

Mathematical Models in Economics and Finance

Homework One

Course instructor: Prof. Y.K. Kwok

- (a) Show that the Alabama Paradox does not occur in Hamilton's method if the number of states $S = 2$.
(b) Is it possible that an increase in house size h can cause a state to lose more than one seat using Hamilton's method?

2. Show that

(a) Hill's method minimizes

$$\sum_{i=1}^S \frac{1}{a_i} (a_i - q_i)^2.$$

(b) Webster's method minimizes

$$\sum_{i=1}^S \frac{1}{q_i} (a_i - q_i)^2.$$

3. Suppose we take $a_i = \max\left(1, \left\lfloor \left\lfloor \frac{p_i}{\lambda} \right\rfloor \right\rfloor\right)$, where

$$\lfloor \lfloor x \rfloor \rfloor = \begin{cases} \lfloor x \rfloor & \text{if } x \text{ is not an integer} \\ x \text{ or } x - 1 & \text{if } x \text{ is an integer} \end{cases},$$

and λ is some positive number chosen such that

$$\sum_{i=1}^S a_i = h, \quad h > 0.$$

(a) Explain why one defines a_i by the above definition instead of the intuitive version: $a_i = \left\lfloor \frac{p_i}{\lambda} \right\rfloor$. Construct a numerical example with two states such that there is no solution to

$$\sum_{i=1}^2 \left\lfloor \frac{p_i}{\lambda} \right\rfloor = h, \quad h > 0.$$

(b) Let \mathcal{S} be the subset of all states for which $a_i > 1$. Show that

$$\max_{\text{for all } i} \frac{p_i}{a_i + 1} \leq \lambda \leq \min_{i \in \mathcal{S}} \frac{p_i}{a_i}.$$

Why does the right hand side inequality have to exclude the case $a_i = 1$?

4. Suppose the rank index is chosen to be $\frac{p}{2a(a+1)/(2a+1)}$, show that the test of inequality is given by $\frac{p_i}{a_i} - \frac{p_j}{a_j}$.

5. Given the following population data for five states:

State	A	B	C	D	E
Population	246	1771	1529	6521	6927

Assume the house size to be 12. Solve the apportionment problem using

- (i) Hamilton's method,
- (ii) Jefferson's method,
- (iii) Webster's method,
- (iv) Hill's method,
- (v) Quota method.

6. The Dean method apportions a_i seats to state i if the absolute difference between the common divisor λ and the state's average district size p_i/a_i is minimized with a_i seats. This apportionment requirement is enforced for all states.

(a) Show that λ satisfies

$$\max_i \frac{p_i}{d(a_i)} \leq \lambda \leq \min_i \frac{p_i}{d(a_i - 1)},$$

$$\text{where } d(a_i) = \frac{(a_i+1)a_i}{a_i+\frac{1}{2}}.$$

- (b) Describe the corresponding recursive scheme of apportioning the seats using the Dean method. Does the Dean method observe the House Monotone Property? Why?
- (c) Does the Dean method automatically satisfy the minimum requirement that every state is allocated at least one seat, given that $h \geq S$? Why?

7. Show that

(a) Jefferson's apportionment solves

$$\min_a \max_i \frac{a_i}{p_i},$$

(ii) Adams' apportionment solves

$$\min_a \max_i \frac{p_i}{a_i}.$$

8. A parametric method is a divisor method ϕ^d based on $d(k) = k + \delta$ for all $k, 0 \leq \delta \leq 1$. For example, Adams suggested $\delta = 0$, Jefferson chose $\delta = 1$ and Webster picked $\delta = 0.5$. Suppose $\mathbf{a} \in M^\alpha(\mathbf{p}, h)$ and $\mathbf{a} \in M^\beta(\mathbf{p}, h)$, show that for all δ such that $\alpha \leq \delta \leq \beta$, we have $\mathbf{a} \in M^\delta(\mathbf{p}, h)$.

9. Show that Webster's method can never produce an apportionment which rounds up for q_i for state i with $q_i - [q_i] < 0.5$ while rounding down q_j for state j with $q_j - [q_j] > 0.5$.

10. This problem is related to the New States Paradox in Hamilton's method. Suppose a population $\mathbf{p} = (p_1 \ p_2 \ p_3)$ apportions h seats to $\mathbf{a} = (a_1 \ a_2 \ a_3)$ in a 3-seat House, this question asks under what condition will the population $\mathbf{p} = (p_1 \ p_2)$ apportion $h - a_3$ seats to $\mathbf{a} = (a_1 + 1 \ a_2 - 1)$. Show that this occurs when

$$\frac{2a_1 + 1}{2a_2 - 1} < \frac{p_1}{p_2} < \frac{2a_1 + 3}{2a_2 - 3}.$$

Hint: The quotas under the 3-seat house and 2-seat house are, respectively,

$$q_i = \frac{p_i}{p_1 + p_2 + p_3} h, i = 1, 2, 3 \quad \text{and} \quad q'_i = \frac{p_i}{p_1 + p_2} (h - a_3), \quad i = 1, 2.$$

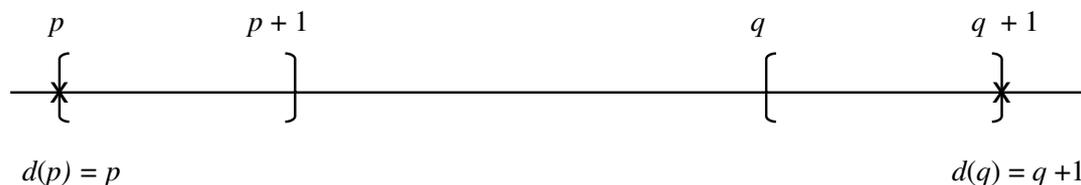
11. Suppose the inequity measure: $\frac{a_i}{a_j} - \frac{p_i}{p_j}$ is used, show that this measure leads to an indefinite cycling of pairwise comparisons when applied to the apportionment problem:

$$\mathbf{p} = (762, 534, 304) \quad \text{and} \quad h = 16.$$

12. Recall that we require the divisor d in the class of divisor methods to satisfy:

$$k \leq d(k) \leq k + 1$$

for any non-negative integer $k \geq 0$. Also, we have to *exclude* the case where there are no two integers $p > 0$ and $q \geq 0$ such that $d(p) = p$ and $d(q) = q + 1$. As an illustration, say $p < q$, it is necessary to exclude the following case:



Suppose the above case is *not* excluded, show that it is possible to have a paradox.

Hint: When p and q are integers, consider the apportionment of $p + q$ seats among two states having respective population p and q .