Mathematical Models in Economics and Finance

Topic 2 – Analysis of powers in voting systems

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2.1 Weighted voting systems and yes-no systems

Voting procedure: "support" or "object" a given motion/bill (no "abstain").

Weighted voting system – individuals/political bodies can cast more ballots than others. For example, voting by stockholders in a corporation, more votes being held by countries with stronger economic powers in the International Monetary Authority.

Question: How to define voting power in a weighted voting system?

- Power is not a trivial function of one's strength as measured by his number of votes.
- Voters with pivotal though small number of votes may have higher influential power per each vote held.

Define two indices that indicate the real distribution of influence.

- Useful in evaluating some of the existing democratic institutions in terms of fairness, concealed biases, etc.

Remark

No mathematical theory will suffice to reveal all of the behind-the-scene nuances - influences of party loyalty, persuasion, lobbying, fractionalism, bribes, gratuities, campaign financing, etc.

A new field called GOVERNMETRICS 計量管治學

Weighted majority voting game is characterized by a voting vector

$$[q; w_1, w_2, \cdots, w_n]$$

where there are n voters, w_i is the voting weight of player i; $N = \{1, 2, \cdots, n\}$ be the set of all n voters; q is the quota (minimum number of votes required to pass a bill).

Let S be a typical coalition, which is a subset of N. A coalition wins a bill (called winning) whenever

$$\sum_{i \in S} w_i \geq q.$$

It is natural to observe "complement of a winning coalition should be losing". As a result, we require the quota to observe $q > \frac{1}{2} \sum_{i \in N} w_i$ in order to avoid two disjoint coalitions that are both winning.

Examples

1. $[51; 28, 24, 24, 24]$; the first voter is much stronger than the last 3 since he needs only one other to pass an issue, while the other three must all combine in order to win.

2. $[51; 26, 26, 26, 22]$, the last player seems powerless since any winning coalition containing him can just as well win without him (a dummy).

3. In the game $[q; 1, 1, \cdots, 1]$, each player has equal power. This is called a pure bargaining game.

4. $[51; 40, 30, 20, 10]$ and $[51; 30, 25, 25, 20]$ seem identical in terms of voting influence, since the same set of coalitions are winning in both voting vectors.

5. Games such as $[3; 2, 2, 1], [8; 7, 5, 3]$ and $[51; 49, 48, 3]$ are identical to $[2; 1, 1, 1]$ in terms of power, since they give rise to the same collection of winning coalitions.

6. If we add to the game $[3; 2, 1, 1]$ the rule that player 2 can cast an additional vote in the case of 2 to 2 tie, then it is effectively $[3; 2, 2, 1]$.

   If player 1 can cast the tie breaker, then it becomes $[3; 3, 1, 1]$ and he is the dictator. He forms a winning coalition by himself.

7. In the game $[50(n-1)+1; 100, 100, \cdots, 100, 1]$, the last player has the same power as the others when n is odd; the game is similar to one in which all players have the same weights. For example, when $n = 5$, we have $[201; 100, 100, 100, 100, 1]$. Any 3 of the 5 players can form a winning coalition.

Dummy players

Any winning coalition that contains such an impotent voter could win just as well without him.

Examples

- Player 4 in $[51; 26, 26, 26, 22]$.
- Player n in $[50(n-1)+1; 100; 100, \cdots, 100, 1]$ is a dummy when n is even. For example, take $n = 4$, we have $[151; 100, 100, 100, 1]$. Obviously, the last player is a dummy.
- In $[10; 5, 5, 5, 2, 1, 1]$, the 4^{th} player with 2 votes is a dummy. The 5^{th} and 6^{th} players with only one vote are sure to be dummies. The collection of dummies remains to be a dummy collection. This is because one cannot turn a losing coalition into a winning coalition by adding a dummy one at a time.

Example

Consider $[16; 12, 6, 6, 4, 3]$, player 5 with 3 votes is a dummy since no subset of the numbers $12, 6, 6, 4$ sums to $13, 14$ or 15. Therefore, player 5 could never be pivotal in the sense that by adding his vote a coalition would just reach or surpass the quota of 16.

Example

If we add the 8^{th} player with one vote into $[15; 5, 5, 5, 5, 2, 1, 1]$ so that the new game becomes $[15; 5, 5, 5, 5, 2, 1, 1, 1]$, the 4^{th} player in the new voting game is not a dummy since sum of votes of some coalition may assume the value of 13.

Notion of Power

- The index should indicate one's relative influence, in some numerical way, to bring about the *passage or defeat of some bill*.

- The index should depend upon the number of players involved, on one's fraction of the total weight, and upon how the remainder of the weight is distributed (a critical swing-man that causes a desired outcome).

- A winning coalition is said to be *minimal winning* if no proper subset of it is winning. A coalition that is not winning is called *losing*. Technically, the one who is the 'last' to join a minimal winning coalition is particularly influential.

- A voter i is a dummy if every winning coalition that contains him is also winning without him, that is, he is in no minimal winning coalition. A dummy has ZERO power.

Example

Party	Leader	No. of seats
Liberals	Pierre E. Trudeau	109
Tories	Robert L. Stanfield	107
New Democrats	David Lewis	31
Others		17

- Though Liberals has the largest number of seats, none of the parties has the majority. Any two of the three leading parties can form a coalition and obtain the majority, so the first three parties had equal power. Other small parties are all dummies.

Veto Power and Dictators

A player or coalition is said to have *veto power* if no coalition is able to win a ballot without his or their consent. A subset S of voters is a blocking coalition or has *veto power* if and only if its complement $N - S$ is not winning. We also require that S itself is losing in order to be a blocking coalition, otherwise S is a dictatorial coalition (see "dictator" below).

Given that $q > 50\%$ of total votes, a player i is a dictator if he forms a winning coalition $\{i\}$ by himself.

- If the dictator says "yes", then the bill is passed. If the dictator says "no", then the bill is not passed (veto power).

- If a dictator exists, then all other players are dummies.

Example

Player 1 has veto power in $[51; 50, 49, 1]$ and $[3; 2, 1, 1]$. In the last case, if he is the chairman with additional power to break ties, then the game becomes $[3; 3, 1, 1]$ and now he becomes a dictator.

Example

The ability of an individual to break tie votes in the pure bargaining game

$$\begin{cases} \left[\frac{n}{2} + 1; 1, 1, \cdots, 1\right] & \text{when } n \text{ is even} \\ \left[\frac{n+1}{2}; 1, 1, \cdots, 1\right] & \text{when } n \text{ is odd} \end{cases}$$

adds power when n is even and adds nothing when n is odd. Actually, when n is odd, tie votes will not occur.

Properties on dummies

A collection of dummies can never turn a losing coalition into a winning coalition. 如足夠多傀儡聯合起來，這團隊可否改變傀儡地位

In other words, it is not possible that $S \cup \{D_1, \cdots, D_m\}$ is winning but S is losing since the dummies can be successively deleted while the coalition remains to be winning.

Corollary

If both "d" and "ℓ" are dummies, then the coalition $\{d, \ell]$ is dummy.

Theorem
In a weighted voting game, let "d" and "ℓ" be two voters with votes x_d and x_ℓ, respectively. Suppose "d" is a dummy and $x_\ell \leq x_d$, then "ℓ" is also a dummy.

Proof
Assume the contrary. Suppose "ℓ" is not a dummy, then there exists a coalition S that does not contain the dummy "d" such that S is losing but $S \cup \{\ell\}$ is winning. As a remark, we just require S not to contain "d" while S may contain dummies other than "d". Now, $n(S) < q$ while $n(S \cup \{\ell\}) \geq q$. Since $x_\ell \leq x_d$, so $n(S \cup \{d\}) \geq q$, contradicting that "d" is a dummy.

Corollary
If the coalition $\{d, \ell\}$ is dummy, then both "d" and "ℓ" are dummies. This is obvious since $n(\{d, \ell\}) \geq \max(x_d, x_\ell)$.

Yes-no voting system

A yes–no voting system is simply a set of rules that specify exactly which collections of "yes" votes yield passage of a bill.

Under what condition that a yes–no voting system is a weighted system (with weights assigned for the voters and quota)?

Example

Bill to be passed: Grades of this course are based on "absolute grading".

Set of players, $N = \{$professor, tutor, Chan, Lee, Cheung, Wong, Ho$\}$.

Yes–no rule requires professor, at least one from "tutor and Chan" and number of students must be at least 3. Each of the following player/coalition, "professor" and "tutor and Chan", has veto power.

United States federal system

- 537 voters in this system: 435 House of Representatives, 100 Senate members, Vice-President and President.
- Vice President plays the role of tie breaker in the Senate
- President has veto power that can be overridden by a two-thirds vote of both the House and the Senate.

To pass a bill, it must be supported by either one of the following three sets of votes

1. 218 or more representatives and 51 or more senators and President
2. 218 or more representatives and 50 senators and both Vice President and President
3. 290 or more representatives and 67 or more senators.

System to amend the Canadian constitution (since 1982)

- In addition to the House of Commons and the Senate, approval by two-thirds majority of provincial legislatures, that is, at least 7 of the 10 Canadian provinces subject to the proviso that the approving provinces have among them at least half of Canada's population.

- Based on 1961 census

  Prince Edward Island (1%)
  Newfoundland (3%)
  New Brunswick (3%)
  Nova Scotia (4%)
  Manitoba (5%)
  Saskatchewan (5%)
  Alberta (7%)
  British Columbia (9%)
  Quebec (29%)
  Ontario (34%)

Swap robustness and trade robustness

Definition (swap robust)

A yes–no voting system is said to be swap robust if a "swap" of one player for one between two winning coalitions leaves at least one of the two coalitions winning.

- Start with two arbitrary winning coalitions X and Y
  - arbitrary player x in X (not in Y) and arbitrary player y in Y (not in X)
  - let X' and Y' be the result of exchanging x and y
  - either X' or Y' is winning for swap robustness

Proposition
Every weighted voting system is swap robust.

Proof

1. If x and y have the same weight, then both X' and Y' are winning.

2. If x is heavier than y, then Y' weighs strictly more than Y. The weight of Y' certainly exceeds the quota, and thus Y' is winning.

3. If y is heavier than x, then a similar argument as in (2) holds.

To show a voting system to be not swap robust, we produce two winning coalitions X and Y and a trade between them that renders both losing. Intuitively, X and Y should both be "almost losing" and make both actually losing by a one-for-one trade. We seek for the appropriate coalitions X and Y among the *minimal winning coalitions*.

Proposition

The US federal system is not swap robust.

Proof

$X = \{$President, 51 shortest senators and 218 shortest House Representatives$\}$

$Y = \{$President, 51 tallest senators and 218 tallest House Representatives$\}$

Let x be shortest senator and y be the tallest House Representative.

Both X and Y are winning coalitions; $x \in X$ but $x \notin Y$; $y \in Y$ but $y \notin X$. After swapping, X' is a losing coalition since it has only 50 senators and Y' is a losing coalition because it has only 217 Representatives.

Corollary The US federal system is not a weighted voting system.

Proposition

The procedure to amend the Canadian Constitution is swap robust (later shown to be not weighted).

Proof

Suppose X and Y are winning coalitions, $x \in X, x \notin Y$ while $y \in Y$ but $y \notin X$. We must show that at least one of X' and Y' is still a winning coalition. That is, at least one of X' and Y' still satisfies both conditions:

(i) It contains at least 7 provinces.

(ii) The provinces represent at least half of the Canadian population.

The first condition is obvious. If x has more population than y, then Y' is a winning coalition. Similar argument when x has a smaller population than y.

Definition (trade robust)

A yes–no voting system is said to be trade robust if an arbitrary exchange of players (a series of trades involving groups of players) among several winning coalitions leaves at least one of the coalitions winning.

1. The exchanges of players are not necessarily one-for-one as they are in swap robustness.
2. The trades may involve more than two coalitions.

