MATH4512 - Fundamentals of Financial Mathematics

Mid-term Test – Spring 2016

Time allowed: 90 minutes *Course instructor: Professor Yue Kuen KWOK*

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1. This question addresses the properties of zero-coupon bearing bonds.
   (a) Explain why the duration of a zero coupon bond is always equal to its maturity. [1 point]
   (b) Suppose we set the horizon of investment H to be the same as the maturity of the zero coupon bond. Explain why the horizon rate of return r_H of the zero coupon bond *always stays the same* under an increase or decrease of the interest rate.
   Hint: Consider the future value of the zero-coupon bond at H. [3 points]

2. This question addresses the properties of duration D and convexity C of a bond. Recall the following formula:
$$convexity = \frac{dispersion + duration\,(duration+1)}{(1 + interest\ rate)^2}$$
   (a) Explain why both the duration and convexity of a coupon bearing bond are decreasing functions of the coupon rate. [2 points]
   (b) Consider two bonds that have the same value at the current level of interest rate. Show graphically that the bond with higher convexity loses less when interest rate increases and gains more when interest rate decreases when compared to the other bond with lower convexity. [1 point]
   (c) Suppose the yield curve is sloping upward and having leveling effect at high maturity.
   (i) Explain why the barbell portfolio has a lower rate of return but higher convexity when compared with its bullet portfolio counterpart when both portfolios have the same value. [2 points]
   (ii) Why the subsequent parallel shift of the yield curve has to be sufficiently strong in order that the barbell portfolio may outperform the bullet portfolio in return? [2 points]

3. This question addresses horizon rate of return r_H and immunization. Let D denote the duration of a bond and H be the investor's horizon rate of return of the bond.
   (a) Explain why r_H is a decreasing function of the interest rate i when $H < D$ and becomes an increasing function of i when $H > D$. [2 points]
   (b) What happen when $H = D$? Why it is desirable to invest in a bond portfolio such that the bond's duration matches with the investor's horizon H? [2 points]
   (c) A bond is said to be immunized from fluctuation in interest rates when its value stays almost the same when the interest rates move up or down. The bond immunization procedure has to be dynamically rebalanced with the passage of the calendar time. Give your explanation. [1 point]

4. This question considers holding period gains / losses and the Orange County Bankruptcy.
   (a) Identify the sources of cash flows and market factors that contribute to the return of a bond investment. Neglecting default and liquidity risk, explain why the fair rate of return of holding a bond is given by
   $$fair\ rate\ of\ return\ of\ bond = \frac{coupon}{bond\ price} + \frac{change\ in\ market\ bond\ price}{bond\ price}.$$
   [2 points]
   (b) What are the key market factors that occurred and the mistakes made by the County Treasurer that triggered the Orange County Bankruptcy? [2 points]

5. Consider a portfolio of two assets with known expected rates of return μ_1 and μ_1, variances σ_1^2 and σ_2^2, and the correlation coefficient of the random rates of return is ρ. Let $1 - \alpha$ and α be the portfolio weight of asset 1 and asset 2, respectively.
   (a) Find the expression for the portfolio variance, σ_P^2. For $-1 < \rho < 1$, determine the portfolio weight α such that the portfolio variance is minimized. [3 points]
   (b) Show that the corresponding covariance matrix M
   $$M = \begin{pmatrix} \sigma_1^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$
   of the 2-asset portfolio is singular when $\rho = -1$. Explain the relation between the singular property of M and existence of asset portfolio with zero portfolio variance. Then, find the corresponding two-asset portfolio with zero variance. [3 points]
   (c) It is easily seen that M is also singular when $\rho = 1$. Explain why in this case we cannot achieve a portfolio of zero variance when short sales of the assets are *not allowed.* [3 points]

6. Recall the *g*-fund and *d*-fund whose portfolio weights are given by
   $$\boldsymbol{w}_g = \frac{M^{-1}\mathbf{1}}{a} \text{ and } \boldsymbol{w}_d = \frac{M^{-1}\boldsymbol{\mu}}{b},$$
   where M is the covariance matrix of the assets' rates of return, $\boldsymbol{\mu}$ is the vector of expected rates of return, $a = \mathbf{1}^T M^{-1}\mathbf{1}$, $b = \mathbf{1}^T M^{-1}\boldsymbol{\mu}$ and $c = \boldsymbol{\mu}^T M^{-1}\boldsymbol{\mu}$.
   (a) Explain the difference between a minimum-variance fund and an efficient fund. Can we find a fund that has *higher expected rate of return and lower portfolio variance* than that of *minimum variance* fund? [2 points]
   (b) Given that $b > 0$, explain why $\boldsymbol{w}_d$ is an efficient fund? Under this case, explain why a convex combination of the *g*-fund and *d*-fund remains to be efficient. [3 points]
   (c) The random rates of return of the *g*-fund and *d*-fund are denoted by r_g and r_d, respectively. Recall that $\text{cov}(r_g, r_d) = \boldsymbol{w}_g^T M \boldsymbol{w}_d$. By virtue of the Two-fund Theorem, any minimum variance fund P can be expressed as $\alpha\boldsymbol{w}_g + (1 - \alpha)\boldsymbol{w}_d$. Find the expected rate of return μ_P and portfolio variance σ_P^2 in terms of α, a, b and c. By eliminating α between μ_P and σ_P^2, show that the equation of the frontier is given by
   $$\sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{ac - b^2}.$$
   [6 points]

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