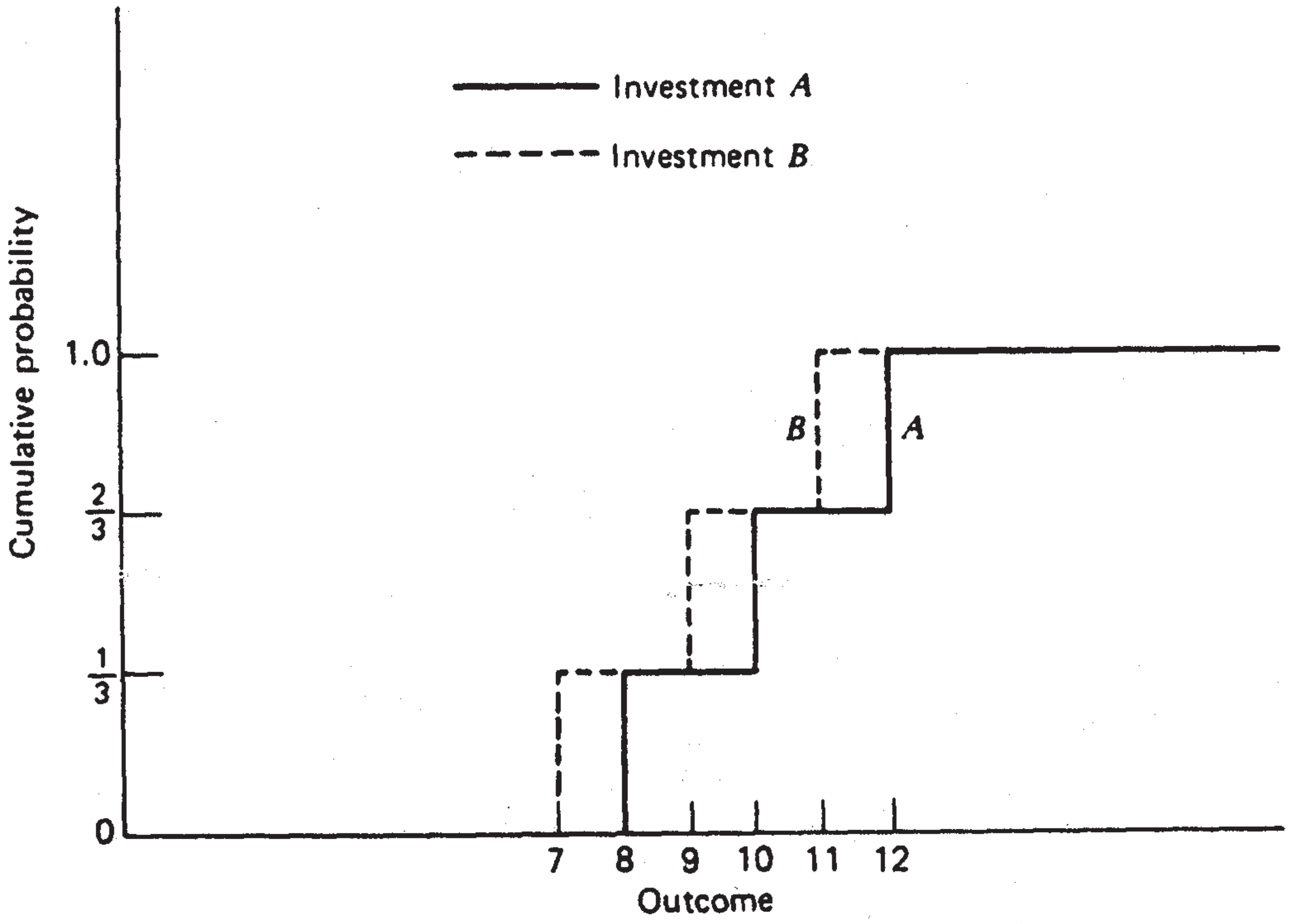
Can we argue that Investment A is better than Investment B? It is still possible that the return from investing in B is 11 units but the return is only 8 units from investing in A.

Comparison of cumulative probability distributions

By looking at the cumulative probability distributions, we observe that for all returns and the odds of obtaining that return or less, B consistently has a higher or at least the same odd.