For example, consider 3 winning coalitions X, Y and Z.

(i) 3 players are exchanged out from X, 2 to Y and 1 to Z;

(ii) 4 players are exchanged out from Y, 1 to X and 3 to Z;

(iii) 3 players are exchanged out from Z, 2 to X and 1 to Z;

Proposition

Every weighted voting system is trade robust.

Proof

- A series of trades among several winning coalitions leaves the total weight of all these coalitions added together unchanged. Hence, the average weight of these coalitions is unchanged.

- Suppose we start with several winning coalitions in a weighted voting system, then their average weight at least meets quota. After the trades, the average weight of the coalitions is unchanged and at least meets quota. Hence, at least one of the coalitions must meet quota (at least one of the resulting coalitions is winning).

Proposition The procedure to amend the Canadian Constitution is not trade robust.

Proof

X	Y
Prince Edward Island (1%)	New Brunswick (3%)
Newfoundland (3%)	Nova Scotia (4%)
Manitoba (5%)	Manitoba (5%)
Saskatchewan (5%)	Saskatchewan (5%)
Alberta (7%)	Alberta (7%)
British Columbia (9%)	British Columbia (9%)
Quebec (29%)	Ontario (34%)

- Let X' and Y' be obtained by trading Prince Edward Island and Newfoundland for Ontario.
- X' is a losing coalition because it has too few provinces (having given up two provinces in exchange for one).
- Y' is a losing coalition because the eight provinces in Y' represent less than half of Canada's population.

Corollary

The procedure to amend the Canadian Constitution is not a weighted voting system.

Theorem (proof omitted)

A yes–no voting system is weighted if and only if it is trade robust.

Minority veto

For example, we have a majority group of 5 voters and a minority group of 3 voters. The passage requires not only approval of at least 5 out of the 8 voters, but also approval of at least 2 of the 3 minority voters. This system is swap robust but not trade robust.

Hint Consider how to make one coalition to have more total number of voters but do not have enough minority voters while the other coalition has less total number of voters but more minority voters after the trades.

Illustration

Majority group of 5 voters   minority group of 3 voters

To form a winning coalition, we must have

- at least 5 out of 8 total voters
- at least 2 out of 3 minority voters

$X = \{M_1, M_2, M_3, m_1, m_2\}$
$Y = \{M_3, M_4, M_5, m_2, m_3\}$

Both $X \& Y$ are winning coalitions

$X' = \{M_1, M_2, M_3, M_4, M_5, m_2\}$
$Y' = \{M_3, m_1, m_2, m_3\}$

Both $X' \& Y'$ are losing coalitons

Hence, the system is NOT trade robust since X' does not satisfy the minority requirement and Y' does not satisfy the "total votes" requirement.

Swap robustness is easily seen since the requirement of 5 votes out of 8 voters is satisfied under one-for-one trade and the minority requirement is always satisfied by at least one of the coalitions after one-for-one trade.

Intersection of weighted voting systems and dimension theory

The procedure to amend the Canadian Constitution can be constructed by "putting together" two weighted systems.

- $W_1 =$ collection of coalitions with 7 or more provinces
- $W_2 =$ collection of coalitions representing at least half of Canada's population

A coalition is winning if and only if it is winning in both System I and System II.

$$W = W_1 \cap W_2.$$

Question How to construct a non-weighted system in terms of intersection of weighted ones?

Dimension of a yes-no voting system

A yes–no voting system is said to be dimension k if and only if it can be represented as the intersection of exactly k weighted voting systems, but not as the intersection of $k-1$ weighted voting systems.

For example, the procedure to amend the Canadian Constitution is of dimension 2 since the passage requires two separate weighted voting systems with regard to "population" and "provinces".

Proposition

The US federal system has dimension 2.

System I: quota $= 67$	System II: quota $= 290$
weight 0 to House Representative	weight 1 to House Representative
weight 1 to Senator	weight 0 to Senator
weight 0.5 to Vice President	weight 0 to Vice President
weight 16.5 to President	weight 72 to President

System I is meant for the Senate (veto power of the President and tie breaker role of the Vice President; sum of weights of P & VP = 17, weight of $P \geq 16$ and weight of VP > 0). System II is meant for the House, with veto power of the President.

Let X be a minimal winning coalition. Then X is one of the following 3 kinds of coalitions:

1. X consists of 218 House members, 51 Senators and the President
2. X consists of 218 House members, 50 Senators, the Vice President and the President
3. X consists of 290 House members and 67 Senators.

All these 3 kinds of coalitions achieve quota in both Systems I and II.

Conversely, how to find the minimal winning coalitions that satisfy both weighted voting systems?

Hint (With the President)

In System I, weight of the President is 16.5, so the other members of X must contribute at least weight 50.5 to the total System I. X must contain either 51 (or more) senators or at least 50 senators and the Vice President.

Looking at System II, which is at least 290 including the 72 contributed by the President. So X must also contain at least $290 - 72 = 218$ House Representatives.

Proposition

Suppose S is a yes–no voting system for the set V of voters, and let m be the number of losing coalitions in S. It is then possible to find m weighted voting systems with the same set V of voters such that a coalition is winning in S if and only if it is winning in every one of these m weighted systems.

Proof

For each losing coalition L in S, we construct an associated weighted voting system. Let $|L|$ be the number of voters in L. Every voter in L is given weight -1. Every voter not in L is given weight 1. Quota is set at $-|L|+1$. The possibility that a coalition is losing if and only if it contains exactly all the voters in this particular losing coalition L. Then every other coalition is a winning coalition in this weighted voting system (including any proper subset of L).

- If a coalition is winning in S, then it is winning in each of these weighted systems.

- Conversely, if a coalition is winning in each of these weighted systems, then it is a winning coalition in S (a losing coalition in S must lose in one of these weighted voting systems).

Unfortunately, this procedure is enormously inefficient since there may be too many losing coalitions.

Puzzle: Since we allow negative values for votes and quota, every proper subset of the losing coalition L also wins in the weighted voting system associated with L.

Alternative proof

- Define a maximal losing coalition to be a losing coalition that becomes winning with the addition of any one voter outside the losing coalition (implicitly all dummies are included in any of these maximal losing coalitions). Any subset of a maximal losing coalition is also losing.

- We identify all these distinct maximal losing coalitions as $L_1, \ldots, L_m$.

- For each maximal losing coalition L_i, we construct a weighted voting system. Let $|L_i|$ be the number of voters in L_i. Every voter in L_i is assigned weight 1, while every voter not in L_i is assigned weight $|L_i| + 1$. The quota is set at $|L_i| + 1$. A subset of L_i is losing under the associated weighted system since the total weight is less than the quota $|L_i| + 1$.

- Under the intersection of these weighted voting systems:

  coalition C in S is losing (one of these L_i's or a subset of some L_i)

  $\Leftrightarrow$ coalition C is losing in at least one of these weighted voting system

Example – Yes-no voting system to decide on grading policy

The professor, tutor and 33 students vote to decide whether "absolute grading" should be applied in assigning the grades of this course.

Voting rule 1

1. The professor, tutor and a particular student A each has veto power.
2. To pass the proposal, the voting rule requires yes votes from the professor, tutor, student A and at least 20 other students.

Note that all winning coalitions must contain the professor, tutor and student A. Any arbitrary trades among the winning coalitions involve only the remaining 32 students.

To show trade robustness, since there are at least 20 other students in the winning coalitions on average, this average remains unchanged after any series of arbitrary trades. There must exist a coalition with at least 20 other students after the trades. This represents a winning coalition.

Hence, the yes-no voting system is trade robust, so it can be represented as a weighted voting system. To find the weighted voting vector, we let w represent the weight of each of the other students is one. The quota q must observe

$$2w + 32 < q \quad \text{and} \quad 3w + 20 \geq q$$

so that $3w + 20 > 2w + 32 \Leftrightarrow w > 12$. Suppose we pick $w = 13$, then

$$59 = 3 \times 13 + 20 \geq q > 2 \times 13 + 32 = 58 \text{ so that } q = 59.$$

A possible representation of the weighted vector is $[59; 13, 13, 13, 1, 1, \ldots, 1, 1]$.

Voting rule 2

We modify voting rule 1 on setting "veto power". Now the veto power is held by the professor or "tutor and student A" together. That is, tutor and student A each does not hold veto power individually. To pass the proposal, the new rule requires yes votes from at least one of tutor and student A, and 20 other students.

It is easily seen that the new voting rule is NOT swap robust. To show the claim, we consider the following pair of winning coalitions:

$$X_1 = \{P, T, S_1, S_2, \ldots, S_{20}\} \quad \text{and} \quad X_2 = \{P, A, S_{13}, S_{14}, \ldots, S_{32}\}.$$

Suppose we transfer S_1 from X_1 to X_2 and A from X_2 to X_1, the two remaining coalitions are both losing. Therefore, this yes-no system is not a weighted voting system.

However, it can be represented as the intersection of 3 weighted voting systems.

System W_1	*System* W_2	*System* W_3
$q_1 = 1$	$q_2 = 1$	$q_3 = 20$
$w_P = 1, w_T = w_A = 0$	$w_P = 0, w_T = w_A = 1$	$w_P = w_T = w_A = 0$
$w_{S_1} = w_{S_2} = \ldots = w_{S_{32}} = 0$	$w_{S_1} = w_{S_2} = \ldots = w_{S_{32}} = 0$	$w_{S_1} = w_{S_2} = \ldots = w_{S_{32}} = 1$

$$W = W_1 \cap W_2 \cap W_3$$

2.2 Power indexes

Shapley-Shubik power index

1. One looks at all possible orderings of the n players, and consider this as all of the potential ways of building up toward a winning coalition. For each one of these permutations, some unique player joins and thereby turns a losing coalition into a winning one, and this voter is called the *pivot*.

2. In the sequence of player $x_1, x_2, \cdots, x_{i-1}, x_i, \cdots, x_n, \{x_1, x_2, \cdots, x_i\}$ is a winning coalition but $\{x_1, x_2, \cdots, x_{i-1}\}$ is losing, then i is in the *pivotal position*.

3. What is the probability that a particular voter is the pivot? The expected frequency with which a voter is the pivot, over all possible orderings of the voters, is taken to be a good indication of his voting power.

Example – 4-player weighted voting game

The 24 permutations of the four players 1, 2, 3 and 4 in the weighted majority game [51; 40, 30, 20, 10] are listed below. The "*" indicates which player is pivotal in the corresponding ordering.

1 2*3 4	2 1*3 4	3 1*2 4	4 1 2*3
1 2*4 3	2 1*4 3	3 1*4 2	4 1 3*2
1 3*2 4	2 3 1*4	3 2 1*4	4 2 1*3
1 3*4 2	2 3 4* 1	3 2*4 1	4 2*3 1
1 4 2* 3	2 4 1* 3	3 4 1* 2	4 3 1* 2
1 4 3* 2	2 4 3* 1	3 4 2* 1	4 3 2* 1

For Player 1 [double-boxed] winning coalitions consisting of 2 players.

[round-boxed] winning coalitions consisting of 3 players.

Shapley-Shubik power index for the i^{th} player is

$$\phi_i = \frac{\text{number of sequences in which player } i \text{ is a pivot}}{n!}$$

and we write $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_n)$.

Here, we assume that each of the $n!$ alignments is *equiprobable*.

The power index can be expressed as

$$\phi_i = \sum \frac{(s-1)!(n-s)!}{n!} \qquad \left(\text{with } \sum_{i \in N} \phi_i = 1\right)$$

where $s = |S|$ = number of voters in set S. The summation is taken over all winning coalitions S for which $S - \{i\}$ is losing.

