Mathematical Models in Economics and Finance

Solution to Homework Two

Course instructor: *Prof. Y.K. Kwok*

---

1. To show the equivalence, it suffices to consider the set of winning coalitions for each of the voting systems and check whether they are the same. It is seen that all of the 3-person games lead to the same set of winning coalitions, namely, $W = \{AB, AC, ABC\}$. Hence, all these 3-person games are equivalent.

2. (a) Show that the voting system is trade robust.
   We let P_i, $i = 1, 2, 3$, be the permanent members and N_j, $j = 1, 2, \ldots, 8$, be the non-permanent members. In order to pass a bill, the winning coalition has to take the following form:
   $$W = \{P_1, P_2, P_3, \text{ at least 4 } N_j\text{'s}\}.$$
   It suffices to show that any arbitrary exchange of players (a series of trades involving groups of players) among several winning coalitions leaves at least one of the coalitions winning. Since P_1, P_2 and P_3 stay in all winning coalitions, they remain in any coalition after a series of trades as it is not sensible to include any one of them in the trades. Also, it is impossible to have all coalitions to have less than 4 N_js; if otherwise, it violates the property that the average of N_j in the winning coalitions before trades is at least 4. Therefore, at least one of the coalitions after trades remains to be winning. Hence, the voting system is trade robust.

   (b) Express the yes-no system into a weighted voting system.
   We let the voting weight of N_j and P_i be 1 and w, respectively. Also, we let q denote the quota. According to the rule of passage of a bill, where each of P_i has veto power, we deduce the following pair of inequalities:
   $$3w + 4 \geq q \quad \text{and} \quad 2w + 8 < q.$$
   Combining the inequalities, we obtain
   $$3w + 4 > 2w + 8 \quad \text{giving} \quad w > 4.$$
   Suppose we set $w = 5$, then q satisfies
   $$19 \geq q > 18,$$
   giving $q = 19$. The corresponding weighted voting system is given by
   $$[19; 5, 5, 5, 1, 1, 1, 1, 1, 1, 1, 1].$$

3. (a) **Reasonable.** Adding more voters in a winning coalition X to form Y gives the enlarged coalition Y to remain winning.

   (b) **Non-reasonable.** Quote a counter-example. Consider the 3-person voting game in which approval is by majority vote. Take $X = \{A, B\}$ and $Y = \{B, C\}$, both are winning. However, $X \cap Y = \{B\}$ is losing.

   (c) **Non-reasonable.** If X and Y are disjoint, then Y is a subset of the complement of X. Suppose both X and Y are winning, the complement of X is also winning [by virtue of (a)]. However, it is not reasonable to have both X and its complement to be both winning.

(d) **Reasonable.** Since $X \cup Y \supseteq X$ and X is winning, by virtue of (a), $X \cup Y$ is winning.

(e) **Non-reasonable.** Quote a counter-example. Consider the 3-person voting game in which approval is by majority vote. Take $X = \{A, B\}$, $Y = \{A\}$ and $Z = \{B\}$. Obviously, both Y and Z are not winning coalitions.

4. (a) S_3 can be represented as the intersection of the following three weighted systems:
   (i) System 1: quota = 1
   weight 1 to A_1 and A_2; zero weight to others.
   (ii) System 2: quota = 1
   weight 1 to A_3 and A_4; zero weight to others.
   (iii) System 3: quota = 1
   weight 1 to A_5 and A_6; zero weight to others.

   (b) We prove by contradiction. Suppose that S_3 can be expressed as the intersection of two weighted systems. Note that both

$$\{A_1, A_3, A_4, A_5\} \quad \text{and} \quad \{A_2, A_3, A_4, A_6\}$$

   are both winning under yes-no voting system S_3. According to the assumption, both coalitions are winning in both of these two weighted systems. Recall that every weighted system is swap robust. Suppose we swap A_5 for A_2 in these two coalitions, the resulting coalitions are

$$\{A_1, A_2, A_3, A_4\} \quad \text{and} \quad \{A_3, A_4, A_5, A_6\}.$$

   Unfortunately, both the resulting new coalitions after the swap are losing. This is a violation of the trade robust property. We have a contradiction.