Cumulative Probability Distribution

	Odds of obtaining a return equal to or less than that shown in Column 1	
Return	A	B
7	0	1/3
8	1/3	1/3
9	1/3	2/3
10	2/3	2/3
11	2/3	1
12	1	1



Cumulative frequency function for gambles A and B.

To compare two investment choices, we examine their corresponding probability distribution, where $F_X(x) = P[X \leq x]$.

Definition

A probability distribution F dominates another probability distribution G according to the first order stochastic dominance if and only if

$$F(x) \leq G(x) \quad \text{for all} \quad x \in C,$$

where C is the set of possible outcomes.

Lemma

F dominates G by FSD if and only if

$$\int_C u(x)\, dF(x) \geq \int_C u(x)\, dG(x)$$

for all monotonically increasing utility functions $u(x)$.

Proof

The utility function u is an increasing function with $u'(x) \geq 0$ (with strict inequality over certain range).

(i) $F(x) \leq G(x) \Rightarrow E_A[(u(x)] \geq E_B[u(x)]$

Let a and b be the smallest and largest values that F and G can take on. Consider

$$\int_a^b u(x)\, d[F(x) - G(x)] = \underbrace{u(x)[F(x) - G(x)]_a^b}_{\substack{\text{zero since } F(a) = G(a) = 0 \\ \text{and } F(b) = G(b) = 1}} - \int_a^b u'(x)[F(x) - G(x)]\, dx;$$

given $F(x) \leq G(x)$ and $u'(x) \geq 0$, so

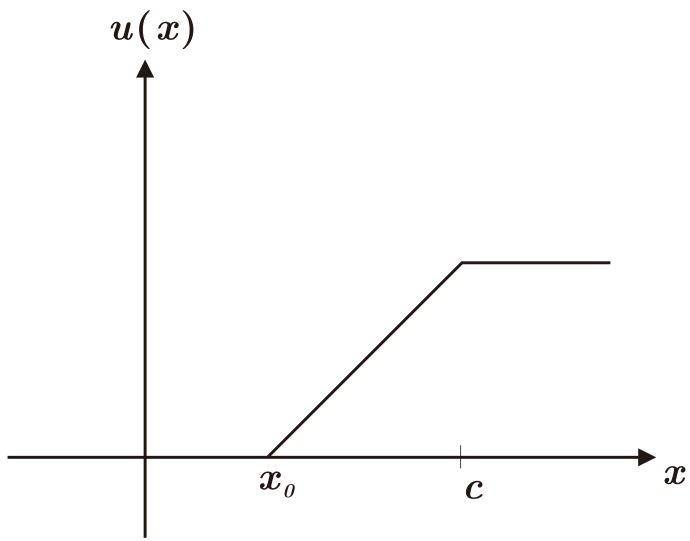
$$-\int_a^b u'(x)[F(x) - G(x)]\, dx \geq 0.$$

Thus, $F(x) \leq G(x) \quad \Rightarrow \quad \int_C u(x)\, dF(x) \geq \int_C u(x)\, dG(x).$

(ii) $E_A[u(x)] \geq E_B[u(x)] \Rightarrow F(x) \leq G(x)$ for all x

We prove by contradiction. Assume the contrary, suppose there exists x_0 such that $F(x_0) > G(x_0)$. Since distribution functions are right continuous, there exists an interval $[x_0, c]$ on the right hand side of x_0 such that $F(x) > G(x)$ for $x \in [x_0, c]$. Define the utility function

$$\begin{aligned} u(x) &= \int_a^x \mathbf{1}_{[x_0,c]}(t)\,dt, \text{ where} \\ \mathbf{1}_{[x_0,c]}(t) &= \begin{cases} 1 & t \in [x_0, c] \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$



Note that $u(x)$ is continuous and monotonically increasing, and

$$u'(x) = \mathbf{1}_{[x_0,c]}(x) \geq 0.$$

Now, consider

$$\begin{aligned} E_A[u(x)] - E_B[u(x)] &= -\int_a^b u'(x)[F(x) - G(x)]\,dx \\ &= -\int_{x_0}^c [F(x) - G(x)]\,dx < 0, \end{aligned}$$

a contradiction is encountered.