Counting permutations for which a player is pivotal in achieving minimal winning coalitions

- Player 1 is pivotal in 3 coalitions (namely $\{1,2,3\},\{1,2,4\},\{1,3,4\}$) consisting of 3 players and in 2 coalitions (namely $\{1,2\},\{1,3\}$) consisting of two players.

$$\phi_1 = 3\frac{(3-1)!(4-3)!}{4!} + 2\frac{(2-1)!(4-2)!}{4!} = \frac{10}{24}.$$

- For player 2, she is pivotal in $\{1,2,3\}$ and $\{1,2,4\}$ with 3 players and $\{1,2\}$ with 2 players. Therefore

$$\phi_2 = 2\frac{(3-1)!(4-3)!}{4!} + \frac{(2-1)!(4-2)!}{4!} = \frac{6}{24}.$$

  We have

$$\boldsymbol{\phi} = \frac{(10,6,6,2)}{24}.$$

Example – A bloc versus singles

Suppose we have n players and that a single block of size b forms. Consider the resulting weighted voting system: $[q; b, \underbrace{1, 1, \cdots, 1}_{n-b \text{ of these}}]$.

- $n - b + 1$ is just the number of distinct orderings. The b bloc will be pivotal precisely when the initial sequence of ones is of length at least $q - b$ but not more than $q - 1$.

- The b bloc is pivotal when the initial sequence of ones is any of the following lengths:

$$q - 1, q - 2, \cdots, q - b.$$

  Note that there are $n - b$ ones available, so the above statement is valid provided that $n - b \geq q - 1$ and $q > b$ (which is equivalent to $b < q \leq n - b + 1$).

- Under the assumption of $b < q \leq n - b + 1$, there are b possible initial sequences of ones that make the bloc pivotal, so

$$\begin{aligned} & \text{Shapley-Shubik index of the block of size } b \\ = & \frac{\text{number of orderings in which } b \text{ is pivotal}}{\text{total number of distinct orderings}} \\ = & \frac{b}{n-b+1}. \end{aligned}$$

The Shapley-Shubik index is higher than the percentage of votes of b/n.

The formation of a block increases the voting power.

Banzhaf index

- Consider all significant combinations of "yes" or "no" votes, rather than permutations of the players as in the Shapley-Shubik index.
- A player is said to be marginal, or a swing or critical, in a given combination of "yes" and "no" if he can change the outcome.
- Let b_i be the number of voting combinations in which voter i is marginal; then $\beta_i = \dfrac{b_i}{\sum b_i}$.

Assuming that all voting combinations are equally probable.

The game is $[51; 40, 30, 20, 10]$. For the second case, if Player 1 changes from Y to N, then the outcome changes from "Pass" to "Fail".

Computation of the Banzhaf Index

Players				Pass/Fail		Marginal			
1	2	3	4	P	F	1	2	3	4
Y	Y	Y	Y	P					
Y	Y	Y	N	P		X			
Y	Y	N	Y	P		X	X		
Y	N	Y	Y	P		X		X	
N	Y	Y	Y	P			X	X	X
Y	Y	N	N	P		X	X		
Y	N	Y	N	P		X		X	

N	Y	Y	N	F	X			X
Y	N	N	Y	F		X	X	
N	Y	N	Y	F	X		X	
N	N	Y	Y	F	X	X		
Y	N	N	N	F		X	X	
N	Y	N	N	F	X			
N	N	Y	N	F	X			
N	N	N	Y	F				
N	N	N	N	F				

$$24 \times \boldsymbol{\beta} = (10, 6, 6, 2)$$

Looking at $YYNN$ (pass) and $NYNN$ (fail), Player 1 can serve as the defector who gives the swing from Pass to Fail in the first case and Fail to Pass in the second case. We expect that the number of swings of winning into losing effected by a particular player is the same as the number of swings of losing into winning by the same player.

Example

Sometimes symmetry can save us writing out all $n!$ orderings. For example, consider the weighted majority game

$$[5; 3, 2, 1, 1, 1, 1].$$

Since the "1" players are all alike, we need to write out only $6 \cdot 5 = 30$ distinct orderings (instead of $6! = 720$):

$3\underline{2}1111$	$2\underline{3}1111$	$21\underline{3}111$	$211\underline{3}11$	$211\underline{1}31$	$211\underline{1}13$
$31\underline{2}111$	$13\underline{2}111$	$12\underline{3}111$	$121\underline{3}11$	$121\underline{1}31$	$121\underline{1}13$
$31\underline{1}211$	$13\underline{1}211$	$11\underline{3}211$	$112\underline{3}11$	$112\underline{1}31$	$112\underline{1}13$
$31\underline{1}121$	$13\underline{1}121$	$11\underline{3}121$	$111\underline{3}21$	$111\underline{2}31$	$111\underline{2}13$
$31\underline{1}112$	$13\underline{1}112$	$11\underline{3}112$	$111\underline{3}12$	$1111\underline{3}2$	$1111\underline{2}3$

Notice that the 1's pivot 12/30 of the time, but since there are four of them, each 1 pivots only 3/30 of the time. We get

$$\begin{aligned}\text{Shapley-Shubik index} &= \phi = \left(\frac{12}{30}, \frac{6}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30}, \frac{3}{30}\right) \\ &= (0.4, 0.2, 0.1, 0.1, 0.1, 0.1).\end{aligned}$$

Power as measured by the Shapley-Shubik index in a weighted voting game is *not* proportional to the number of votes cast. For instance, the player with $3/9 = 33\frac{1}{3}\%$ of the votes has 40% of the power.

Use the same game for the computation of the *Banzhaf index*

Types of winning coalitions with	Number of ways this can occur	Number of swings for		
		3	2	1
5 votes: $\underline{32}$	1	1	1	
$\underline{311}$	$6 = {}_4C_2$	6		12
$\underline{2111}$	$4 = {}_4C_3$		4	12
6 votes: $\underline{32}1$	$4 = {}_4C_1$	4	4	
$\underline{3}111$	$4 = {}_4C_3$	4		
$\underline{2}1111$	$1 = {}_4C_4$		1	
7 votes: $\underline{3}211$	$6 = {}_4C_2$	6		
$\underline{3}1111$	$1 = {}_4C_4$	1		
		22	10	24

We do not need to include those winning coalitions of 8 or 9 votes, since not even the player with 3 votes can be critical to them.

Remark

It suffices to consider the swings only in winning coalitions in the calculation of the Banzhaf index. A defector that turns a winning coalition into a losing coalition also gives the symmetric swing that turns a losing coalition into a winning coalition.

The numbers in the second column are derived from the theory of combinations. For instance, the number of ways that you could choose 311 from 321111 is ${}_4C_2 = 6$.

$$\begin{aligned}\boldsymbol{\beta} &= \left(\frac{22}{56}, \frac{10}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56}\right) \\ &\approx (0.392, 0.178, 0.107, 0.107, 0.107, 0.107).\end{aligned}$$

Comparing this with $\boldsymbol{\phi}$, we see that the two indices turn out to be quite close in this case, with $\boldsymbol{\beta}$ giving slightly less power to the two large players and slightly more to the small players.

2.3 Case studies of power indexes calculations

United Nations Security Council

1. Big "five" – permanent member 常任理事國 each has veto power; ten "small" countries whose (non-permanent) membership rotates.
2. It takes 9 votes, the "big five" plus at least 4 others to carry an issue.

   For simplicity, we assume no "abstain" votes. The game is $[39; 7, 7, 7, 7, 7, 1, 1, \cdots, 1]$. Why? Let x be the weight of any of the permanent member and q be the quota. Then

   $$4x + 10 < q \quad \text{and} \quad q \leq 5x + 4$$

   so that $4x + 10 < 5x + 4$ giving $x > 6$. Taking $x = 7$, we then have $38 < q \leq 39$. We take $q = 39$.

3. A "small" country i can be pivotal in a winning coalition if and only if S contains exactly 9 countries including the big "five". There are ${}_9C_3$ such different S that contain i since the remaining 3 "small" countries are chosen from 9 "small" countries (other than country i itself). For each such S, the corresponding coefficient in the Shapley-Shubik formula for this 15-person game is $\dfrac{(9-1)!(15-9)!}{15!}$. Hence, $\phi_S = {}_9C_3 \times \dfrac{8!6!}{15!} \approx 0.001863$. Any "big-five" has index $\phi_b = \dfrac{1-10\phi_S}{5} = 0.1963$.

4. Old Security Council before 1963, which was

$$[27; 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1].$$

What is the corresponding yes-no voting system?

Answer for $\phi : \phi_b = \dfrac{1}{5} \cdot \dfrac{76}{77}; \phi_S = \dfrac{1}{6} \cdot \dfrac{1}{77}$.

Canadian Constitutional Amendment

Investigate the voting powers exhibited in a 10-person game between the provinces, and to compare the results with the provincial populations.

The winning coalitions or those with veto power can be described as follows. In order for passage, approval is required of

(a) any province that has (or ever had) 25% of the population,

(b) at least two of the four Atlantic provinces, and

(c) at least two of the four western provinces that currently contain together at least 50% of the total western population.

Veto power

Recall that a blocking coalition (holding veto power) is a subset of players whose complement is not winning. Using the current population figures, the veto power is held by

(i) Ontario (O) and Quebec (Q),

(ii) any three of the four Atlantic (A) provinces [New Brunswick (NB), Nova Scotia (NS), Prince Edward Island (PEI), and Newfoundland (N)],

(iii) British Columbia (BC) plus any one of the three prairie (P) provinces [Alberta (AL), Saskatchewan (S), and Manitoba (M)], and

(iv) the three prairie provinces taken together.

In calculating the Shapley-Shubik index of Quebec or Ontario, it is necessary to list all possible winning coalitions since any of these winning coalitions must contain Quebec and Ontario.

Winning Provincial Coalitions

Type	S	s	No. of such S	
1	$1P$, $2A$, BC, Q, O	6	18	
2	$2P$, $2A$, BC, Q, O	7	18	
3	$3P$, $2A$, Q, O	7	6	36
4	$1P$, $3A$, BC, Q, O	7	12	
5	$3P$, $2A$, BC, Q, O	8	6	
6	$2P$, $3A$, BC, Q, O	8	12	25
7	$3P$, $3A$, Q, O	8	4	
8	$1P$, $4A$, BC, Q, O	8	3	
9	$3P$, $3A$, BC, Q, O	9	4	
10	$2P$, $4A$, BC, Q, O	9	3	8
11	$3P$, $4A$, Q, O	9	1	
12	$3P$, $4A$, BC, Q, O	10	1	
		Total:	88	

Ontario's Shapley-Shubik index

$$\varphi_O = \frac{[18(5!4!) + 36(6!3!) + 25(7!2!) + 8(8!1!) + 1(9!0!)]}{10!} = \frac{53}{168}$$

- There are 18 winning coalitions that contain 6 provinces. In order that Ontario serves as the pivotal player, 5 provinces are in front of her and 4 provinces are behind her. This explains why there are altogether 18(5!4!) permutations in these 6-province winning coalitions.

- Ontario and Quebec are equivalent in terms of influential power (though their populations are different).

British Columbia

Listing of all winning coalitions that upon deleting British Columbia the corresponding coalition becomes losing. These are the winning coalitions that British Columbia can serve as the pivotal player.

Type	S	s	No. of such S
1	1P, 2A, BC, Q, O	6	${}_3C_1 \times {}_4C_2 = 18$
2	1P, 3A, BC, Q, O	7	${}_3C_1 \times {}_4C_3 = 12$
3	1P, 4A, BC, Q, O	8	${}_3C_1 \times {}_4C_4 = 3$
4	2P, 2A, BC, Q, O	7	${}_3C_2 \times {}_4C_2 = 18$
5	2P, 3A, BC, Q, O	8	${}_3C_2 \times {}_4C_3 = 12$
6	2P, 4A, BC, Q, O	9	${}_3C_2 \times {}_4C_4 = 3$

- Note that we exclude those coalitions with 3 prairie provinces since the deletion of British Columbia does not cause the coalition to become losing.

$$\phi_{BC} = \frac{18(5!4!) + 30(6!3!) + 15(7!2!) + 3(8!1!)}{10!}.$$

Atlantic provinces

We consider winning coalitions that contain a particular Atlantic province and one of the three other Atlantic provinces.

