Mathematics and Social Choice Theory

Homework Four

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1. If there are only three alternatives, is the Hare elimination procedure equivalent to the plurality voting with a runoff?

2. Verify that the Majority criterion is weaker than the Condorcet winner criterion, that is, any voting rule which satisfies that Condorcet criterion must also satisfy the Majority criterion.

3. Prove or disprove each of the following statements:

   (a) Plurality voting always yields a unique social choice.
   (b) The Borda count always yields a unique social choice.
   (c) The Hare system always yields a unique social choice.
   (d) Sequential pairwise voting with a fixed agenda always yields a unique social choice.
   (e) A dictatorship always yields a unique social choice.

4. If we have a sequence of individual preference lists, and r and s are two of the alternatives, then "Net($r > s$)" is defined to be the number of voters who prefer r to s minus the number of voters who prefer s to r. Suppose we change the way that we assign points in computing the Borda score of an alternative so that these scores are symmetric about zero. That is, for three alternatives, the first place will be worth 2 points, the second place will be worth 0 points, and the third place will be worth -2 points. We let "$B(r)$" denote the Borda score of the alternative r computed using these points. Consider the following preference lists:

$$\begin{array}{ccc} x & x & y \\ y & z & z \\ z & y & x \end{array}$$

   (a) Evaluate $\text{Net}(x > y)$, $\text{Net}(x > z), B(x)$, and $B(z)$.
   (b) Prove each of the following results for this example:

$$\text{Net}(x > y) + \text{Net}(x > z) = B(x).$$
$$\text{Net}(y > z) + \text{Net}(y > x) = B(y).$$
$$\text{Net}(z > x) + \text{Net}(z > y) = B(z).$$

5. Suppose we have a social choice procedure that satisfies monotonicity. Suppose that for the four alternatives a, b, c, d, we have a sequence of individual preference lists that yields d as the social choice. Suppose person one changes his list:

$$\begin{array}{cccc} \text{from:} & a & \text{to:} & d \\ & b & & a \\ & c & & b \\ & d & & c \end{array}$$

   Show that d is still the social choice, or at least tied for such. The procedure one uses to show this is called "iterating the definition."

6. Suppose we have three voters and four alternatives, and the individual preference lists are as follows:

$$\begin{array}{ccc} a & c & b \\ b & a & d \\ d & b & c \\ c & d & a \end{array}$$

   Show that if the social choice procedure being used is sequential pairwise voting with a fixed agenda, and suppose you have agenda setting power (i.e., you get to choose the order), then you can arrange for whichever alternative you want to be the social choice.

7. Prove that for a given social choice procedure and a given sequence of individual preference lists, a Condorcet winner, if it exists, must be unique.

8. Show that for a fixed sequence of individual preference lists and an odd number of voters, an alternative is a Condorcet winner if and only if it emerges as the social choice in sequential pairwise voting with a fixed agenda regardless of the agenda.

9. An alternative is said to be a *Condorcet loser* if it would be defeated by every other alternative in the kind of one-on-one contest that takes place in sequential pairwise voting with a fixed agenda. Further, say that a social choice procedure satisfies the *Condorcet loser criterion* provided that a Condorcet loser is never among the social choices. Does the Condorcet loser criterion hold for:

   (a) plurality voting?
   (b) the Borda count?
   (c) the Hare system?
   (d) sequential pairwise voting with a fixed agenda?
   (e) a dictatorship?

   Prove or disprove: a Condorcet loser, if it exists, is unique.

10. Prove that for three alternatives and an arbitrary sequence of individual preference lists, there is no Condorcet loser if and only if for each alternative there is an agenda under which that alternative wins in sequential pairwise voting. Your proof should not involve producing three particular preference lists.

11. Consider the following social choice procedure. If there is a Condorcet winner, it is the social choice. Otherwise, the alternative on top of the first person's list is the social choice. (That is, if there is no Condorcet winner, then person one acts as a dictator.) Give an example with three people and three alternatives showing that this procedure does not satisfy independence of irrelevant alternatives. (Hint: start with the same sequence of lists that produces the voting paradox, and then move one alternative that should be irrelevant to the social choice). Prove that this social choice procedure satisfies the Pareto condition.

12. Say that a social choice procedure satisfies the "top condition," provided that an alternative is never among the social choices unless it occurs on top of at least one individual preference list. Prove or disprove each of the following:

   (a) Plurality voting satisfies the top condition.
   (b) The Borda count satisfies the top condition.
   (c) The Hare system satisfies the top condition.
   (d) Sequential pairwise voting satisfies the top condition.
   (e) A dictatorship satisfies the top condition.
   (f) If a procedure satisfies the top condition, then it satisfies the Pareto condition.

13. Consider the following sequence of individual preference lists:

$$\begin{array}{ccccccc} a & a & a & c & c & b & e \\ b & d & d & b & d & c & c \\ c & b & b & d & b & d & d \\ d & e & e & e & a & a & b \\ e & c & c & a & e & e & a \end{array}$$

(a) Consider the social welfare function arrived at by iterating the plurality procedure. Write down the social preference list that results from applying this function to the above sequence of individual preference lists.
(b) Do the same for the social welfare function arrived at by iterating the Borda count.
(c) Do the same for the social welfare function arrived at by iterating the Hare procedure.
(d) Do the same for the social welfare function arrived at by iterating sequential pairwise voting with a fixed agenda.
(e) Do the same for the social welfare function arrived at by iterating the procedure where the last person on the right is a dictator.

14. Without using Arrow's theorem, show that if a social welfare function satisfies both Pareto and independence of irrelevant alternatives, then

(a) the set P of all individuals is a dictating set, and
(b) a set $\{p\}$ consisting of only one person is a dictating set if and only if p is a dictator.

15. Suppose $A = \{a, b, c\}$ and a given social welfare function produces: output $\begin{array}{c} a \\ b \\ c \end{array}$ when the confront with input $\begin{array}{ccc} c & a & b \\ b & c & c \\ a & b & a \end{array}$

(a) If neutrality is satisfied, what is the output when confronted with input:

$$\begin{array}{ccc} a & c & b \\ b & a & a \\ c & b & c \end{array}$$

(b) What input would definitely yield c over a over b as output?