Properties of efficient and inefficient sets under FSD

- We do not require an inefficient investment to be dominated by all efficient investments. In order that an investment is relegated into the inefficient set, it is sufficient to have one investment that dominates the inefficient investment.

- Dominance or non-dominance among investment choices within the inefficient set is irrelevant since all investments included in this set are inferior.

- The distribution functions of all investments within FSD efficient set must intercept. If otherwise, one distribution would dominate the other, a contradiction to non-dominance.

Second order stochastic dominance

Two Investment Alternatives: Outcomes and Associated Probabilities

Investment A		Investment B	
Outcome	Probability	Outcome	Probability
6	1/4	5	1/4
8	1/4	9	1/4
10	1/4	10	1/4
12	1/4	12	1/4

If both investments turn out the worst, the investor obtains outcome of 6 from A and only outcome of 5 from B. If the second worst return occurs, the investor obtains 8 from A rather than 9 from B. If he is risk averse, then he should be willing to forfeit 1 unit in return at a higher level of return in order to obtain an extra 1 unit at a lower return level. If risk aversion is assumed, then A is preferred to B.

Definition

A probability distribution F dominates another probability distribution G according to the second order stochastic dominance if and only if for all $x \in C$

$$\int_{-\infty}^{x} F(y)\,dy \leq \int_{-\infty}^{x} G(y)\,dy.$$

Theorem

F dominates G by SSD if and only if

$$\int_C u(x)\,dF(x) \geq \int_C u(x)\,dG(x)$$

for all increasing and concave utility functions $u(x)$.

The Sum of the Cumulative Probability Distribution

	Cumulative Probability		Sum of Cumulative Probability	
Return	A	B	A	B
4	0	0	0	0
5	$\boxed{0}$	1/4	0	1/4
6	$\boxed{1/4}$	1/4	1/4	1/2
7	1/4	1/4	1/2	3/4
8	1/2	$\boxed{1/4}$	1	1
9	1/2	$\boxed{1/2}$	1 1/2	1 1/2
10	3/4	3/4	2 1/4	2 1/4
11	3/4	3/4	3	3
12	1	1	4	4

According to SSD, A is preferred to B since the sum of cumulative probability for A is always less than or equal to that for B.

Write $I_A(x) = \int_{-\infty}^{x} F_A(y)\,dy$ and observe that $F_A(8)$ is constant within $[8, 9)$

$$
\begin{aligned}
I_A(8.6) &= I_A(8) + F_A(8) \times 0.6 = 1 + \frac{1}{2} \times 0.6 = 1.3 \\
I_A(13.5) &= I_A(12) + F_A(12) \times 1.5 = 4 + 1 \times 1.5 = 5.5.
\end{aligned}
$$

Note that $F_A(x)$ has discrete jumps at those discrete values that can be taken by the random outcome of investment A, a feature that is typical for the distribution function of a discrete-valued random variable.

Proof (i) "if" part

$$\begin{aligned}
\int_a^b u(x)\,d[F(x)-G(x)] &= -\int_a^b u'(x)[F(x)-G(x)]\,dx \\
&= -u'(x)\int_a^x [F(y)-G(y)]\,dy\Big|_a^b \\
&\quad + \int_a^b u''(x)\int_a^x [F(y)-G(y)]\,dydx \\
&= -u'(b)\int_a^b [F(y)-G(y)]\,dy \\
&\quad + \int_a^b u''(x)\int_a^x [F(y)-G(y)]\,dydx.
\end{aligned}$$