Type	S	s	No. of such S
1	$A_{sp}, 1A, 1P, BC, Q, O$	6	${}_3C_1 \times {}_3C_1 = 9$
2	$A_{sp}, 1A, 2P, BC, Q, O$	7	${}_3C_1 \times {}_3C_2 = 9$
3	$A_{sp}, 1A, 3P, BC, Q, O$	8	${}_3C_1 = 3$
4	$A_{sp}, 1A, 3P, Q, O$	7	${}_3C_1 = 3$

$$\phi_{A_{sp}} = \frac{9(5!4!) + 12(6!3!) + 3(7!2!)}{10!}.$$

Prairie provinces

We consider winning coalitions that contain

(i) a particular prairie province and British Columbia

(ii) a particular prairie province and two other prairie provinces

Type	S	s	No. of such S
1	P_{sp}, 2A, BC, Q, O	6	6
2	P_{sp}, 3A, BC, Q, O	7	4
3	P_{sp}, 4A, BC, Q, O	8	1
4	P_{sp}, 2P, 2A, Q, O	7	6
5	P_{sp}, 2P, 3A, Q, O	8	4
6	P_{sp}, 2P, 4A, Q, O	9	1

$$\phi_{P_{sp}} = \frac{6(5!4!) + 10(6!3!) + 5(7!2!) + 8!1!}{10!}.$$

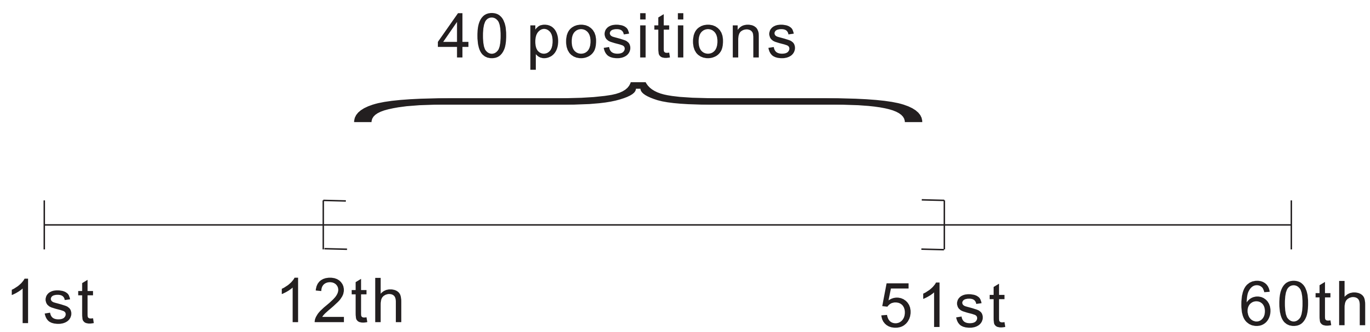
Shapley-Shubik Index Provinces

Province	φ (in %)	% Population	φ/Population
BC	12.50	9.38	1.334
AL	4.17	7.33	0.570
S	4.17	4.79	0.872
M	4.17	4.82	0.865
(4 Western)	(25.01)	(26.32)	(0.952)
O	31.55	34.85	0.905
Q	31.55	28.94	1.092
NB	2.98	3.09	0.965
NS	2.98	3.79	0.786
PEI	2.98	0.54	5.53
N	2.98	2.47	1.208
(4 Atlantic)	(11.92)	(9.89)	(1.206)

- British Columbia has a higher index value per capita compared to other Western provinces.

Power of the major stockholders

- Consider a corporation with one major stockholder X who controls 40% of the stock, and suppose the remainder is split evenly among 60 other stockholders, each having 1%.
- There are 61 players. Since the 60 minor stockholders are symmetric, there are only 61 distinct orderings, depending only on the position of X.
- Of these 61 orderings, X will pivot if he appears in positions $12 - 51$ inclusive (if we assume that approval must be by an amount strictly over 50%), i.e. 40/61 of the time.



60 singles as the 60 minor stockholders

- Now suppose X still controls 40% of the stock, but the remainder is split evenly among 600 other stockholders, each controlling 0.1%.
- Of the 601 distinct orderings, X will pivot if he appears in positions $102 - 501$, i.e., 400/601 of the time. Clearly, as the number of minor stockholders gets very large, X's share of the power (as measured by the Shapley-Shubik index) approaches 2/3.

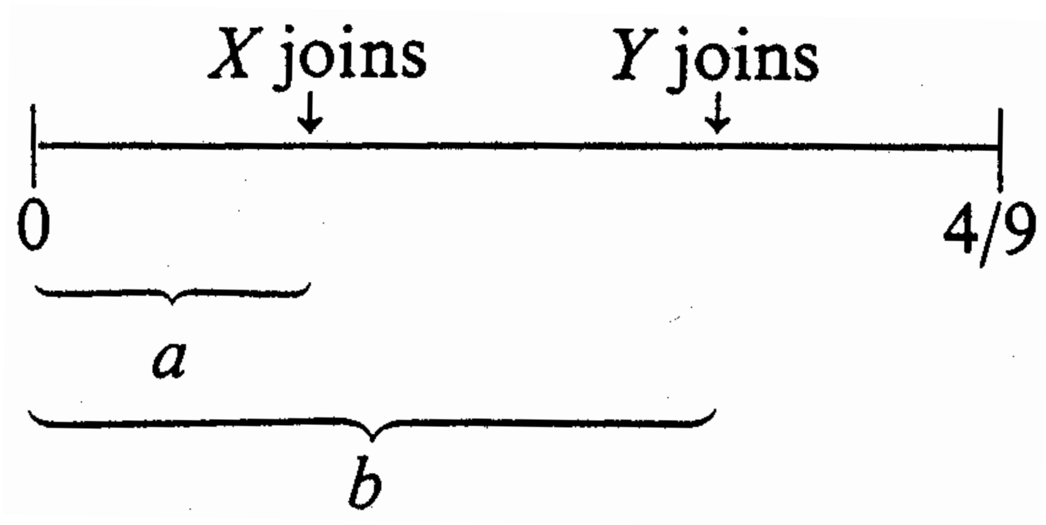
Oceanic weighted voting game

- Let there be one major player X controlling 40% of the vote, with the remaining 60% held by an infinite "ocean" of minor voters.
- Think of the minor voters lined up as points in a line segment of length 0.6, as they come to join a coalition in support of some proposal.
- Voter X can join at any point along this line segment. He will pivot if he joins after 0.1 and before (or at) 0.5. His Shapley-Shubik index is

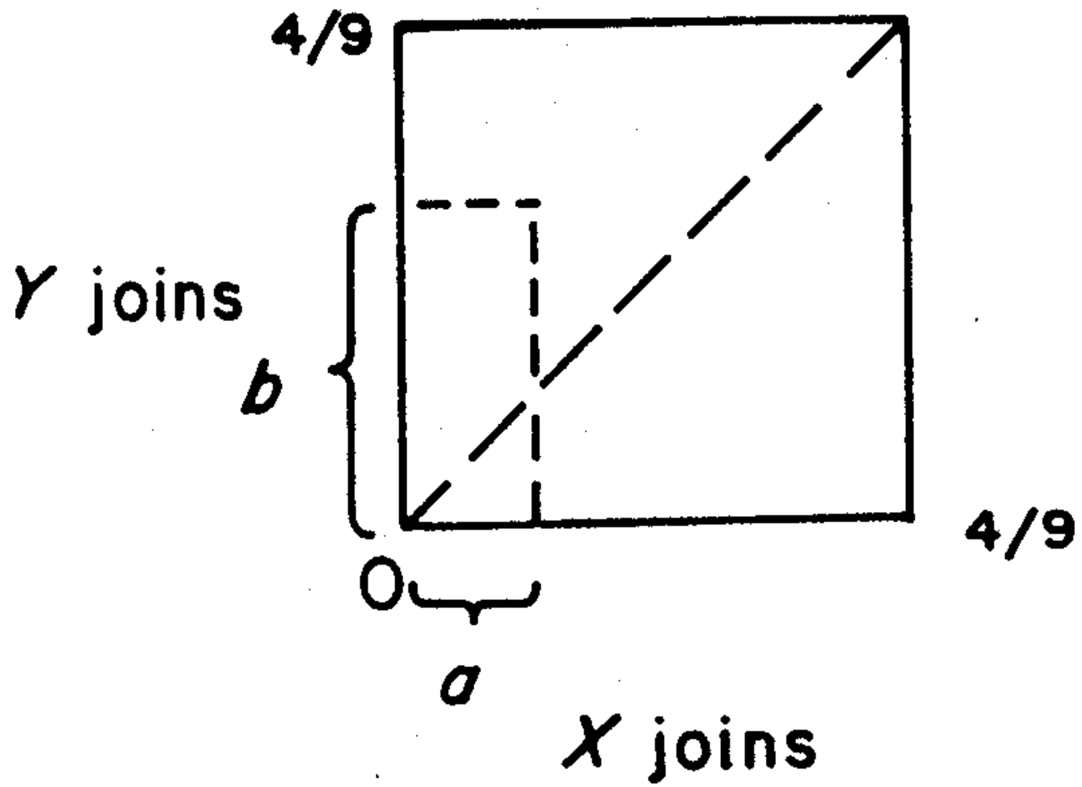
$$\phi_X = \frac{\text{Length of segment in which } X \text{ pivots}}{\text{Total length of segment}} = \frac{0.5 - 0.1}{0.6} = \frac{2}{3}.$$

Example

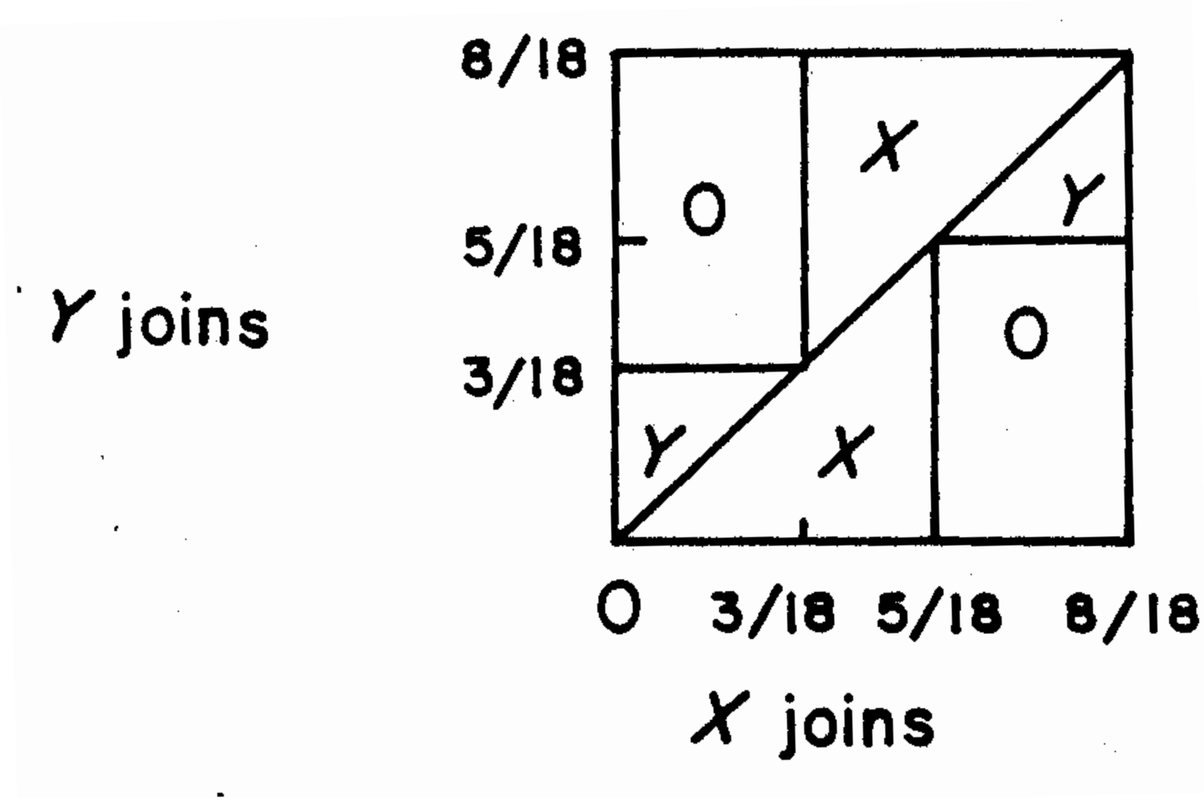
There are two major voters and an ocean of minor voters. Suppose voter X holds 3/9 of the total vote, and voter Y holds 2/9, with the other 4/9 held by the ocean of minor voters. The minor voters line up along a line segment of length 4/9. X and Y can join at any point along this line segment:



- We can represent geometrically the positions at which X and Y join by giving a single point in a square of side 4/9, whose horizontal coordinate is X's position and whose vertical coordinate is Y's:
- The point is *above* the diagonal of the square if X joins before Y, and below the diagonal if Y joins before X.