5. (a) The voting system is swap robust. Consider any pair of winning coalitions W_1 and W_2, and let W_1^* and W_2^* be the coalitions after 1-1 swap. First, the number of voters in each of W_i^*, $i = 1, 2$, remains to be at least 4. The average number of minority voters in W_1 and W_2 is at least one. The average remains unchanged after the 1-1 swap. Hence, at least one of W_1^* and W_2^* contains one or more minority voters. Therefore, at least one of W_1^* and W_2^* remains to be winning.
   However, the voting system is not trade robust. Consider the following two winning coalitions:

$$W_1 = \{M_1, M_2, M_3, m_1\} \quad \text{and} \quad W_2 = \{M_3, M_4, M_5, m_2\}.$$

   Suppose we move $\{M_1, M_2\}$ to W_2 and m_2 to W_1, the resulting new coalitions after this trade are

$$W_1^* = \{M_3, m_1, m_2\} \quad \text{and} \quad W_2^* = \{M_1, M_2, M_3, M_4, M_5\}.$$

   It is seen that both W_1^* and W_2^* are losing. Hence, the system is not trade robust.

   (b) We consider two voting systems:

$$S_1 : \text{winning if the coalition contains at least 4 voters;}$$
$$S_2 : \text{winning if the coalition contains at least 1 minority voter.}$$

   It is easily seen that a coalition in the given yes-no voting system is winning if and only if it wins in both S_1 and S_2. Both S_1 and S_2 are weighted voting systems, where

$$S_1 = [4 : 1, 1, 1, 1, 1, 1, 1] \quad \text{and} \quad S_2 = [1 : 0, 0, 0, 0, 0, 1, 1].$$

   The given yes-no voting system is the intersection of the weighted voting systems S_1 and S_2.

6. (a) Consider the weighted voting system $[5; 4, 2, 1, 1, 1]$:

(i) Shapley-Shubik indexes

The 4-vote player is pivotal if there are one to three other player entering into a coalition before he enters. The number of such orderings is

$$O_4 = \sum_{n=1}^{3} c_n^4 n!(4-n)! = 72.$$

The 2-vote player is pivotal if the 4-vote player joins earlier or all three 1-vote player join earlier in a coalition. The number of such orderings is

$$O_2 = 1!3! + 3!1! = 12.$$

The 1-vote player is pivotal if either (1) the 4-vote player and one 1-vote player join earlier, or (2) the 2-vote player and two 1-vote players join earlier in a coalition. The number of such orderings is

$$O_1 = c_1^2 2!2! + 2!2! = 12.$$

The individual Shapley-Shubik indexes are

$$\Phi_4 = \frac{72}{5!} = \frac{3}{5}, \Phi_2 = \frac{12}{120} = \frac{1}{10} \text{ and } \Phi_1 = \frac{12}{120} = \frac{1}{10}.$$

Surprisingly, the 2-vote player and the three 1-vote players are equally powerful under the Shapley-Shubik power index.

(ii) Banzhaf indexes

The 4-vote player is marginal in the winning coalition if the winning coalition contains one to three other players. The number of such coalitions is

$$B_4 = \sum_{n=1}^{3} c_n^4 = 14.$$

The 2-vote player is marginal in the winning coalition if the winning coalition contains either (1) the 4-vote player only, or (2) all the three 1-vote players. The number of such coalitions is $B_2 = 2$. Lastly, any one of the 1-vote players is marginal if the winning coalition contains either (1) the 4-vote player and one of the other two 1-vote players, or (2) the 2-vote player and both of the other two 1-vote players. The number of such coalitions is $B_1 = c_1^2 + 1 = 3$. The individual Banzhaf indexes are given by

$$\beta_4 = \frac{14}{14 + 2 + 3 \times 3} = \frac{14}{23}, \beta_2 = \frac{2}{23} \text{ and } \beta_1 = \frac{3}{23}.$$

Surprisingly, the 2-vote player is less powerful than any one of the three 1-vote players under the Banzhaf power index.