Given that $u'(b) \geq 0$ and $u''(x) \leq 0$, we obtain

$$\int_C u(x)\,dF(x) \geq \int_C u(x)\,dG(x)$$

if F dominates G by SSD, where

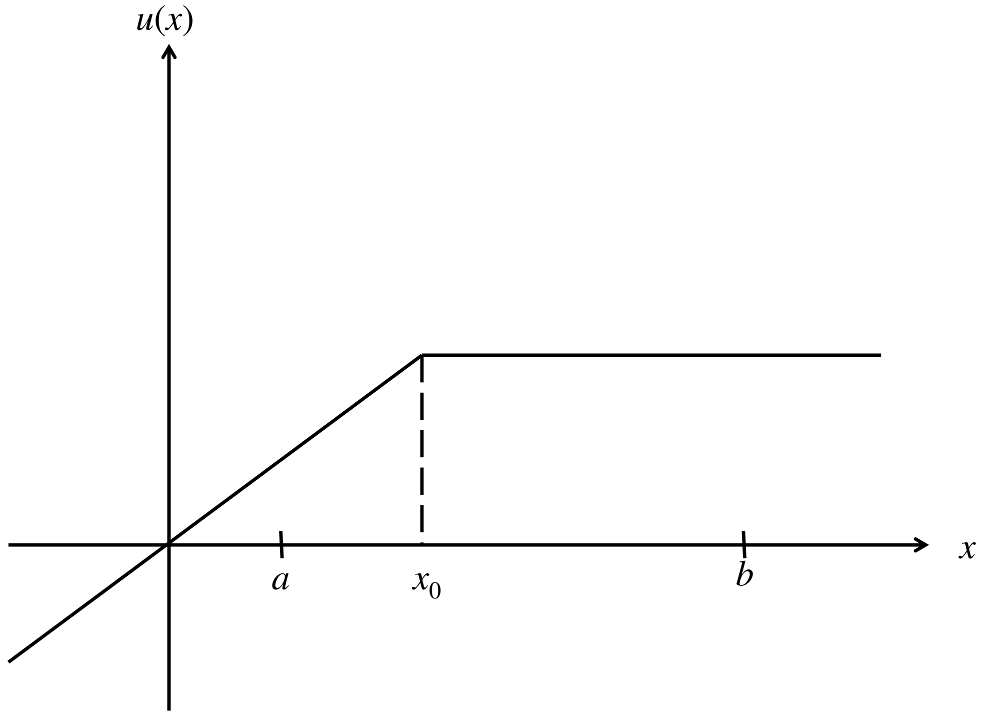
$$\int_a^x [F(y)-G(y)]\,dy \leq 0,$$

for all x.

(ii) "only if" part

We prove by contradiction. Suppose $\int_a^{x_0} F(x)\,dx > \int_a^{x_0} G(x)\,dx$ for some $x_0 \in [a,b]$. Consider the choice of the following utility function:

$$u(x) = \begin{cases} x_0 & \text{if } x \geq x_0 \\ x & \text{if } x < x_0 \end{cases}.$$



Obviously, $u(x)$ is increasing and concave, so $u \in \boldsymbol{U}_2$. It suffices to show that this choice of utility function leads to the violation of the property: $U(F) \geq U(G)$. Recall $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$, and consider

$$\begin{aligned}
& \int_a^b u(x)\, dF(x) - \int_a^b u(x)\, dG(x) \\
= \;& \int_a^{x_0} x\, d[F(x) - G(x)] + \int_{x_0}^b x_0\, d[F(x) - G(x)] \\
= \;& x[F(x) - G(x)]\Big|_a^{x_0} - \int_a^{x_0} [F(x) - G(x)]\, dx - x_0[F(x_0) - G(x_0)] \\
= \;& -\int_a^{x_0} [F(x) - G(x)]\, dx < 0,
\end{aligned}$$

so there is a contradiction.