- Which points in the square correspond to orderings for which X or Y pivots?
- Divide the square into regions where X pivots, Y pivots, or voters in the ocean (O) pivot:

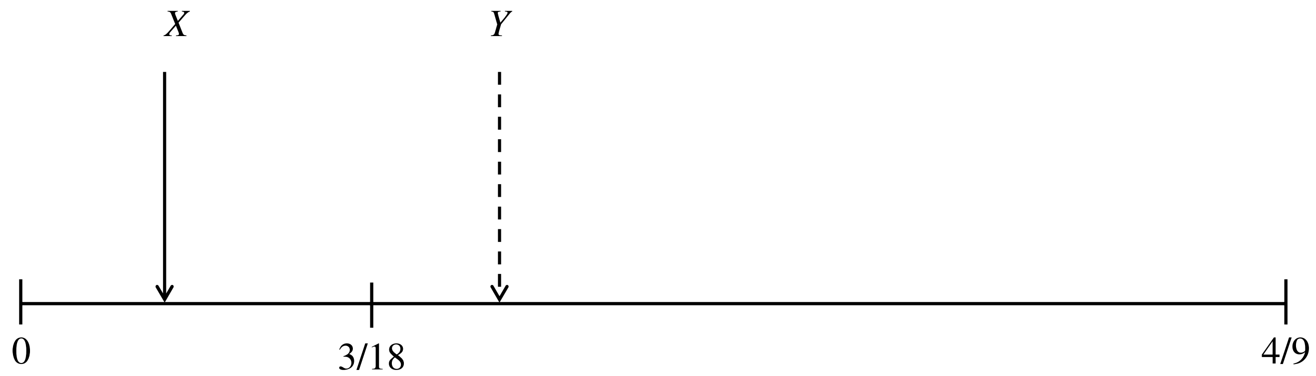


X *joins before* Y (points that lie above the diagonal $\Leftrightarrow a < b$)

1. If X joins before 3/18, it can never be pivotal since

$$a + 3/9 < 1/2 \quad \text{for } a < 3/18.$$

   Y can be pivotal provided that $b < 3/18$.



   When Y joins after X and the point of joining is after 3/18, then the oceanic voters have pivoted already.

2. The oceanic voters all combined together cannot pass the bill since they hold 4/9 which is less than 50% of the votes. If X joins after $1/2 - 3/9 = 3/18$, then X pivots.

Calculate the Shapley-Shubik index for X or Y by calculating the *area* of the region in which X or Y pivots and dividing by the total area of the square. We have

$$\phi_X = \frac{(5/18)^2}{(4/9)^2} = \frac{25}{64} \approx 0.391$$
$$\phi_Y = \frac{(3/18)^2}{(4/9)^2} = \frac{9}{64} \approx 0.141;$$

with the other $1 - \frac{25}{64} - \frac{9}{64} = \frac{30}{64}$ being shared by the oceanic players. Interestingly, the major stockholder X has a higher power relative to his percentage holding. The gain comes at the expense of Y. (做老大總是好一些)

- If there are three major players in an oceanic game, we represent orderings as points in a cube. We then calculate the volumes of the regions where each of the major players pivots.

Shapley-Shubik index of the President

Actual federal system (with the Vice President ignored)

When the number of Representatives ≥ 290 and the number of Senator ≥ 67, the President cannot be pivotal.

House	Senate	
218	51 to 100	Without the two-thirds majority
⋮	⋮	in both the Senate and the House,
289	51 to 100	the President can veto.
290	51 to 66	Once the House has the two-thirds majority, the
⋮	⋮	President is pivotal only when the Senate lies
435	51 to 66	between one-half and two-thirds majority

$$
\begin{aligned}
&\binom{435}{218}\Bigg[\binom{100}{51}(218+51)!(535-218-51)!+\cdots \\
&\qquad\qquad +\binom{100}{100}(218+100)!(535-218-100)!\Bigg] \\
&+\cdots \\
&+\binom{435}{289}\Bigg[\binom{100}{51}(289+51)!(535-289-51)!+\cdots \\
&\qquad\qquad +\binom{100}{100}(289+100)!(535-289-100)!\Bigg] \\
&+\binom{435}{290}\Bigg[\binom{100}{51}(290+51)!(535-290-51)!+\cdots \\
&\qquad\qquad +\binom{100}{66}(290+66)!(535-290-66)!\Bigg] \\
&+\cdots \\
&+\binom{435}{435}\Bigg[\binom{100}{51}(435+51)!(535-435-51)!+\cdots \\
&\qquad\qquad +\binom{100}{66}(435+66)!(535-435-66)!\Bigg]
\end{aligned}
$$

When divided by 536!, we obtain the Shapley-Shubik index of the President as $\phi = 0.16047$.

Inclusion of the Vice President

We need to add the case where 50 Senators and the Vice President say "yes". The following terms should be added:

$$\begin{aligned} & \binom{435}{218}\binom{100}{50}(218+51)!(535-218-50)! \\ + \quad & \binom{435}{219}\binom{100}{50}(219+51)!(535-219-50)! \\ +\cdots+ \quad & \binom{435}{435}\binom{100}{50}(435+51)!(535-435-50)! \end{aligned}$$

The denominator is modified to be 537!

- For the first term, we choose 218 Representatives from 435 of them and 50 Senators from 100 of them. There are $218 + 50 + 1$ "yes" voters and $535 - 218 - 50$ "no" voters. We have $(218 + 50 + 1)!$ orderings before the President and $(535 - 218 - 50)!$ orderings after the President.

Banzhaf index of the President

Let S denote the number of coalitions within the Senate that contain more than two-thirds of the members of the Senate:

$$S = \binom{100}{67} + \cdots + \binom{100}{100}.$$

Let s denote the number of coalitions within the Senate that contain equal and more than one-half of the members of the Senate:

$$s = \binom{100}{50} + \binom{100}{51} + \cdots + \binom{100}{100}.$$

Let H denote the number of coalitions within the House that contain more than two-thirds of the members of the House:

$$H = \binom{435}{290} + \cdots + \binom{435}{435}.$$

Let h denote the number of coalitions within the House that contain more than one-half of the members of the House:

$$h = \binom{435}{218} + \cdots + \binom{435}{435}.$$

- We count the number of winning coalitions with the President such that the defection of the President turns winning into losing. Write N_P = total number of winning coalitions in which the president is critical.

$$\begin{aligned} N_P &= \left[\binom{435}{218} + \cdots + \binom{435}{289}\right]\left[\binom{100}{50} + \binom{100}{51} + \cdots + \binom{100}{100}\right] \\ &\quad + \left[\binom{435}{290} + \cdots + \binom{435}{435}\right]\left[\binom{100}{50} + \binom{100}{51} + \cdots + \binom{100}{66}\right] \\ &= (h - H) \times s + H \times (s - S) = h \times s - H \times S. \end{aligned}$$

- The total number of winning coalitions in which the Vice President is critical $= N_V = \binom{100}{50} \times h$.

- Given a particular senator, we find the number of winning coalitions such that this senator is critical.

  Without the President, the number of critical swings effected by this senator is $\binom{99}{66} \times H$; and with the President, (but without the Vice president), the number of critical swings is $\binom{99}{50} \times h$. The last term corresponds to the presence of both the President and Vice President. The total number of winning coalitions to which the chosen senator is critical $= N_S = \binom{99}{66} \times H + \binom{99}{50} \times h + \binom{99}{49} \times h$.

- In a similar manner, the total number of winning coalitions to which a particular Representative is critical $= N_R = \binom{434}{289} \times S + \binom{434}{217} \times s$.

Banzhaf indexes calculations

$$
\begin{aligned}
& \text{Total number of critical swings by all players} \\
= \;& N = 100 \times N_S + 435 \times N_R + N_P + N_V
\end{aligned}
$$

$$
\begin{aligned}
& \text{Banzhaf index of any senator} \\
= \;& \frac{\text{number of winning coalitions to which the chosen senator is critical}}{\text{total number of critical swings}} \\
= \;& N_S/N
\end{aligned}
$$

Banzhaf index for the President $= N_P/N$.

2.4 Probabilistic characterization of power indexes

Question of Individual Effect. *What is the probability that my vote will make a difference, that is, that a proposal will pass if I vote for it, but fail if I vote against it?*

Question of Individual-Group Agreement. *What is the probability that the group decision will agree with my decision on a proposal?* (如己所願)

- The answers depend on both the decision rule of the body and the probabilities that various members will vote for or against a proposal. In some particular political example, we might also be able to estimate voting probabilities of the players for some particular proposal or class of proposals.
- If we are interested in general theoretical questions of power, we cannot reasonably assume particular knowledge about individual players or proposals. We should only make assumptions about voting probabilities which do not discriminate among the players.

Homogeneity Assumption. *Every proposal to come before the decision-making body has a certain probability* p *of appealing to each member of the body. The homogeneity is among members: they all have the same probability* p *of voting for a given proposal, but* p *varies from proposal to proposal.*

The homogeneity assumption does not assume that members will all vote the same way, but it does say something about their similar criteria for evaluating proposals. For instance, some bills that came before a legislature seem to have a high probability of appealing to all members, and pass by large margins: those have high p. Others are overwhelmingly defeated (low p) or controversial (p near 1/2).

Remark For the Shapley-Shubik index, we further assume the common p to be uniformly distributed between 0 and 1.

Shapley-Shubik index focuses on the *order* in which a winning coalition forms, and defines the power of a player to be proportional to the number of orderings in which she is pivotal. If voters have a certain degree of homogeneity, then ϕ is most appropriate.

Theorem 1. *The Shapley-Shubik index ϕ gives the answer to the question of individual effect under the homogeneity assumption about voting probabilities.*

Remark A swing for player i occurs if a coalition S_i exists such that

$$\sum_{j \in S_i} w_j < q \quad \text{and} \quad w_i + \sum_{j \in S_i} w_j \geq q.$$

Proof of Theorem 1

We randomize the probabilities $p_1, \ldots, p_N$ and invoke the conditional independence assumption. Given the realization of $\boldsymbol{p}_i = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N)$, the conditional probability that player i's vote will make a difference is given by

$$\pi_i(\boldsymbol{p}_i) = \sum_{S_i} \prod_{j \in S_i} p_j \prod_{j \notin S_i} (1 - p_j),$$

where p_j is the voting probability of player j. The sum is taken over all such coalitions where player i is pivotal. The expected frequency where player i is pivotal is obtained by integrating over the probability distribution:

$$E[\pi_i(\boldsymbol{p}_i)] = \int_0^1 \int_0^1 \cdots \int_0^1 \pi_i(\boldsymbol{p}_i) f_i(\boldsymbol{p}_i)\, dp_1 \ldots dp_{i-1}\, dp_{i+1} \ldots dp_N$$

where $f_i(\boldsymbol{p}_i)$ is the joint density function of $\boldsymbol{p}_i$. The voting probabilities are randomized.

