MATH 4512 – Fundamentals of Mathematical Finance

Mid-term Test, 2017

Time allowed: 90 minutes *Instructor: Prof. Y. K. Kwok*

[points]

1. Consider the formula for the duration of D for a coupon bearing bond paying annual coupons c

$$D = 1 + \frac{1}{i} + \frac{T(i - \frac{c}{B_T}) - (1+i)}{\frac{c}{B_T}[(1+i)^T - 1] + i},$$

where i is the interest rate, T is the maturity date and B_T is the par value of the bond.

   (a) Explain why when $c/B_T < i$, D first increases with T, reaches a maximum, then tends to an asymptotic limit when $T \to \infty$. **[3]**

   (b) With an increase in c/B_T, deduce from the formula for D that D is always a decreasing function of c/B_T. Give a financial interpretation of this decreasing property. **[2]**

   (c) The bankruptcy of Savings and Loan Associations in the US in early 1980s was attributed to the mismatch of durations of assets and liabilities. They owned deposits with short maturities while their loans to mortgage developers had very long durations. Explain why these Associations suffered great losses when interest rates climbed sharply. **[2]**

   (d) For the horizon rate of return r_H, use financial argument to explain why when the duration of a bond portfolio matches with the horizon H, then the bond portfolio observes immunization with respect to interest rate fluctuations. Also, with the passage of calendar time, explain why one has to rebalance the composition of a bond portfolio in order to achieve immunization against interest rate fluctuations. **[3]**

   (e) Recall the following equation for the definition of the horizon rate of return r_H, where

$$B_0(1+r_H)^H = B(i)(1+i)^H.$$

   Here, B_0 is the initial bond price at initial interest rate i_0, while $B(i)$ is the bond price at the new interest rate level i.

   (i) For any horizon H, explain why r_H remains to be i_0 if i does not move. **[1]**

   (ii) When $H \to \infty$, show mathematically that $r_H \to i$. Give a financial interpretation of this result. **[2]**

2. Recall the following formula for the relative change in bond price $\Delta B/B$ with respect to the small change of interest rate Δi, where

$$\frac{\Delta B}{B} \approx \frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}i}\Delta i + \frac{1}{2}\frac{1}{B}\frac{\mathrm{d}^2 B}{\mathrm{d}i^2}(\Delta i)^2.$$

We define the convexity C to be

$$C = \frac{1}{B}\frac{\mathrm{d}^2 B}{\mathrm{d}i^2}.$$

(a) Explain why a bond with a higher convexity would lose less when interest rate increases and gain more when interest rate decreases when compared with another bond with the same duration but lower convexity. **[2]**

(b) Recall the formula:

$$\text{convexity} = \frac{\text{dispersion} + \text{duration}(\text{duration} + 1)}{(1 + \text{interest rate})^2}.$$

Find the convexity of a zero coupon bond with maturity 10 years and interest rate 10%. Leave your answer in fraction. **[2]**

Hint: Explain why the dispersion of a zero coupon bond is zero.

(c) Explain why a barbell portfolio has a higher convexity compared to that of a bullet portfolio. When the yield curve is upward sloping, show that the barbell strategy gives up yield in order to achieve a higher convexity. **[3]**

3. Suppose we modify the Markowitz mean-variance formulation by introducing the target portfolio rate of return μ_P into the objective function. The new formulation reads as follows:

$$\text{maximize } \tau\mu_P - \frac{\sigma_P^2}{2}, \text{ with } \tau \geq 0, \text{ subject to } \mathbf{1}^T\boldsymbol{w} = 1.$$

Here, τ is called the risk tolerance parameter.