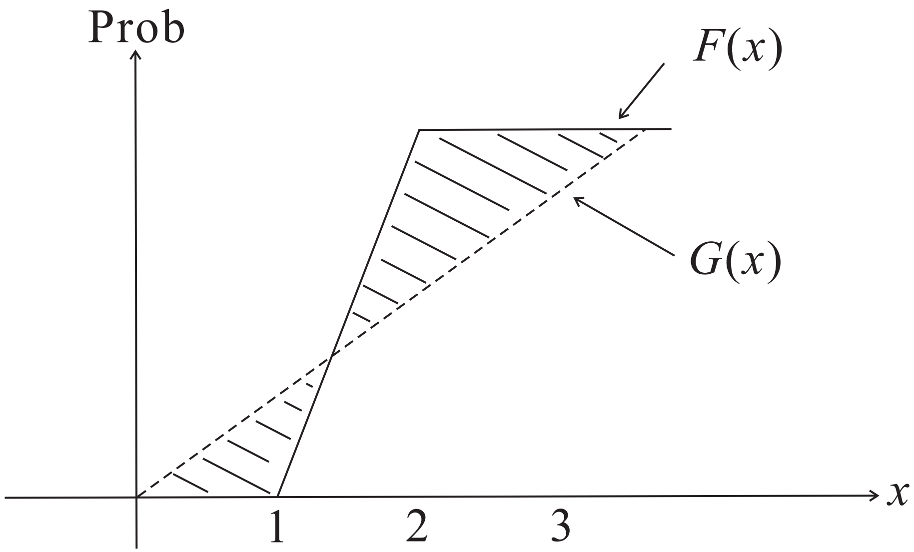
Example

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ x - 1 & \text{if } 1 \le x \le 2 \\ 1 & \text{if } x \ge 2 \end{cases}, \qquad G(x) = \begin{cases} 0 & \text{if } x < 0 \\ x/3 & \text{if } 0 \le x \le 3 \\ 1 & \text{if } x \ge 3 \end{cases}.$$

F dominates G by SSD since

$$\int_{-\infty}^{x} F(y)\,dy \le \int_{-\infty}^{x} G(y)\,dy.$$

$F(x)$ is seen to be more concentrated (less dispersed).



In this example, $F(x) \le G(x)$ is not valid for all x.

Sufficient rules and necessary rules for second order stochastic dominance

Sufficient rule 1 FSD rule is sufficient for SSD

Proof: If F dominates G by FSD, then $F(x) \leq G(x), \forall x$.

This implies $\int_a^x [G(y) - F(y)]\, dy \geq 0$.

Remark

The inefficient set according to FSD is a subset of that of SSD.

Proof: Suppose G lies in the inefficient set of FSD, say, it is dominated by F by FSD. Then F dominates G by SSD so that G must lie in the inefficient set of SSD.

Sufficient rule 2

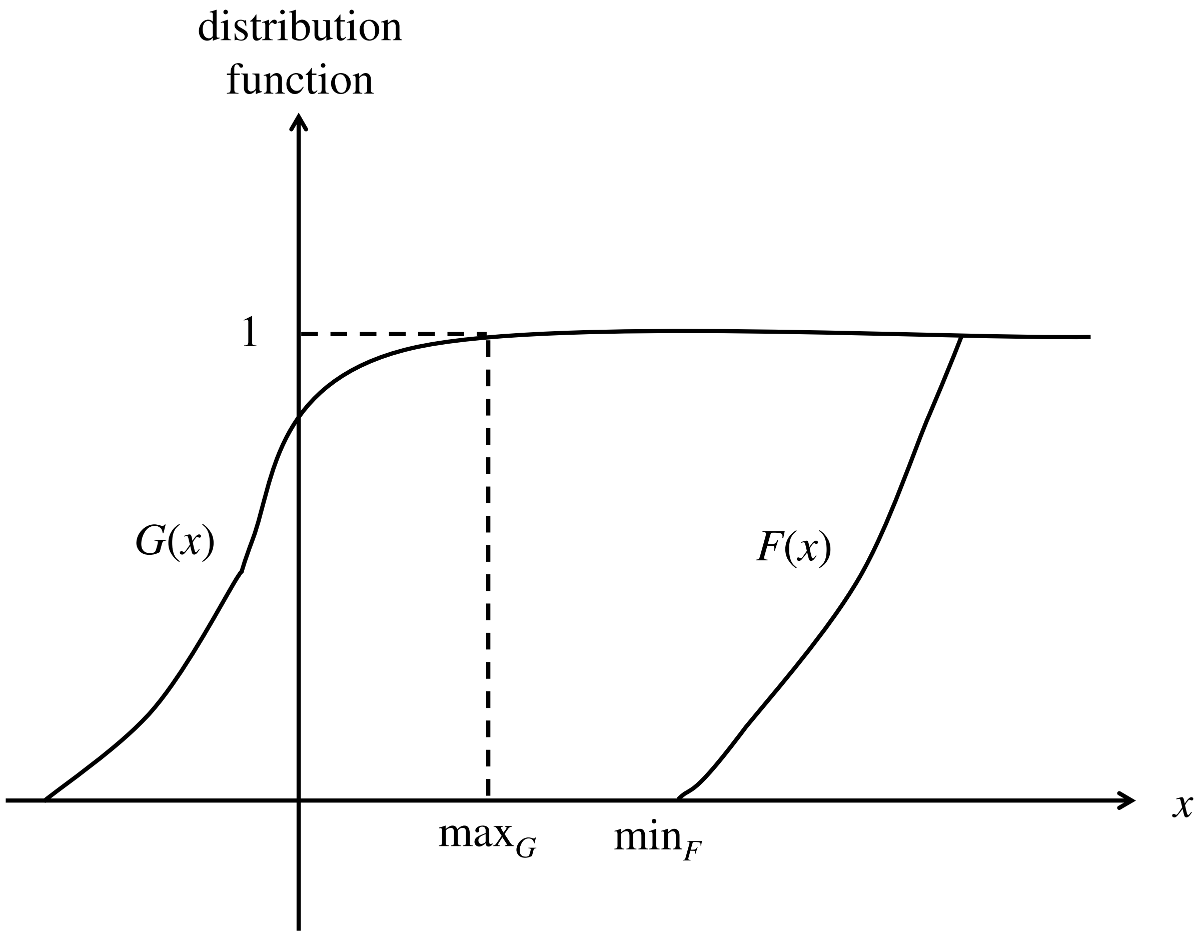
$\text{Min}_F \geq \text{Max}_G$ is a sufficient rule for SSD. Note that $\text{Min}_F \geq \text{Max}_G$ is a very strong requirement.

