Mathematics and Social Choice Theory

Homework One

Course instructor: *Prof. Y.K. Kwok*

---

1. Two simple games are *equivalent* (or *isomorphic*) if the players can be labeled in such a way that the winning coalitions are the same in both games. Show that the following three-person games are all equivalent.

   (i) $N = \{A, B, C\}$. Approval is by majority vote, but A has a *veto*.

   (ii) $[3; 2, 1, 1]$.

   (iii) $[5; 3, 2, 2]$.

   (iv) $[17; 16, 1, 1]$.

2. Suppose the U.N. Security Council has eight nonpermanent members and three permanent members with passage of a bill requiring a total of seven votes subject to the veto power of each of the permanent members. Prove that this is a weighted voting system. (Include a discussion showing how you found the appropriate weights and quota.)

3. Suppose we have a four-person weighted voting system with positive weights a,b,c, and d for the voters named A, B, C, and D, respectively. Assume the quota is the number q. Now suppose we create a new yes-no voting system by adding a clause that gives voters A veto power. Show that this is also a weighted voting system. In general, if a weighted voting system is altered by giving veto power to some of the voters, then the resulting yes-no voting system is again a weighted voting system.

   Hint: To get started notationally, assume the voters are named $v_1, \ldots, v_k, p_{k+1}, \ldots, p_n$ with weights $w_1, \ldots, w_n$, and suppose we want to give veto power to $v_1, \ldots, v_k$.

4. Which of the following properties about "winning coalition" are considered to be reasonable? Give your justification.

   (a) If X is a winning coalition and every voter in X is also in Y, then Y is also a winning coalition.

   (b) If X and Y are winning coalitions, then so is the coalition consisting of voters in both X and in Y.

   (c) If X is a winning coalition every voter in Y is also in X, then Y is also a winning coalition.

   (d) If X and Y are disjoint (that is, have no voters in common), then at least one fails to be a winning coalition.

   (e) If X and Y are winning coalitions, then so is the coalition consisting of voters in either X or in Y.

   (f) If X is the set of all players, then X is a winning coalition.

   (g) If X is the empty set, then X is a winning coalition.

   (h) If X is a winning coalition and X is split into two sets Y and Z so that every voter in X is in exactly one of Y and Z, then either Y is a winning coalition or Z is a winning coalition.

5. Suppose we have six voters: A_1, A_2, A_3, A_4, A_5, and A_6. Let S_3 denote the yes-no voting system wherein a coalition X is winning if and only if

   (i) at least one of A_1 and A_2 is in X,

   (ii) at least one of A_3 and A_4 is in X, and

   (iii) at least one of A_5 and A_6 is in X.

   (a) Prove that S_3 has dimension at most 3.

   (b) Suppose, for contradiction, that S_3 can be expressed as the intersection of two weighted systems. Since $\{A_1, A_2, A_3, A_4\}$, $\{A_1, A_2, A_5, A_6\}$, and $\{A_3, A_4, A_5, A_6\}$ are all losing, each ont must be losing in at least one of the two weighted systems. Hence, two of these – say $\{A_1, A_2, A_3, A_4\}$ and $\{A_3, A_4, A_5, A_6\}$ – are losing in the same weighted system. Show that the desired contradiction now can be achieved by an appropriate trade.

6. Consider the "yes-no" voting system with minority veto where the 7 voters are classified into the majority group of 5 voters and the minority group of 2 voters. A bill is passed by requiring at least 4 votes from all voters and at least 1 vote from the minority voters.

   (a) Show that the above "yes-no" system is swap robust but not trade robust.

   (b) Show that this "yes-no" system is the intersection of two weighted voting systems, that is, a coalition is winning in the "yes-no" system if and only if it is winning in both weighted voting systems.

   (c) Find the corresponding weights assigned to the voters and the quotas in the two weigthed voting systems.

   (d) Find the Shapley-Shubik index and the Banzhaf index for the voters in the majority group and the minority group.

7. Calculate the Shapley-Shubik and Banzhaf indices for the following weighted voting games:

   (a) $[4; 3, 1, 1, 1]$.

   (b) $[7; 4, 3, 2, 1]$.

   (c) $[5; 4, 2, 1, 1, 1]$.

   (d) $[9; 5, 4, 3, 2, 1]$.

8. A seven-person legislature has a three-person committee. Approval must be achieved by a majority of *both* the committee and the entire legislature. Denote the members by *AAAbbbb*. Compute power indices. What is the ratio of power between a committee member and a noncommittee member?

9. Calculate the Shapley-Shubik and Banzhaf indices for the large stockholder with 40% of the shares if the remaining shares are split evenly among

   (a) five other stockholders;

   (b) seven other stockholders.

   Why is the large stockholder less powerful in (b) than he is in (a)?

10. Show that in an oceanic majority game where there is just one major player X who holds a fraction x of the total vote, we have

$$\phi_X = \begin{cases} \frac{x}{1-x}, & \text{if } x \leq \frac{1}{2}, \\ 1, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

11. Consider a voting system consisting of 3 big states and 6 small states, passage of a bill requires "yes" vote from all the big states and at least 2 "yes" votes from the 6 small states.

    (a) Find the weighted voting vector of the above game, specifying the quota and the number of votes held by each of the big states and small states.

    (b) Consider one of the big states, assuming the homogeneity assumption on voting probabilities among all 9 states, find the probability $\pi_b(p)$ that the vote of this particular big state makes a difference between approval or rejection of a bill. Here, p denotes the common homogeneous voting probability.

    (c) Using the above $\pi_b(p)$, or otherwise, compute the Shapley-Shubik index and Banzhaf index for any one of the big states. Then deduce the values for the above two power indexes for any one of the small states.

    (d) Suppose the 3 big states vote independently while the set of 6 smaller states vote as a homogeneous group. Based on the Shapley-Shubik index calculations, determine how the power is shared among the big states and small states?

12. Consider $[5; 3, 2, 1, 1]$.
    $\quad\quad\;\; A \; B \; C \; D$

    Show that

$$\begin{aligned} \pi_A(p) &= \quad p + (1-p)p^2 &&= p + p^2 - p^3 \\ \pi_B(p) &= \quad p(1-p^2) &&= p - p^3 \\ \pi_C(p) &= \quad p(1-p)p &&= p^2 - p^3. \end{aligned}$$

    Hence, check that

$$\begin{aligned} \beta &= \left(\frac{5}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right) \\ \phi &= \left(\frac{7}{12}, \frac{3}{12}, \frac{1}{12}, \frac{1}{12}\right). \end{aligned}$$