(a) By using the first order condition in the Lagrangian formulation, show that the optimal portfolio weight is given by

$$\boldsymbol{w}^* = \tau\Omega^{-1}\boldsymbol{\mu} + \lambda\Omega^{-1}\mathbf{1},$$

where Ω is the invertible covariance matrix and $\boldsymbol{\mu}$ is the vector of expected rates of return of the risky assets in the portfolio and λ is the Lagrangian multiplier. **[3]**

(b) By imposing the constraint: $\mathbf{1}^T\boldsymbol{w} = 1$, show that

$$\boldsymbol{w}^* = \boldsymbol{w}_g + \tau\boldsymbol{z},$$

where $\boldsymbol{w}_g = \dfrac{\Omega^{-1}\mathbf{1}}{a}$, $\boldsymbol{z} = b(\boldsymbol{w}_d - \boldsymbol{w}_g)$, $\boldsymbol{w}_d = \dfrac{\Omega^{-1}\boldsymbol{\mu}}{b}$, $a = \mathbf{1}^T\Omega^{-1}\mathbf{1}$, $b = \mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}$, $c = \boldsymbol{\mu}^T\Omega^{-1}\boldsymbol{\mu}$, $\Delta = ac - b^2$. **[4]**

(c) The expected rates of return of $\boldsymbol{w}_g$ and $\boldsymbol{w}_d$ are known to be

$$\begin{aligned}\mu_g &= \boldsymbol{\mu}^T\boldsymbol{w}_g = \frac{\boldsymbol{\mu}^T\Omega^{-1}\mathbf{1}}{a} = \frac{b}{a},\\ \mu_d &= \boldsymbol{\mu}^T\boldsymbol{w}_d = \frac{\boldsymbol{\mu}^T\Omega^{-1}\boldsymbol{\mu}}{b} = \frac{c}{b}.\end{aligned}$$

Also, the portfolio variances of $\boldsymbol{w}_g$ and $\boldsymbol{w}_d$ are given by

$$\begin{aligned}\sigma_g^2 &= \boldsymbol{w}_g^T\Omega^{-1}\boldsymbol{w}_g = \frac{1}{a},\\ \sigma_d^2 &= \boldsymbol{w}_d^T\Omega^{-1}\boldsymbol{w}_d = \frac{c}{b^2}.\end{aligned}$$

Use these relations to show that μ_P and σ_P^2 corresponding to the optimal weight: $\boldsymbol{w}^* = \boldsymbol{w}_g + \tau\boldsymbol{z}$ are given by

$$\mu_P = \frac{b}{a} + \frac{\Delta}{a}\tau \text{ and } \sigma_P^2 = \frac{1}{a} + \frac{\Delta}{a}\tau^2.$$ **[6]**

Hint:

$$\begin{aligned}
\mu_P &= \boldsymbol{\mu}^T(\boldsymbol{w}_g + \tau \boldsymbol{z}) = \mu_g + \tau\mu_z \\
\sigma_P^2 &= \text{cov}(r_g + \tau r_z, r_g + \tau r_z) \\
&= \sigma_g^2 + 2\tau \text{cov}(r_g, r_z) + \tau^2\sigma_z^2,
\end{aligned}$$

where r_g and r_z are the random rate of return of $\boldsymbol{w}_g$ and $\boldsymbol{z}$, respectively, σ_g^2 and σ_z^2 are the portfolio variance of $\boldsymbol{w}_g$ and $\boldsymbol{z}$, respectively.

(d) (i) By eliminating τ in the pair of equations for μ_P and σ_P^2 in part (c), show that

$$\sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}, \ \ \Delta = ac - b^2.$$ **[2]**

(ii) Explain why the optimal portfolios are all efficient. **[2]**

Hint: Efficient portfolios are those which give the highest expected rate of return for a given portfolio variance and the lowest portfolio risk for a given target expected rate of return.

(e) Recall the theoretical result that $\Delta = ac - b^2$ becomes zero when $\boldsymbol{\mu} = h\mathbf{1}$, where h is a positive scalar. Financially, this corresponds to the scenario where all risky assets share the same expected rate of return h. In this case, we cannot prescribe μ_P arbitrarily since μ_P must be set equal to h. Interpret and analyze the mean-variance formulation in this problem under this degenerate case and explain why the set of optimal portfolios reduces to only one portfolio. Deduce this singleton optimal portfolio by financial argument. **[3]**

— *End* —