(b) Consider the weighted voting system $[9; 5, 4, 3, 2, 1]$:

(i) Shapley-Shubik indexes

The 5-vote player is pivotal if players "4" or "4, 1" or "4, 2" or "4, 3" or "4, 1, 2" or "4, 1, 3" or "3, 1" or "3, 2" or "3, 1, 2" entering into a coalition before he enters. The number of such orderings is

$$O_5 = 1!3! + 5 \times 2!2! + 3 \times 3!1! = 44.$$

The 4-vote player is pivotal if players "5" or "5, 1" or "5, 2" or "5, 3" or "5, 1, 2" or "3, 2" or "3, 1, 2" entering into a coalition before he enters. The number of such orderings is

$$O_4 = 1!3! + 4 \times 2!2! + 2 \times 3!1! = 34.$$

The 3-vote player is pivotal if players "5, 1" or "5, 2" or "5, 1, 2" or "4, 2" or "4, 1, 2" entering into a coalition before he enters. The number of such orderings is

$$O_3 = 3 \times 2!2! + 2 \times 3!1! = 24.$$

(ii) Banzhaf indexes
The 5-vote player is marginal in a winning coalition if the winning coalition contains either "4" or "4, 1" or "4, 2" or "4, 3" or "4, 1, 2" or "4, 1, 3" or "3, 1" or "3, 2" or "3, 1, 2". The number of such coalitions is $B_5 = 9$.
The 4-vote player is marginal in a winning coalition if the winning coalition contains either "5" or "5, 1" or "5, 2" or "5, 3" or "5, 1, 2" or "3, 2" or "3, 1, 2". The number of such coalition is $B_4 = 7$.
The 3-vote player is marginal in a winning coalition if the winning coalition contains either "5, 1" or "5, 2" or "5, 1, 2" or "4, 2" or "4, 1, 2". The number of such coalitions is $B_3 = 5$.
The 2-vote player is marginal in a winning coalition if the winning coalition contains either "5, 3" or "4, 3" or "4, 3, 1". The number of such coalitions is $B_2 = 3$.
The 1-vote player is marginal in a winning coalition if the winning coalition contains either "5, 3" only. The number of such coalitions is $B_1 = 1$.
The individual Banzhaf indexes are given by

$$\beta_5 = \frac{9}{9+7+5+3+1} = \frac{9}{25}, \quad \beta_4 = \frac{7}{25}, \quad \beta_3 = \frac{5}{25} = \frac{1}{5},$$

$$\beta_2 = \frac{3}{25} \quad \text{and} \quad \beta_1 = \frac{1}{25}.$$

7. According to the rule, the passage of a bill requires support from at least 4 members in the entire legislature and at least 2 votes from the 3-person committee.

(i) Shapley-Shubik indexes
A committee member A is pivotal if there are either (1) two other "A"s and exactly one "b", or (2) exactly one "A" and at least two "b"s entering into a coalition before A enters. The number of such orderings is

$$O_A = c_1^4 3!3! + \sum_{n=2}^{4} c_1^2 c_n^4 (n+1)!(4-n+1)! = 4! \times 50.$$

The Shapley-Shubik indexes of A and b are given by

$$\Phi_A = \frac{4! \times 50}{7!} = \frac{5}{21} \quad \text{and} \quad \Phi_b = \frac{1}{4}(1 - \Phi_A) = \frac{1}{14}.$$

Note that we have used the relation: sum of all Shapley-Shubik indexes equals one. The ratio of power (Shapley-Shubik) between a committee member and a non-committee member is $\dfrac{5/21}{1/14} = \dfrac{10}{3}$.

(ii) Banzhaf indexes
A committee member A is marginal in a winning coalition if the coalition contains either (1) other two "A"s and exactly one "b", or (2) exactly one "A" and at least two "b"s. The number of such coalitions is

$$B_A = c_1^4 + \sum_{n=2}^{4} c_1^2 c_n^4 = 26.$$

A non-committee member is marginal in the winning coalition if the coalition also contains n "A"s and exactly $3-n$ other "b"s, where $n \geq 2$. The number of such coalitions is

$$B_b = \sum_{n=2}^{3} c_n^3 c_{3-n}^3 = 10.$$

The individual Banzhaf indexes are given by

$$\beta_A = \frac{26}{26 \times 3 + 10 \times 4} = \frac{13}{59} \quad \text{and} \quad \beta_b = \frac{10}{26 \times 3 + 10 \times 4} = \frac{5}{59}.$$

The ratio of power (Banzhaf) between a committee member and a non-committee member is $\frac{13}{5}$.

8. (a) With 5 other equally split stockholders, the proportion of shares held by each of these 5 stockholders is $\frac{100\% - 40\%}{5} = 12\%$.