Example

F		G	
x	$p(x)$	x	$p(x)$
5	1/2	2	3/4
10	1/2	4	1/4

$\text{Min}_F = 5 \geq \text{Max}_G = 4$. Note that $F(x) = 0$ for $x \leq \text{min}_F$ while $G(x) = 1$ for $x \geq \text{max}_G$. Since $F(x)$ and $G(x)$ are non-decreasing functions in x, so $F(x) \leq G(x)$.

$$\text{Min}_F \geq \text{Max}_G \Rightarrow \text{FSD} \Rightarrow \text{SSD} \Rightarrow E_F u(x) \geq E_G u(x), \forall u \in \mathbf{U}_2.$$



$$F(x) = 0 \text{ for } x \leq \text{min}_F$$
$$G(x) = 1 \text{ for } x \geq \text{max}_G.$$

Obviously, $F(x) < G(x), \forall x \in C$.

Necessary rule 1 (Geometric means)

Given a risky project with the distribution $(x_i, p_i), i = 1, \cdots, n$, the geometric mean, $\overline{X}_{geo}$, is defined as

$$\overline{X}_{geo} = x_1^{p_1} \cdots x_n^{p_n} = \prod_{i=1}^{n} x_i^{p_i}, x_i \geq 0.$$

Taking logarithm on both sides

$$\ln \overline{X}_{geo} = \Sigma p_i \ln x_i = E[\ln X].$$

$\boxed{\overline{X}_{geo}(F) \geq \overline{X}_{geo}(G) \text{ is a necessary condition for dominance of } F \text{ over } G \text{ by SSD.}}$

Proof

Suppose F dominates G by SSD, we have

$$E_F u(x) \geq E_G u(x), \text{ for all } u \in \mathbf{U}_2.$$

Since $\ln x = u(x) \in \mathbf{U}_2$,

$$E_F \ln x = \ln \overline{X}_{geo}(F) \geq E_G \ln x = \ln \overline{X}_{geo}(G);$$

we obtain $\ln \overline{X}_{geo}(F) \geq \ln \overline{X}_{geo}(G)$.

Since the logarithm function is an increasing function, we deduce $\overline{X}_{geo}(F) \geq \overline{X}_{geo}(G)$.

Therefore, F dominates G by SSD $\Rightarrow \overline{X}_{geo}(F) \geq \overline{X}_{geo}(G)$.

