Mathematical Models in Economics and Finance

Homework Two

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1. Two simple games are *equivalent* (or *isomorphic*) if the players can be labeled in such a way that the winning coalitions are the same in both games. Show that the following three-person games are all equivalent.

   (i) $N = \{A, B, C\}$. Approval is by majority vote, but A has a *veto*.

   (ii) $[3; 2, 1, 1]$.

   (iii) $[5; 3, 2, 2]$.

   (iv) $[17; 16, 1, 1]$.

2. Suppose the U.N. Security Council has eight nonpermanent members and three permanent members with passage of a bill requiring a total of seven votes subject to the veto power of each of the permanent members. Prove that this is a weighted voting system. Include a discussion showing how you found the appropriate weights and quota.

3. Which of the following properties about "winning coalition" are considered to be reasonable? Give your justification.

   (a) If X is a winning coalition and every voter in X is also in Y, then Y is also a winning coalition.

   (b) If X and Y are winning coalitions, then so is the coalition consisting of voters in both X and in Y.

   (c) If X and Y are disjoint (that is, have no voters in common), then at least one fails to be a winning coalition.

   (d) If X and Y are winning coalitions, then so is the coalition consisting of voters in either X or in Y.

   (e) If X is a winning coalition and X is split into two sets Y and Z so that every voter in X is in exactly one of Y and Z, then either Y is a winning coalition or Z is a winning coalition.

4. Suppose we have six voters: A_1, A_2, A_3, A_4, A_5, and A_6. Let S_3 denote the yes-no voting system wherein a coalition X is winning if and only if

   (i) at least one of A_1 and A_2 is in X,

   (ii) at least one of A_3 and A_4 is in X, and

   (iii) at least one of A_5 and A_6 is in X.

   To show that the yes-no voting system has dimension equals 3, it is necessary to establish the following two results.

   (a) Show that S_3 can be represented as the intersection of three weighted voting systems.

   (b) Show that S_3 cannot be represented as the intersection of two weighted voting systems.

Hint: We prove by contradiction. Suppose that S_3 can be expressed as the intersection of two weighted systems. Since $\{A_1, A_2, A_3, A_4\}$, $\{A_1, A_2, A_5, A_6\}$, and $\{A_3, A_4, A_5, A_6\}$ are all losing, each one must be losing in at least one of the two weighted systems. Hence, two of these – say $\{A_1, A_2, A_3, A_4\}$ and $\{A_3, A_4, A_5, A_6\}$ – are losing in the same weighted system. Show that the desired contradiction can be achieved by an appropriate swap.

5. Consider the "yes-no" voting system with minority veto where the 7 voters are classified into the majority group of 5 voters and the minority group of 2 voters. A bill is passed by requiring at least 4 votes from all voters and at least 1 vote from the minority voters.

   (a) Show that the above "yes-no" system is swap robust but not trade robust.

   (b) Show that this "yes-no" system is the intersection of two weighted voting systems, that is, a coalition is winning in the "yes-no" system if and only if it is winning in both weighted voting systems.

   (c) Find the Shapley-Shubik index and the Banzhaf index for the voters in the majority group and the minority group.

6. Calculate the Shapley-Shubik and Banzhaf indices for the following weighted voting games:

   (a) $[5; 4, 2, 1, 1, 1]$;

   (b) $[9; 5, 4, 3, 2, 1]$.

7. A seven-person legislature has a three-person committee. Approval must be achieved by a majority of *both* the committee and the entire legislature. Denote the members by $AAAbbbb$. Compute the Shapley-Shubik and Banzhaf power indices. What is the ratio of power between a committee member and a noncommittee member?

8. The passage of a bill requires majority of votes. Calculate the Shapley-Shubik and Banzhaf indices for the large stockholder L with 40% of the shares if the remaining shares are split evenly among

   (a) five other stockholders;

   (b) seven other stockholders.

   Give the intuition why the large stockholder is less powerful in (b) than he is in (a)?

9. Consider a voting system consisting of 3 big states and 6 small states, passage of a bill requires "yes" vote from all the big states and at least 2 "yes" votes from the 6 small states.

   (a) Find the weighted voting vector of the above game, specifying the quota and the number of votes held by each of the big states and small states.

   (b) Consider one of the big states, assuming the homogeneity assumption on voting probabilities among all 9 states, find the probability $\pi_b(p)$ that the vote of this particular big state makes a difference between approval or rejection of a bill. Here, p denotes the common homogeneous voting probability.

   (c) Using the above $\pi_b(p)$, or otherwise, compute the Shapley-Shubik index and Banzhaf index for any one of the big states. Then deduce the values for the above two power indexes for any one of the small states.

(d) Suppose the 3 big states vote independently while the set of 6 smaller states vote as a homogeneous group. Show how to compute the Shapley-Shubik indexes.

10. Consider $\underset{A\ B\ C\ D}{[5; 3, 2, 1, 1]}$.

    Show that

$$\begin{array}{rcl} \pi_A(p) = & p + (1-p)p^2 & = p + p^2 - p^3 \\ \pi_B(p) = & p(1-p^2) & = p - p^3 \\ \pi_C(p) = & p(1-p)p & = p^2 - p^3. \end{array}$$

    Hence, check that the Banzhaf index and Shapley-Shubik index are given by

$$\begin{aligned} \beta &= \left(\frac{5}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right), \\ \phi &= \left(\frac{7}{12}, \frac{3}{12}, \frac{1}{12}, \frac{1}{12}\right), \end{aligned}$$

    respectively.

11. The two power indexes ϕ and β need not agree about the effect of quarreling. Illustrate this phenomenon by considering a BC quarrel in $\underset{A\ B\ C\ D}{[5; 3, 2, 1, 1]}$.

12. In $\underset{A\ B\ C\ D}{[7; 4, 3, 2, 1]}$, consider the effect of one-way quarrels involving A and C, or A and B.

    For each case, note carefully who is helped and who is hurt.

13. Consider the following weighted voting game:

$$[10; 7, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1].$$

    Show that there exists a bandwagon effect for any one of the uncommitted voters (each with "1" vote) to join the major voter with "7" votes using the Shapley-Shubik index.

14. Analyze the possibility of a bandwagon effect in $[5; 3, 2, 1, 1, 1]$ using the Banzhaf index, where the first two players are not allowed to appear together in a winning coalition.

15. The United States federal system has 537 voters in the system: 435 House of Representatives, 100 Senate members, the Vice President and the President. The Vice President plays the role of tie breaker in the Senate. The President has veto power that can be overridden by two-thirds vote of both the House and the Senate.

    (a) Which of the following pairs are equally desirable (equivalent)? Give the full justification of your answers.
        Vice President and a Senator
        President and a House Representative

    (b) Which of the following pairs are incomparable? Give the full justification of your answers.
        President and a Senator
        Vice President and a House Representative

16. Suppose that x and y are voters in a yes-no voting system and that $x \approx y$. Suppose that Z' is a winning coalition to which both x and y belong. Assume that x's defection from Z' is critical. Prove that y's defection from Z' is also critical.

    Hint: Assume, for contradiction, that y's defection from Z' is not critical. Consider the coalition Z obtained by deleting x and y from Z'.

17. Suppose that x and y are voters in a yes-no voting system and that $x \approx y$. Suppose that Z' is a coalition that contains x but not y. Let Z'' be the coalition resulting from replacing x by y in Z'.

    (a) Prove that if Z' is winning, then Z'' is also winning.

    (b) Prove that if Z' is losing, then Z'' is also losing.

    Hint: Let Z be the result of deleting x from Z' and argue by contradiction.