# Mathematics and Social Choice Theory

## Homework Three

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1. Show that the Alabama Paradox does not occur in Hamilton's method if the number of states $S = 2$. Is it possible that an increase in house size $h$ can cause a state to lose more than one seat using Hamilton's method?

2. Show that

   (a) Hill's method minimizes
   $$\sum_{i=1}^{S} \frac{1}{a_i}(a_i - q_i)^2.$$

   (b) Webster's method minimizes
   $$\sum_{i=1}^{S} \frac{1}{q_i}(a_i - q_i)^2.$$

3. Suppose we take $a_i = \max\left(1, \left\lfloor\left\lfloor \frac{p_i}{\lambda} \right\rfloor\right\rfloor\right)$, where
   $$\lfloor\lfloor x \rfloor\rfloor = \begin{cases} \lfloor x \rfloor & \text{if } x \text{ is not an integer} \\ x \text{ or } x-1 & \text{if } x \text{ is an integer} \end{cases},$$
   and $\lambda$ is some positive number chosen such that
   $$\sum_{i=1}^{S} a_i = h, \quad h > 0.$$

   (a) Explain why one defines $a_i$ by the above definition instead of the intuitive version: $a_i = \left\lfloor \frac{p_i}{\lambda} \right\rfloor$. Construct a numerical example with two states such that there is no solution to
   $$\sum_{i=1}^{2} \left\lfloor \frac{p_i}{\lambda} \right\rfloor = h, \quad h > 0.$$

   (b) Let $\mathcal{S}$ be the subset of all states for which $a_i > 1$. Show that
   $$\max_{\text{for all } i} \frac{p_i}{a_i + 1} \leq \lambda \leq \min_{i \in \mathcal{S}} \frac{p_i}{a_i}.$$

   Why does the right hand side inequality have to exclude the case $a_i = 1$?

4. Suppose the rank index is chosen to be $\dfrac{p}{2a(a+1)/(2a+1)}$, show that the test of inequality is given by $\dfrac{p_i}{a_i} - \dfrac{p_j}{a_j}$.

5. Given the following population data for five states:

| State | $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|---|
| Population | 246 | 1771 | 1529 | 6521 | 6927 |

Assume the house size to be 12. Solve the apportionment problem using

(i) Hamilton's method,

(ii) Jefferson's method,

(iii) Webster's method,

(iv) Hill's method,

(v) Quota method.

6. The Dean method apportions $a_i$ seats to state $i$ if the absolute difference between the common divisor $\lambda$ and the state's average district size $p_i/a_i$ is minimized with $a_i$ seats. This apportionment requirement is enforced for all states.

(a) Show that $\lambda$ satisfies
$$\max_i \frac{p_i}{d(a_i)} \leq \lambda \leq \min_i \frac{p_i}{d(a_i-1)},$$
where $d(a_i) = \frac{(a_i+1)a_i}{a_i+\frac{1}{2}}$.

(b) Describe the corresponding recursive scheme of apportioning the seats using the Dean method. Does the Dean method observe the House Monotone Property? Why?

(c) Does the Dean method automatically satisfy the minimum requirement that every state is allocated at least one seat, given that $h \geq S$? Why?

7. Show that

(a) Jefferson's apportionment solves
$$\min_a \max_i \frac{a_i}{p_i},$$

(ii) Adams' apportionment solves
$$\min_a \max_i \frac{p_i}{a_i}.$$

8. A parametric method is a divisor method $\phi^d$ based on $d(k) = k + \delta$ for all $k, 0 \leq \delta \leq 1$. For example, Adams suggested $\delta = 0$, Jefferson chose $\delta = 1$ and Webster picked $\delta = 0.5$. Suppose $\boldsymbol{a} \in M^\alpha(\boldsymbol{p}, h)$ and $\boldsymbol{a} \in M^\beta(\boldsymbol{p}, h)$, show that for all $\delta$ such that $\alpha \leq \delta \leq \beta$, we have $\boldsymbol{a} \in M^\delta(\boldsymbol{p}, h)$.

9. Show that Webster's method can never produce an apportionment which rounds up for $q_i$ for state $i$ with $q_i - \lfloor q_i \rfloor < 0.5$ while rounding down $q_j$ for state $j$ with $q_j - \lfloor q_j \rfloor > 0.5$.

10. This problem is related to the New States Paradox in Hamilton's method. Suppose a population $\boldsymbol{p} = (p_1 \quad p_2 \quad p_3)$ apportions $h$ seats to $\boldsymbol{a} = (a_1 \quad a_2 \quad a_3)$ in a 3-seat House, this question asks under what condition will the population $\boldsymbol{p} = (p_1 \quad p_2)$ apportion $h - a_3$ seats to $\boldsymbol{a} = (a_1 + 1 \quad a_2 - 1)$. Show that this occurs when
$$\frac{2a_1+1}{2a_2-1} < \frac{p_1}{p_2} < \frac{2a_1+3}{2a_2-3}.$$

*Hint*: The quotas under the 3-seat house and 2-seat house are, respectively,

$$q_i = \frac{p_i}{p_1 + p_2 + p_3} h, i = 1, 2, 3 \quad \text{and} \quad q_i' = \frac{p_i}{p_1 + p_2}(h - a_3), \quad i = 1, 2.$$

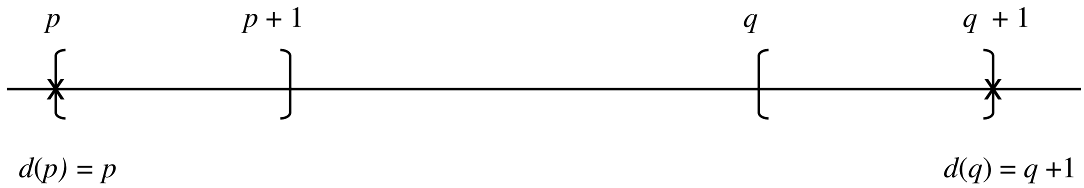
11. Suppose the inequity measure: $\frac{a_i}{a_j} - \frac{p_i}{p_j}$ is used, show that this measure leads to an indefinite cycling of pairwise comparisons when applied to the apportionment problem:

$$\boldsymbol{p} = (762, 534, 304) \quad \text{and} \quad h = 16.$$

12. Recall that we require the divisor $d$ in the class of divisor methods to satisfy:

$$k \leq d(k) \leq k + 1$$

for any non-negative integer $k \geq 0$. Also, we have to *exclude* the case where there are no two integers $p > 0$ and $q \geq 0$ such that $d(p) = p$ and $d(q) = q + 1$. As an illustration, say $p < q$, it is necessary exclude the following case:




Suppose the above case is *not* excluded, show that it is possible to have a paradox.

*Hint*: When $p$ and $q$ are integers, consider the apportionment of $p + q$ seats among two states having respective population $p$ and $q$.