(i) Shapley-Shubik indexes
L is pivotal if there are one, two, three or four other small stockholders (S) entering into the coalition before L enters. The number of such orderings is

$$O_L = \sum_{n=1}^{4} c_n^5 n!(5-n)! = 4 \times 5!.$$

The individual Shapley-Shubik indexes are given by

$$\Phi_L = \frac{4 \times 5!}{6!} = \frac{2}{3}, \quad \Phi_S = \frac{1}{5}(1 - \Phi_L) = \frac{1}{15}.$$

(ii) Banzhaf indexes
L is marginal in the winning coalition if the coalition also contains $1 \sim 4$ other stockholders. The number of such coalitions is $b_L = \sum_{n=1}^{4} c_n^5 = 30$.
S is marginal in the winning coalition if the coalition also contains either (1) L only or (2) exactly 4 S's. The number of such coalitions is $b_S = 1 + c_4^4 = 2$. The Banzhaf indexes are given by

$$\beta_L = \frac{30}{30 + 5 \times 2} = \frac{3}{4}, \quad \beta_S = \frac{2}{30 + 5 \times 2} = \frac{1}{20}.$$

(b) With 7 other equally split stockholders, the proportion of shares held by each of these 7 stockholders is $\frac{100\% - 40\%}{7} = 8.57\%$.

(i) Shapley-Shubik indexes
L is pivotal if there are two, three, four or five other small stockholders (S) entering into the coalition before L enters. The number of such orderings is

$$O_L = \sum_{n=2}^{5} c_n^7 n!(7-n)! = 4 \times 7!.$$

The individual Shapley-Shubik indexes are given by

$$\Phi_L = \frac{4 \times 7!}{8!} = \frac{1}{2}, \quad \Phi_S = \frac{1}{7}(1 - \Phi_L) = \frac{1}{14}.$$

(ii) Banzhaf indexes
L is marginal in a winning coalition if the coalition also contains two, three, four or five small stockholders. The number of such coalitions is $B_L = \sum_{n=2}^{5} c_n^7 = 112$.
S is marginal in a winning coalition if the coalition also contains either (1) L and exactly one other "S", or (2) exactly five other small stockholders. The number of such coalition is $B_5 = c_1^6 + c_5^6 = 12$.
The individual Banzhaf indexes are given by

$$\beta_L = \frac{112}{112 + 7 \times 12} = \frac{4}{7} \quad \text{and} \quad \beta_S = \frac{12}{112 + 7 \times 12} = \frac{3}{49}.$$

When the remaining proportion of shares are split among a large number of stockholders, the chance of forming a winning coalitions among the small stockholders against the major stockholder is less, so the major stockholder is more powerful when there are more equally split small stockholders.

9. (a) Let the voting weight of each small state be 1 and the voting weight of each big state be x. The quota q must satisfy

$$3x + 2 \geq q \quad \text{and} \quad q > 2x + 6.$$

Solving the inequalities yields $x > 4$. Suppose we take $x = 5$, then q satisfies $17 \geq q > 16$, so $q = 17$. The yes-no voting system can be written as the weighted voting system with voting vector specified as $[17; 5, 5, 5, 1, 1, 1, 1, 1, 1]$.

(b) $\pi_b(p) = P(\text{other 2 big states say "yes" and at least 2 small states say "yes"})$

$$= p^2 \left[\sum_{k=2}^{6} c_k^6 p^k (1-p)^{6-k} \right] = 15p^4 - 40p^5 + 45p^6 - 24p^7 + 5p^8.$$

(c) The Shapley-Shubik index and Banzhaf index of any of the big states are given by

$$\Phi_b = \int_0^1 \pi_b(p)\, dp = \int_0^1 (15p^4 - 40p^5 + 45p^6 - 24p^7 + 5p^8)\, dp = \frac{20}{63};$$

$$\beta_b = \pi_b\left(\frac{1}{2}\right) = \frac{51}{256}.$$

The Shapley-Shubik index and Banzhaf index of any of the small states are given by

$$\Phi_s = \frac{1 - 3\Phi_b}{6} = \frac{1}{126} \quad \text{and} \quad \beta_s = \frac{1 - 3\beta_b}{6} = \frac{103}{1536}.$$

(d) Assume that the 3 big states vote independently and the 6 smaller states vote as a homogeneous group. Let p_1, p_2 and p_3 be the voting probabilities of the 3 big states, respectively, and p be the common voting probability of the small states. We first compute $\pi_{b_k}(p, p_1, p_2, p_3)$, and $\pi_s(p, p_1, p_2, p_3)$ in terms of p_1, p_2, p_3, p as follows:

