MATH 4512 – Fundamentals of Mathematical Finance

Mid-term Test, 2015

Time allowed: 90 minutes *Instructor: Prof. Y. K. Kwok*

[points]

1. Consider a portfolio of two assets with known expected values $\overline{r}_1$ and $\overline{r}_2$, variances σ_1^2 and σ_2^2, of the random rates of return r_1 and r_2, together with $\text{cov}(r_1, r_2) = \rho\sigma_1\sigma_2$. Let $1 - \alpha$ and α be the weights of asset 1 and asset 2 in this two-asset portfolio.

   (a) Find the covariance matrix Ω associated with this two-asset portfolio and the corresponding portfolio variance. **[2]**

   (b) When the two assets are completely negatively correlated with $\rho = -1$, show that it is possible to construct a portfolio with zero portfolio variance. Determine α for this zero-variance portfolio. **[3]**

   (c) Show that the covariance matrix when $\rho = -1$ is singular; that is, $\det \Omega = 0$. Explain why zero-variance portfolio can be constructed when Ω is singular. **[3]**

2. Consider the n-asset portfolio whose risky assets have the same mean rate of return $\overline{r}$ and different variances σ_i^2, $i = 1, 2, \ldots, n$. Furthermore, we assume that their random rates of return are uncorrelated.

   (a) Describe the efficient set of this n-asset portfolio. **[3]**

   (b) Find the minimum variance portfolio. **[3]**

3. We consider the special case of a portfolio with a riskfree asset and n risky assets, where the deterministic rate of return of the riskfree asset r is equal to the expected rate of return μ_g of the global minimum variance portfolio of the risky assets. If we let Ω be the covariance matrix of the risky assets, $\boldsymbol{\mu}$ be the expected rate of return vector, and

$$a = \mathbf{1}^T\Omega^{-1}\mathbf{1}, \; b = \boldsymbol{\mu}^T\Omega^{-1}\mathbf{1} \text{ and } c = \boldsymbol{\mu}^T\Omega^{-1}\boldsymbol{\mu},$$

   we have

$$r = \frac{b}{a} = \mu_g.$$

   (a) Explain why the tangency portfolio does not exist under this special scenario. **[2]**

   (b) Explain why an efficient fund would invest 100% of the investment value on the riskfree asset and set sum of weights of the risky assets to be zero. **[4]**

   (c) Consider the z-fund whose weight vector $\boldsymbol{w}_z$ is given by

$$\boldsymbol{w}_z = \Omega^{-1}\boldsymbol{\mu} - \frac{b}{a}\Omega^{-1}\mathbf{1},$$

   explain how this z-fund plays an important role in the One-fund Theorem under this special scenario of $r = \dfrac{b}{a} = \mu_g$. **[4]**

4. Consider the optimal portfolio problem of n risky assets (no riskfree asset) with the inclusion of a risk tolerance factor τ, where the objective function is

$$\max_{\boldsymbol{w}\in\mathbb{R}^n} \tau\mu_p - \frac{\sigma_p^2}{2},$$

subject to $\mathbf{1}^T\boldsymbol{w} = 1$ and $\tau > 0$.

(a) Show that the weight vector $\boldsymbol{w}^*$ of the optimal portfolio can be expressed as

$$\boldsymbol{w}^* = \boldsymbol{w}_g + \tau\boldsymbol{w}_z,$$

where $\boldsymbol{w}_g = \dfrac{\Omega^{-1}\mathbf{1}}{a}$ and $\boldsymbol{w}_z = \Omega^{-1}\boldsymbol{\mu} - \dfrac{b}{a}\Omega^{-1}\mathbf{1}$, a, b and c are defined in Question 3. **[4]**

(b) Deduce an interpretation of the Two-fund Theorem from the solution of the optimal portfolio. What are the two funds? **[2]**

5. (a) The Capital Asset Pricing Model states that

$$\overline{r}_i - r = \beta_i(\overline{r}_M - r),$$

where $\beta_i = \dfrac{\sigma_{iM}}{\sigma_M^2}$. Show that for any pair of risky assets i and j, we have

$$\frac{\overline{r}_i - r}{\beta_i} = \frac{\overline{r}_j - r}{\beta_j}.$$ **[3]**

(b) Show that the beta value of an efficient portfolio is equal to the proportional weight α of the market portfolio in the efficient portfolio. **[3]**

(c) What is the difference between the capital market line and security market line? Under the equilibrium condition assumed by the Capital Asset Pricing Model, should every asset fall on the capital market line or security market line. Give your explanation. **[4]**

— *End* —