Necessary rule 2 (left-tail rule for SSD)

Suppose F dominates G by SSD, then

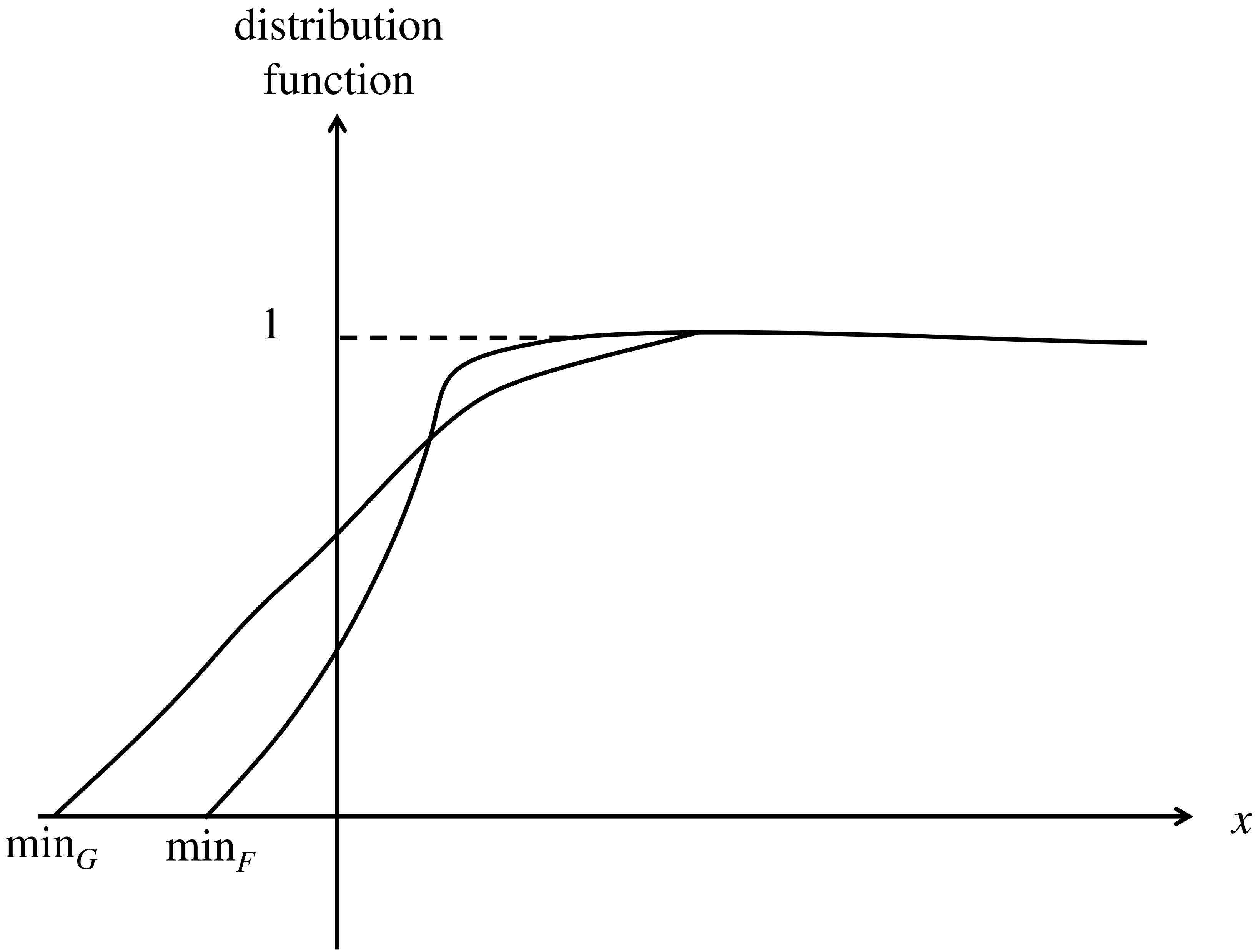
$$\text{Min}_F \geq \text{Min}_G,$$

that is, the left tail of G must be "thicker".

Proof by contradiction: Suppose $\text{Min}_F < \text{Min}_G$, and write $x_k = \text{Min}_G$. At x_k, G will still be zero but F will be positive. Observe that

$$\int_{-\infty}^{x_k} [G(y) - F(y)]\, dy = \int_{-\infty}^{x_k} [0 - F(y)]\, dy < 0,$$

implying that F is not dominated by G by SSD. Hence, if F dominates G by SSD, then $\text{Min}_F \geq \text{Min}_G$.



The distribution G has a "thicker" tail at the left end.