(i) $\pi_{b_1}(p, p_1, p_2, p_3) = P(\text{other 2 big states say "yes" and at least 2 small states say "yes"})$

$$= p_2 p_3 \left[\sum_{k=2}^{6} c_k^6 p^k (1-p)^k \right];$$

(ii) $\pi_{b_2}(p,p_1,p_2,p_3) = p_1p_3\left[\sum_{k=2}^{6} c_k^6 p^k(1-p)^k\right]$;

(iii) $\pi_{b_3}(p,p_1,p_2,p_3) = p_1p_2\left[\sum_{k=2}^{6} c_k^6 p^k(1-p)^k\right]$;

(iv) $\pi_s(p,p_1,p_2,p_3) = P(3$ big states say "yes" and exactly one small state say "yes"$)$
$= p_1p_2p_3[c_1^5 p(1-p)^4]$.

The Shapley-Shubik indexes are given by

$$\Phi_{b_1} = E[\pi_{b_1}(p,p_1,p_2,p_3)] = \int_0^1\int_0^1\int_0^1 p_2p_3\left[\sum_{k=2}^{6} c_k^6 p^k(1-p)^k\right] dp_2dp_3dp;$$

$$\Phi_s = E[\pi_s(p,p_1,p_2,p_3)] = \int_0^1\int_0^1\int_0^1\int_0^1 p_1p_2p_3\left[c_1^5 p(1-p)^4\right] dp_1dp_2dp_3dp.$$

10. Let p be the voting probability of each of the four players, assuming homogeneity. We have

$$\begin{aligned}
\pi_A(p) &= P(B \text{ say "yes", zero, one or two of } C \text{ and } D \text{ say "yes"})\\
&\quad + P(B \text{ say "no", both of } C \text{ and } D \text{ say "yes"})\\
&= p[(1-p)^2 + 2p(1-p) + p^2] + (1-p)p^2 = p + p^2 - p^3;\\
\pi_B(p) &= P(A \text{ say "yes" and zero or one of } C \text{ and } D \text{ say "yes"});\\
&= p[(1-p)^2 + 2p(1-p)] = p - p^3;\\
\pi_C(p) &= P(A \text{ say "yes", } B \text{ say "no", } D \text{ say "yes"}) = p^2(1-p) = p^2 - p^3;\\
\pi_D(p) &= \pi_C(p) = p^2 - p^3.
\end{aligned}$$

The Shapley-Shubik indexes and Banzhaf indexes of the players are found to be

$$\beta = \left(\pi_A\left(\frac{1}{2}\right), \pi_B\left(\frac{1}{2}\right), \pi_C\left(\frac{1}{2}\right), \pi_D\left(\frac{1}{2}\right)\right) = \left(\frac{5}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right);$$

$$\Phi = \left(\int_0^1 \pi_A(p)\ dp, \int_0^1 \pi_B(p)\ dp, \int_0^1 \pi_C(p)\ dp, \int_0^1 \pi_D(p)\ dp\right) = \left(\frac{7}{12}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}\right).$$

11. Let us consider the weighted voting system: $[5; \underset{A}{3}, \underset{B}{2}, \underset{C}{1}, \underset{D}{1}]$. Without quarrel, the power indexes are

$$\Phi = \left(\frac{7}{12}, \frac{3}{12}, \frac{1}{12}, \frac{1}{12}\right) \quad \text{and} \quad \beta = \left(\frac{5}{10}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10}\right).$$

However, when B and C quarrel, we obtain

$$\Phi_{BC}^Q = \left(\frac{3}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}\right) \quad \text{and} \quad \beta_{BC}^Q = \left(\frac{3}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}\right).$$

It is quite disquieting to observe that the Shapley-Shubik index shows that B increases in power while the reverse result occurs under the Banzhaf index.

