Mathematics and Social Choice Theory

Homework Two

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1. Construct an example of one-way quarrel in a weighted voting game such that the hostile player helps (possibly against her wish) the opponent player being hated by her.

2. The two power indexes ϕ and β need not agree about the effect of quarreling. Illustrate this phenomenon by considering a BC quarrel in $\underset{A\ B\ C\ D}{[5; 3, 2, 1, 1]}$.

3. In $\underset{A\ B\ C\ D}{[7; 4, 3, 2, 1]}$, consider the effect of one-way quarrels involving A and C, or A and B. For each case you consider, note carefully who is helped and who is hurt.

4. Consider the following weighted voting game:

$$[10; 7, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1].$$

   Show that there exists a bandwagon effect for any one of the uncommitted voters (each with "1" vote) to join the major voter with "7" votes using the Shapley-Shubik index.

5. Analyze the possibility of a bandwagon effect in $[5; 3, 2, 1, 1, 1]$ using the Banzhaf index, where the first two players are not allowed to appear together in a winning coalition.

6. The United States federal system has 537 voters in the system: 435 House of Representatives, 100 Senate members, the Vice President and the President. The Vice President plays the role of tie breaker in the Senate. The President has veto power that can be overridden by two-thirds vote of both the House and the Senate.

   (a) Which of the following pairs are equally desirable (equivalent)? Give the full justification of your answers.
   Vice President and a Senator
   President and a House Representative

   (b) Which of the following pairs are incomparable? Give the full justification of your answers.
   President and a Senator
   Vice President and a House Representative

7. Consider the yes-no voting system in which there are six voters: a, b, c, d, e, f. Suppose the winning coalitions are precisely the ones containing at least two of a, b and c and at least two of d, e and f.

   (a) Show that a and b are equally desirable.

   (b) Show that the desirability of a and d is incomparable.

8. Suppose that x and y are voters in a yes-no voting system and that $x \approx y$. Suppose that Z' is a winning coalition to which both x and y belong. Assume that x's defection from Z' is critical. Prove that y's defection from Z' is also critical.

Hint: Assume, for contradiction, that y's defection from Z' is not critical. Consider the coalition Z arrived at by deleting x and y from Z'.

9. Suppose that x and y are voters in a yes-no voting system and that $x \approx y$. Suppose that Z' is a coalition that contains x but not y. Let Z'' be the coalition resulting from replacing x by y in Z'.

   (a) Prove that if Z' is winning, then Z'' is also winning.

   (b) Prove that if Z' is losing, then Z'' is also losing.

   Hint: Let Z be the result of deleting x from Z' and argue by contradiction.

10. Prove that, in a weighted voting system, we have $x > y$ if and only if x has strictly more weight than y in every weighting that realizes the system.