Remark

Density function $f_X(x)$ of a uniform distribution over $[a, b]$ is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}.$$

Under the homogeneity assumption, a number p is selected from the uniform distribution on $[0, 1]$ and p_j is set equal to p for all j. In this case, $f_i(p) = 1$ since $a = 0$ and $b = 1$ so that

$$E[\pi_i(p)] = \int_0^1 \pi_i(p)\, dp \quad \text{where} \quad \pi_i(p) = \sum_{S_i} p^{s_i}(1-p)^{N-s_i-1}, \quad s_i = n(S_i).$$

Lastly, making use of the Beta integral:

$$\frac{s_i!(N - s_i - 1)!}{N!} = \int_0^1 p^{s_i}(1 - p)^{N - s_i - 1}\, dp,$$

we obtain

$$E[\pi_i(p)] = \sum_{S_i} \frac{s_i!(N - s_i - 1)!}{N!} = \phi_i = \text{ Shapley-Shubik index for player } i.$$

The Beta integral links the probability of being pivotal under the homogeneity assumption of voting probabilities with the expected frequency of being pivotal in various orderings of voters.

Independence Assumption. *Every proposal has a probability p_i of appealing to the i^{th} member. Each of the p_i is chosen independently from the interval* $[0, 1]$*. Here how one member feels about the proposal has nothing to do with how any other member feels.*

Banzhaf index ignores the question of ordering and looks only at the final coalition which forms in support of some proposal. The power of a player is defined to be proportional to the number of such coalitions. If the voters in some political situation behave completely independently, then β is the most appropriate index.

Theorem 2.

The absolute Banzhaf index β' gives the answer to the question of individual effect under the independence assumption about voting probabilities.

- The absolute Banzhaf index β_i' can be interpreted as assuming that each player votes randomly and independently with a probability of $1/2$. It can be shown mathematically that this is equivalent to assume that voting probabilities are selected randomly and independently from a distribution with mean $1/2$ without regard for the forms of those distributions.

- Each player can be thought of as having probability $1/2$ of voting for any given proposal, so we can think of all coalitions to be equally likely to form. Therefore, the probability of player i's vote making a difference is exactly the probability that player i will be a swing voter. This is precisely the absolute Banzhaf index.

Proof of Theorem 2

Under the independence assumption, the voting probabilities are selected independently from distributions (not necessarily uniform) on $[0,1]$ with $E[p_j] = 1/2$. Since p_j are independent, the joint density is

$$f_i(\boldsymbol{p}) = \prod_{j \neq i} f_j(p_j)$$

where $f_j(p_j)$ is the marginal density for p_j. Consider

$$\begin{aligned}
E[\pi_i(\boldsymbol{p})] &= \sum_{S_i} \int_0^1 \cdots \int_0^1 \prod_{j \in S_i} p_j \prod_{j \notin S_i} (1 - p_j) \prod_{j \neq i} f_j(p_j)\, dp_1 \cdots dp_N \\
&= \sum_{S_i} \prod_{j \in S_i} \int_0^1 p_j f_j(p_j)\, dp_j \prod_{j \notin S_i} \int_0^1 (1 - p_j) f_j(p_j)\, dp_j \\
&= \pi_i\left(\frac{1}{2}\right) = \sum_{S_i} \frac{1}{2^{N-1}} = \frac{\eta_i}{2^{N-1}} = \beta_i' \\
&= \text{absolute Banzhaf index for player } i,
\end{aligned}$$

where η_i is the number of swings for player i. Note that the sum of the absolute Banzhaf indexes for all players is not equal to 1.

Example

$$[3; 2, 1, 1]$$
$$A\ B\ C$$

- Each voter will vote for a proposal with probability p. What is the probability that A's vote will make a difference between approval and rejection?
- If both B and C vote against the proposal, A's vote will *not* make a difference, since the proposal will fail regardless of what he does.
- If B or C or both vote for the proposal, A's vote will decide between approval and rejection.

An alternative approach is shown here to compute $\phi_i\left(\beta_i\right)$ without resort to counting of pivotal orderings (swings).

The probability that A's vote will make a difference is given by

$$\pi_A(p) = \underset{B \text{ for, } C \text{ against}}{p(1-p)} + \underset{B \text{ against, } C \text{ for}}{(1-p)p} + \underset{\text{both for}}{p^2} = 2p - p^2.$$

Similarly, B's vote will make a difference only if A votes for, and C votes against. If they both voted for, the proposal would pass regardless of what B did.

$$\pi_B(p) = \underset{A \text{ for, } C \text{ against}}{p(1-p)} = p - p^2.$$

By symmetry, we also have $\pi_C(p) = p - p^2$.

- Shapley-Shubik index: voting probabilities are chosen by players from a common uniform distribution on the unit interval.
- Banzhaf index: voting probabilities are selected independently from any set of distributions which have a common mean of 1/2.

1. *Homogeneity assumption*

   We must average the probability of making a difference $\pi_A(p)$ over all p between 0 and 1.

$$
\begin{aligned}
\text{for } A: \quad & \int_0^1 \pi_A(p)\,dp = \int_0^1 (2p - p^2)\,dp = \frac{2}{2} - \frac{1}{3} = \frac{2}{3} = \phi_A \\
\text{for } B: \quad & \int_0^1 \pi_B(p)\,dp = \int_0^1 (p - p^2)\,dp = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \phi_B \\
\text{for } C: \quad & \int_0^1 \pi_C(p)\,dp = \int_0^1 (p - p^2)\,dp = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \phi_C
\end{aligned}
$$

2. *Independence assumption*

   Assume that all players vote with probability 1/2 for or against a proposal. We obtain

$$
\begin{aligned}
\pi_A\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{3}{4} = \beta'_A \\
\pi_B\left(\frac{1}{2}\right) &= \pi_C\left(\frac{1}{2}\right) = \frac{1}{4} = \beta'_B = \beta'_C,
\end{aligned}
$$

   thus verifying Theorem 2. Finally, $\beta_A = \frac{3}{5}$, $\beta_B = \frac{1}{5}$, $\beta_C = \frac{1}{5}$.

Theorem 3

The answer to player i's question of individual-group agreement, under the independence assumption about voting probabilities, is given by $(1+\beta_i')/2$.

Theorem 2 says that β_i' gives the probability that player i's vote will make the difference between approval and rejection. Since his vote makes the difference, in this situation the group decision always agrees with his.

- With probability $1-\beta_i'$ player i's vote will *not* make a difference, but in this case the group will still agree with him, by chance, half the time.
- Hence the total probability that the group decision will agree with player i's decision is

$$(\beta_i')(1) + (1-\beta_i')\left(\frac{1}{2}\right) = \frac{1+\beta_i'}{2}.$$

Example

Consider the weighted voting game: $[51; 40, 30, 20, 10]$. We list all the marginal cases where defection of a player changes losing to winning.

Players				marginal (losing to winning)			
1	2	3	4	1	2	3	4
N	Y	Y	N	×			×
Y	N	N	Y		×	×	
N	Y	N	Y	×		×	
N	N	Y	Y	×	×		
Y	N	N	N		×	×	
N	Y	N	N	×			
N	N	Y	N	×			

For player 1, we have $\eta_1 = 5$, where the 5 coalitions are

$$S_1^{(1)} = \{2,3\}, S_1^{(2)} = \{2,4\}, S_1^{(3)} = \{3,4\}, S_1^{(4)} = \{2\}, S_1^{(5)} = \{3\}.$$

The conditional probability that player 1 makes a difference:

$$\begin{aligned}\pi_1(p_2, p_3, p_4) = {} & p_2 p_3 (1 - p_4) + p_2(1 - p_3)p_4 + (1 - p_2)p_3 p_4 \\ & + p_2(1 - p_3)(1 - p_4) + (1 - p_2)p_3(1 - p_4).\end{aligned}$$

Under the independence assumption and expected probabilities all equal $\frac{1}{2}$, the absolute Banzhaf index of Player 1 is given by

$$
\begin{aligned}
E[\pi_1(p_2,p_3,p_4)] = &\int_0^1 p_2 f_2(p_2)\ dp_2 \int_0^1 p_3 f_3(p_3)\ dp_3 \int_0^1 (1-p_4) f_4(p_4)\ dp_4 \\
&+ \int_0^1 p_2 f_2(p_2)\ dp_2 \int_0^1 (1-p_3) f_3(p_3)\ dp_3 \int_0^1 p_4 f_4(p_4)\ dp_4 \\
&+ \int_0^1 (1-p_2) f_2(p_2)\ dp_2 \int_0^1 p_3 f_3(p_3)\ dp_3 \int_0^1 p_4 f_4(p_4)\ dp_4 \\
&+ \int_0^1 p_2 f_2(p_2)\ dp_2 \int_0^1 (1-p_3) f_3(p_3)\ dp_3 \int_0^1 (1-p_4) f_4(p_4)\ dp_4 \\
&+ \int_0^1 (1-p_2) f_2(p_2)\ dp_2 \int_0^1 p_3 f_3(p_3)\ dp_3 \int_0^1 (1-p_4) f_4(p_4)\ dp_4 \\
= &\ \pi_1(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \frac{5}{2^3} = \beta_1'.
\end{aligned}
$$

Similarly, we obtain

$$\beta_2' = \frac{3}{8},\ \beta_3' = \frac{3}{8},\ \beta_4' = \frac{1}{8}.$$

The Banzhaf index is

$$\beta = (\frac{5}{12}\ \frac{3}{12}\ \frac{3}{12}\ \frac{1}{12}).$$

Player 1 - group agreement

Out of $2^4 = 16$ cases, there are $2\eta_1 = 2 \times 5 = 10$ cases where Player 1 is marginal. In the remaining 6 cases (out of 16 cases), Player 1 does not make a difference.

players				pass/fail	
1	2	3	4		
Y	Y	Y	Y	P	with Y for players 2,3&4 gives "Pass" already,
N	Y	Y	Y	P	player 1 has equal probability to say Y or N
Y	N	N	Y	F	with N for players 2&3 gives "Fail" already,
N	N	N	Y	F	player 1 has equal probability to say Y or N
Y	N	N	N	F	with N for players 2,3&4 gives "Fail" already,
N	N	N	N	F	player 1 has equal probability to say Y or N

Probability of player 1-group agreement $= \frac{1}{2} \times (1 - \frac{5}{8}) + 1 \times \frac{5}{8} = \frac{13}{16}$.

Example

Look again at $\underset{A\ \ B\ \ C}{[3;2,1,1]}$. What is the probability that, under the independence assumption, the group decision will agree with A's preference?

- With probability $1/2, A$ will support a proposal. It will then pass *unless* B and C both oppose it, which will happen with probability $1/4$.
- If A opposes the proposal (probability $1/2$), it will always fail.
- The probability of agreement with A is thus

$$\frac{1}{2}\left(1-\frac{1}{4}\right)+\frac{1}{2}(1)=\frac{7}{8}=\frac{1+\frac{3}{4}}{2}=\frac{1+\beta'_A}{2}.$$

- Similarly, if B supports a proposal (probability 1/2), it will pass if and only if A supports it (probability 1/2).

- If B opposes the proposal (probability 1/2), it will fail unless both A and C support it (probability 1/4):

$$\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}\left(1-\frac{1}{4}\right)=\frac{5}{8}=\frac{1+\frac{1}{4}}{2}=\frac{1+\beta'_B}{2}.$$

Example

Consider $\underset{A\ \ B\ \ C\ D}{[5; 3, 2, 1, 1]}$. Let $\rho_i(p)$ be the probability that the group decision agrees with player i's decision, given that all players (including i) vote for a proposal with probability p. Note that A has veto power.