12. Consider the weighted voting system: $[7; 4, 3, 2, 1]$, the Shapley-Shubik indexes are found to be

$$\Phi = \left(\frac{7}{12}, \frac{3}{12}, \frac{1}{12}, \frac{1}{12}\right).$$

Consider the power indexes under one-way quarrel, where

$$\Phi^Q_{A\to C} = \left(\frac{4}{12}, \frac{5}{12}, \frac{2}{12}, \frac{1}{12}\right) \qquad \text{helps the hated}$$

$$\Phi^Q_{C\to A} = \left(\frac{14}{20}, \frac{5}{20}, 0, \frac{1}{20}\right) \qquad \text{helps the hated, } C \text{ becomes powerless}$$

$$\Phi^Q_{A\to B} = \left(\frac{2}{7}, \frac{3}{7}, \frac{1}{7}, \frac{1}{7}\right) \qquad \text{helps the hated}$$

$$\Phi^Q_{B\to A} = \left(\frac{7}{9}, 0, \frac{1}{9}, \frac{1}{9}\right) \qquad \text{helps the hated, } B \text{ becomes powerless.}$$

13. Before any 1-vote player joins the 7-vote player, the Shapley-Shubik indexes are

$$\Phi_7 = \frac{91}{180} = 50.6\% \quad \text{and} \quad \Phi_1 = \frac{13}{1260} = 1.03\%.$$

When the 7-vote player is joined with any 1-vote player, the Shapley-Shubik index becomes $\Phi_8 = \dfrac{11}{18} = 61.1\%$.
The 1-vote player rises the power of the 7-vote player from 50.6% to 61.1%, an increment of 10.5%. This is far more than 1.03%, which is the power index of the uncommitted 1-vote player. Obviously, a bandwagon effect is observed.

14. Consider the weighted voting system: $[5; 3, 2, 1, 1, 1]$, the Banzhaf indexes are found to be

$$\beta = \left(\frac{4}{14}, \frac{1}{14}, \frac{3}{14}, \frac{3}{14}, \frac{3}{14}\right) = (0.29, 0.071, 0.21, 0.21, 0.21).$$

Suppose the 3-vote player is joined with any 1-vote player, the Banzhaf indexes become

$$\beta' = \left(\frac{3}{5}, 0, \frac{1}{5}, \frac{1}{5}\right) = (0.6, 0, 0.2, 0.2).$$

The 1-vote player rises the power of the 3-vote player from 0.29 to 0.6, an increment of 0.31. This is far more than 0.2, so a bandwagon effect is observed.
However, when the 2-vote player is joined with any 1-vote player, the Banzhaf indexes become

$$\beta'' = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{6}, \frac{2}{6}\right) = (0.167, 0.167, 0.33, 0.33).$$

The 1-vote player rises the power of the 2-vote player by an increment of 0.096, which is less than 0.21. Therefore, there is no bandwagon effect.

15. Let P be president, VP be vice president, S be a senator and HR be a house representative.

 (a) Vice President and a Senator are not equally desirable
 Consider the coalition

$$Z = \{291HR + 66S\},$$

 it is obvious that $Z \cup \{VP\}$ is losing and $Z \cup \{S\}$ is winning.

 President and a House Representative are not equally desirable
 Consider the coalition

$$Z = \{218HR + 51S\},$$

 it is seen that $Z \cup \{P\}$ is winning and $Z \cup \{HR\}$ is losing.

(b) Vice President and a House representative are incomparable
To see this, consider the coalitions:

$$Z = \{218HR, 50S, P\} \quad \text{and} \quad Z' = \{290HR, 67S\},$$

we see that $Z \cup \{VP\}$ is winning and $Z \cup \{HR\}$ is losing; while $Z' \cup \{VP\}$ is losing and $Z' \cup \{HR\}$ is winning.

President and a Senator are not incomparable
For any coalition, without P and S, we always have

$$Z \cup \{S\} \text{ is winning and } Z \cup \{P\} \text{ is winning.}$$

On the other hand, consider $Z' = \{218HR, 51S\}$, we see that $Z' \cup \{P\}$ is winning but $Z' \cup \{S\}$ is losing. Hence, P is more desirable than S.

16. It suffices to show that $Z' \backslash \{y\}$ is losing. Since $Z' \backslash \{x, y\}$ is the coalition without x and y and x, y are equally desirable, so $Z' \backslash \{x\}$ is winning $\Leftrightarrow$ $Z' \backslash \{y\}$ is winning. Since $Z' \backslash \{x\}$ is losing, then $Z' \backslash \{y\}$ is losing.

17. (a) Suppose Z' is winning, $Z' \backslash \{x\}$ is the coalition without x, y and x, y are equally desirable, then $(Z' \backslash \{x\}) \cup \{x\} = Z'$ is winning $\Leftrightarrow$ $(Z' \backslash \{x\}) \cup \{y\} = Z''$ is winning. Therefore, Z'' is winning.

    (b) Suppose Z' is losing, using a similar argument as in (a), we can deduce that Z'' is losing.