Remark

In the calculation procedure, it is convenient to set $p_A = p_B = p_C = p_D = p$. This is because under the independence assumption and common mean of probabilities of 1/2, we may set $p = 1/2$ apparently in the calculation of $E[\rho_i(p_A, p_B, p_C, p_D)]$.

(a) It can be shown easily that

$$\rho_A(p) = \underset{A\text{ yes}}{p}\ \underset{B\text{ yes}}{[p} + \underset{B\text{ no, }C+D\text{ yes}}{(1-p)\,p^2]} + \underset{A\text{ no}}{(1-p)(1)} = 1 - p + p^2 + p^3 - p^4.$$

$$\rho_B(p) = \underset{B\text{ yes}}{p}\ \underset{A\text{ yes}}{(p)} + \underset{B\text{ no, not all of }A,\,C,\,D\text{ yes}}{(1-p)\,(1-p^3)} = 1 - p + p^2 - p^3 + p^4$$

$$\rho_C(p) = \underset{C\text{ yes}}{p}\ \underset{A\text{ yes}}{[p}\ \underset{B\text{ yes}}{(p +}\ \underset{B\text{ no, }D\text{ yes}}{(1-p)\,p)]} + \underset{C\text{ no}}{(1-p)}\ \underset{A\text{ no}}{[(1-p)} + \underset{A\text{ yes, }B\text{ no}}{p\,[(1-p)]}$$

$$= 1 - p - p^2 + 3p^3 - p^4.$$

(b) Now calculate $\rho_A(1/2), \rho_B(1/2)$, and $\rho_C(1/2)$ and show that these are $(1+\beta'_A)/2, (1+\beta'_B)/2$, and $(1+\beta'_C)/2$, thus verifying Theorem 3 for this case.

Example

Consider the majority-minority voting system with 7 voters, where 5 of them are in the majority group and the remaining 2 voters are in the minority group. The passage of a bill requires at least 4 votes from all voters and at least 1 vote from the minority group. Suppose the 5 members in the majority group vote as a homogeneous group and the 2 members in the minority group vote as another homogeneous group.

(a) Compute the probability that a majority player's vote decides the passage of a bill.

(b) Compute the probability that a minority player's vote decides the passage of a bill.

Solution

Under the homogeneity assumption, we let p and q denote the homogeneous voting probability of the majority group and minority group, respectively.

(a) Consider a particular majority member, her vote can decide the passage of a bill if

   (i) 1 minority member and 2 other majority members say "yes" and other members say "no";

   (ii) 2 minority member and 1 other majority members say "yes" and other members say "no".

$$
\begin{aligned}
& P[\text{majority player's vote can decide the passage}|p,q] \\
= \; & C_1^2 C_2^4 q(1-q)p^2(1-p)^2 + C_1^4 q^2 p(1-p)^3 \\
= \; & 12q(1-q)p^2(1-p)^2 + 4q^2p(1-p)^3.
\end{aligned}
$$

Assuming independence of the random probabilities p and q, and both of them follow the uniform distribution, we obtain

$$
\begin{aligned}
& P[\text{majority player's vote can decide the passage}] \\
= \; & \int_0^1 \int_0^1 \left[12q(1-q)p^2(1-p)^2 + 4q^2p(1-p)^3 \right] dp dq \\
= \; & 12 \int_0^1 p^2(1-p)^2 \; dp \int_0^1 q(1-q) \; dq + 4 \int_0^1 p(1-p)^3 \; dp \int_0^1 q^2 \; dq.
\end{aligned}
$$

(b) Consider a particular minority member, her vote can decide the passage of a bill if

(i) 3 or more majority members say "yes" and other members say "no";

(ii) 2 majority members and the other minority member say "yes" and other members say "no".

Using similar assumptions on p and q, we obtain

$$
\begin{aligned}
& P[\text{minority player's vote can decide the passage}] \\
= \; & \int_0^1 \int_0^1 \left[\sum_{k=3}^{5} C_k^5 p^k (1-p)^{5-k} (1-q) + C_2^5 p^2 (1-p)^3 q \right] dpdq \\
= \; & 10 \left[\int_0^1 p^3 (1-p)^2 \; dp \int_0^1 (1-q) \; dq + \int_0^1 p^2 (1-p)^3 \; dp \int_0^1 q \; dq \right] \\
& + 5 \int_0^1 p^4 (1-p) \; dp \int_0^1 (1-q) \; dq + \int_0^1 p^5 \; dp \int_0^1 (1-q) \; dq.
\end{aligned}
$$

Example

Consider the voting game: $[2; 1, 1, 1]$.

Let p_A, p_B and p_C be the probabilities that A, B and C will vote for a proposal. Assuming independence of the random voting probabilities, we calculate the probabilities of a player's vote making a difference:

$$\begin{aligned}
\pi_A &= p_B(1-p_C)+(1-p_B)p_C, \\
\pi_B &= p_A(1-p_C)+(1-p_A)p_C, \\
\pi_C &= p_A(1-p_B)+(1-p_A)p_B.
\end{aligned}$$

- If the p_is are all independent (β') or all equal (ϕ) as they vary between 0 and 1, then the players have equal power.

Suppose B and C are homogeneous ($p_B = p_C$), but A is independent. Then the answers to the question of individual effect are

$$\text{for } A: \quad \int_0^1 2p_B(1-p_B)\,dp_B = \frac{1}{3}$$

$$\text{for } B \text{ or } C: \quad \left(\int_0^1 p_A\,dp_A\right)\left(\int_0^1 (1-p_B)\,dp_B\right) + \left(\int_0^1 (1-p_A)\,dp_A\right)\left(\int_0^1 p_B\,dp_B\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}.$$

With the pair sharing homogeneity in voting probabilities, B and C both have more power than A. In particular, we could normalize $(1/3, 1/2, 1/2)$ to $(1/4, 3/8, 3/8)$ and compare that to $(1/3, 1/3, 1/3)$.

Canadian Constitutional Amendment Scheme revisited

B_1	$\otimes$	B_2	$\otimes$	$M_{4,2}$	$\otimes$	$[3; 2, 1, 1, 1]$
Quebec		Ontario		Atlantic		British Columbia and Central.

Intersection of 4 weighted voting systems.

	Percentage of power			
Province	Shapley-Shubik index	Banzhaf index	Percentage of population	
Ontario	31.55	21.78	34.85	average 31.90 (Ontario, Quebec)
Quebec	31.55	21.78	2.94	
Bristish Columbia	12.50	16.34	9.38	
Central				
Alberta	4.17	5.45	7.33	average 5.65 (Alberta, Saskatchewan, Manitoba)
Saskatchewan	4.17	5.45	4.79	
Manitoba	4.17	5.45	4.82	
Atlantic				
New Brunswick	2.98	5.94	3.09	average 2.47 (New Brunswick, Nova Scotia, P.E.I., Newfoundland)
Nova Scotia	2.98	5.94	3.79	
P.E.I.	2.98	5.94	0.54	
Newfoundland	2.98	5.94	2.47	

Observations

- Based on the Shapley-Shubik index calculations, the scheme "produces a distribution of power that matches the distribution of population surprisingly well".
- However, based on the Banzhaf analysis, the scheme would seriously under-represent Ontario and Quebec and seriously over-represent British Columbia and the Atlantic provinces.
- It is disquieting that the two power indexes actually give different orders for the power of the players. ϕ says the Central Provinces are more powerful than the Atlantic provinces, and β says the opposite.

Which index is more applicable?

- Use ϕ if we believe there is a certain kind of homogeneity among the provinces.
- Use β if we believe there are more likely to act independently of each other.

Actual behavior

- Quebec and British Columbia would likely to behave independently.
- The four Atlantic provinces would more likely to satisfy the homogeneity assumption.

Hybrid approach

If a group of provinces is homogeneous, assign the members of that group the same p, which varies between 0 and 1 (independent of the p assigned to other provinces or groups of provinces).

Now calculate the probability that Quebec's vote, say, will make a difference:

$$\begin{array}{rcl} \pi_Q & = & p_O[6p_A^2(1-p_A)^2 + 4p_A^3(1-p_A) + p_A^4] \\ & & O \text{ yes} \qquad\qquad\qquad\qquad \text{2 or more } A\text{'s yes} \\ & & \cdot\{p_B[3p_C(1-p_C)^2 + 3p_C^2(1-p_C) + p_C^3] + (1-p_B)p_C^3\} \\ & & B \text{ yes} \quad \text{1 or } 2C\text{'s yes or} \qquad\qquad 3C\text{'s yes} \end{array}$$

We now compute the expectation of π_Q as p_O, p_A, p_B, and p_C vary independently between 0 and 1. Technically, that involves a "fourfold multiple integral."

$$E[\pi_Q] = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \pi_Q \, dp_C \, dp_B \, dp_A \, dp_O.$$

Note that the joint density function of p_C, p_B, p_A and p_D reduces to 1 since it is the product of the marginal functions of p_C, p_B, p_A and p_D (due to independence assumption) and each of these marginal density functions equals 1 since they are uniform density functions over $[0,1]$.

We obtain $E[\pi_Q] = E[\pi_O] = 24/160, E[\pi_C] = 8/160, E[\pi_B] = 12/160, E[\pi_A] = 5/160$. There are $3C$'s and $4A$'s, the π's sum to 104/160, so we normalize by multiplying the factor 160/104. The final power indexes under this scenarios are tabulated below under "A_s homogeneous and C_s homogeneous".

Table 2

Provinces	All homogeneous (ϕ)	*A*s homogeneous *C*s and *B* homogeneous	*A*s homogeneous *C*s homogeneous	All independent (β)	Average % of population
Quebec or Ontario	31.55	26.09	23.08	21.78	31.90
British Columbia	12.50	13.04	11.54	16.34	9.38
Central province	4.17	4.35	7.69	5.45	5.65
Atlantic province	2.98	5.43	4.81	5.94	2.47

Alternative homogeneity assumption

Quebec seems often to consider itself an island of French culture in the sea of English Canada. Treat all 9 other provinces as homogeneous among themselves, and Quebec as independent.

Quebec: 38.69	British Columbia: 11.61
Ontario: 25.84	Central provinces: 3.87
	Atlantic province: 3.07

Quebec's veto gives it considerable power. Alternatively, by staying homogeneous with other provinces, Ontario loses her power when compared to Quebec.

- Consider the effect of British Columbia's possible homogeneity with the Central provinces. Is it obvious that such homogeneity should give Quebec and Ontario more power?

Apparently, the previous power index calculations indicate that a higher level of homogeneity of other players gives more influential power to the province with veto power.

2.5 Potential blocs, quarelling paradoxes and bandwagon effects

Power of Potential blocs

- Groups of voters with similar interests and values who might consider joining together and casting their votes in common.

Question: If a potential bloc decides to organize and vote as an actual bloc, does it really gain power?

Under the Shapley-Shubik model, we are comparing the chance that the organized bloc will pivot against the chance that one of the unorganized members will pivot.

- If we are in a majority game where each player has just one vote, the answer is "yes", provided that no other potential blocs organize.

- Union does not always mean strength in other simple games.

Example – disunity adds strength

Consider the weighted majority game $[5; 3, 3, 1, 1, 1]$

Here $\phi = \left(\frac{9}{30}, \frac{9}{30}, \frac{4}{30}, \frac{4}{30}, \frac{4}{30}\right)$ and $\boldsymbol{\beta} = \left(\frac{2}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right)$.

If the 3 small players unite to form a bloc of three, that bloc will have power $1/3$, which is less than the total of what the members originally had, measured by either index.

Example

In a 40-member council, 11 members from Town B, 14 from Town J and 15 from other towns and rural areas.

Shapley-Shubik power index calculations

1. Both towns do not organize
   If the 11 members in Town B vote independently, then each supervisor will have $1/40$ of the total power and Town B together will have $\frac{11}{40} = 27.5\%$.

2. Town B organizes but Town J does not organize

- If the 11 members in Town B organize, then there are effectively 30 voters. Since 21 votes are needed to pass a measure, Town B will pivot if it joins a coalition after the 10th through 20th i.e. 11/30 of the time. Town B will have $11/30 = 36\frac{2}{3}\%$ of the power.
- Town J who had $14/40 = 35\%$ of the power before Town B organized, would have $(14/29)(19/30) \approx 30.5\%$ of the power after Town B organizes. Note that 29 council members from Town J and other towns share the remaining power of $1 - \dfrac{11}{30} = \dfrac{19}{30}$. Since the power of $\dfrac{19}{30}$ is shared equally among 29 members, so the total power of the 14 independent councils from Town $J = \dfrac{14}{29} \times \left(1 - \dfrac{11}{30}\right)$.

3. Both town organize

   As a result, we have a game of 17 voters: 15 casting a single vote. Town J pivots 100/272 of the time, for 37% of the power, while Town B pivots 49/272 of the time, for 18% of the power.

There are 15 other members, other than Town B and Town J.

Let Town B joins right after position i and Town J joins right after position j, $0 \leq i \leq 15$ and $0 \leq j \leq 15$.

(a) When Town J enters first, what is the number of possible orderings?
Since $i \geq j$, we have $j = 0, 1, 2, \cdots, 15, i = j, j + 1, \cdots, 15$, so number of orderings $= 16 + 15 + \cdots + 1 = \frac{16 \times 17}{2} = 136$.

- Suppose $j \geq 7$, then J always pivots since $[j+1, j+2, \cdots, j+14]$ contains the pivotal 21^{st} position.

- On the other hand, with $j < 7$, and Town B enters after position i with $i \geq j$, then B pivots when $[i+14, i+15, \cdots, i+24]$ contains the pivotal 21^{st} position. This occurs when $i < 7$.

Summary

(i) J pivots when $j = 7, 8, \cdots, 15, i = j, j+1, \cdots, 15$;

number of orderings $= 9 + 8 + \cdots + 1 = \frac{9 \times 10}{2} = 45$.

(ii) B pivots when $j < 7$, $i \leq j$ and $i < 7$. We have

$$j = 0, 1, 2, \cdots, 6; i = j, j+1, \cdots, 6;$$

the number of orderings $= 7 + 6 + \cdots + 2 + 1 = 28$.

(b) We consider the another case where Town B enters earlier, where $i \leq j$.

The total number of possible orderings remains to be 136. Following similar arguments, we can show that

(i) the number of orderings that J pivots is 55;

(ii) the number of orderings that B pivots is 21.

Finally, combining the two cases, we have

$$\phi_B = \frac{28 + 21}{272} \simeq 18\% \text{ and } \phi_J = \frac{45 + 55}{272} \approx 37\%$$

First entry is the power index for B, second entry is the power index for J.

	Town J does not organize		Town J organizes	
Town B does not organize	27.5	35	20	52
Town B organizes	36.7	30.5	18	37

- Town J will prefer to organize regardless of what Town B does since Town J increases its power upon organizing. "Organize" is a dominant strategy of Town J.

- Once Town J organizes, Town B is actually better off not organizing.

- Thus the natural outcome is Town B not organizes while Town J organizes. Town B supervisors should be "cunning" to choose uncooperative behavior. Town B should be very happy not to rock the boat.

Example

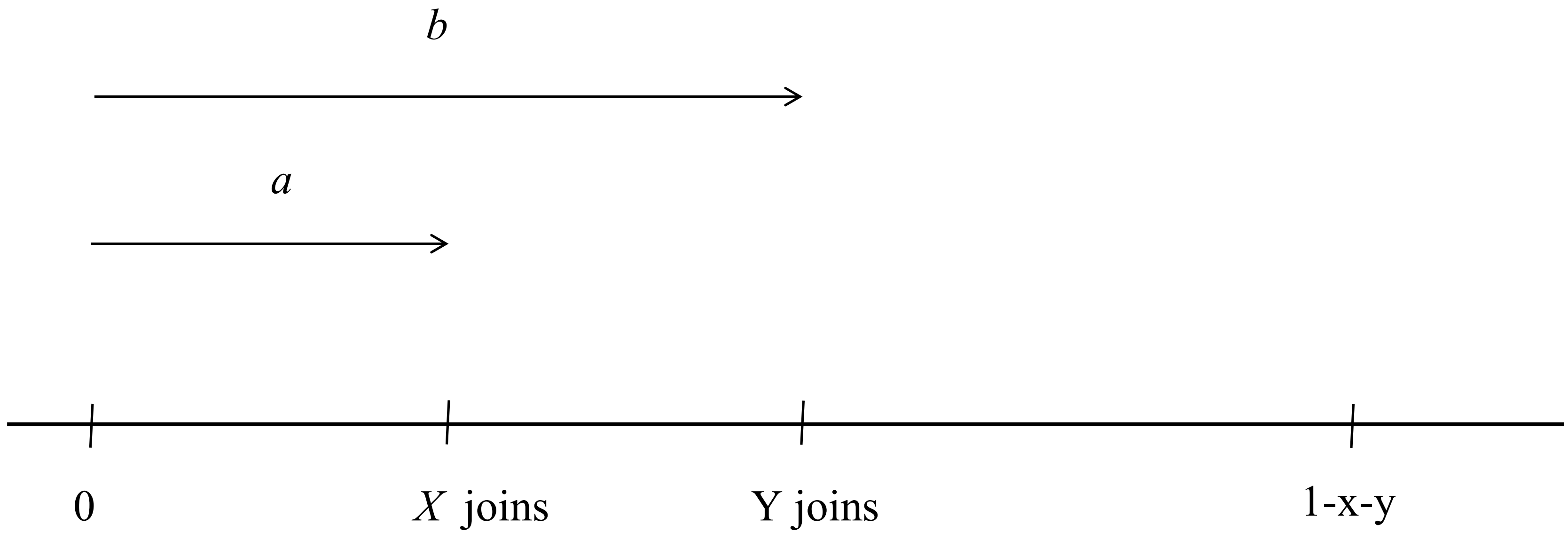
Assuming $0 < x < 1/2$ and $0 < y < 1/2$. If Player X controls a fraction of x of the vote and Player Y controls a fraction y, then

$$\phi_X(x,y) = \begin{cases} \frac{\left(\frac{1}{2}-y\right)^2}{(1-x-y)^2}, & \text{if } x+y \geq \frac{1}{2} \\ \frac{x(1-x-2y)}{(1-x-y)^2}, & \text{if } x+y \leq \frac{1}{2} \end{cases}.$$

The earlier example on P.65 (Topic 1) corresponds to $x = 3/9$ and $y = 2/9$, where $x + y > 1/2$. The area of the two triangles that correspond to X being pivotal is $\left(\frac{1}{2} - y\right)^2$. The area of the square is $(1 - x - y)^2$, so

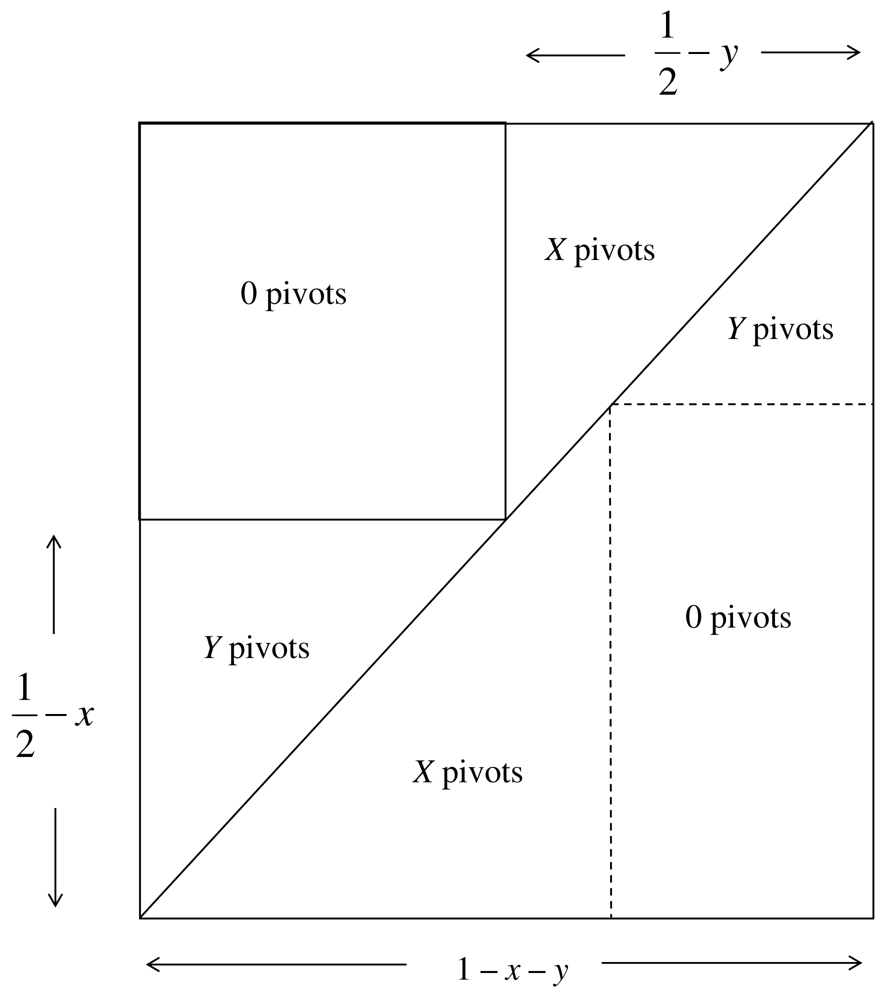
$$\phi_X(x,y) = \frac{\left(\frac{1}{2} - y\right)^2}{(1-x-y)^2}, \quad x + y \geq \frac{1}{2}.$$

If the members of X and Y vote independently, the members of X will have power equal to their fraction of the vote, namely, x.

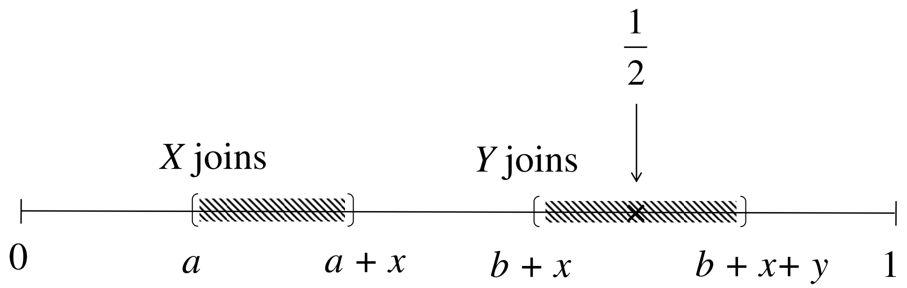


Consider the case where X enters earlier than Y, so that $a < b$; the total length of segment for oceanic voters is $1 - x - y$.

(i) $x+y \geq \frac{1}{2}$



Distinguish the various cases to determine which one of the two intervals $[a, a+x]$ or $[b+x, b+x+y]$ includes the pivotal point $\frac{1}{2}$.



For $x + y \geq \frac{1}{2}$; by assuming $a < b, X$ pivots when $a + x \geq \frac{1}{2}$ i.e. $a \geq \frac{1}{2} - x$. Otherwise, for $a + x < \frac{1}{2}$, we have

$$\begin{cases} Y \text{ pivots when } b + x \leq \frac{1}{2} & \text{i.e. } b \leq \frac{1}{2} - x \\ O \text{ pivots when } b + x > \frac{1}{2} & \text{i.e. } b > \frac{1}{2} - x. \end{cases}$$

The power index of X and Y are, respectively,

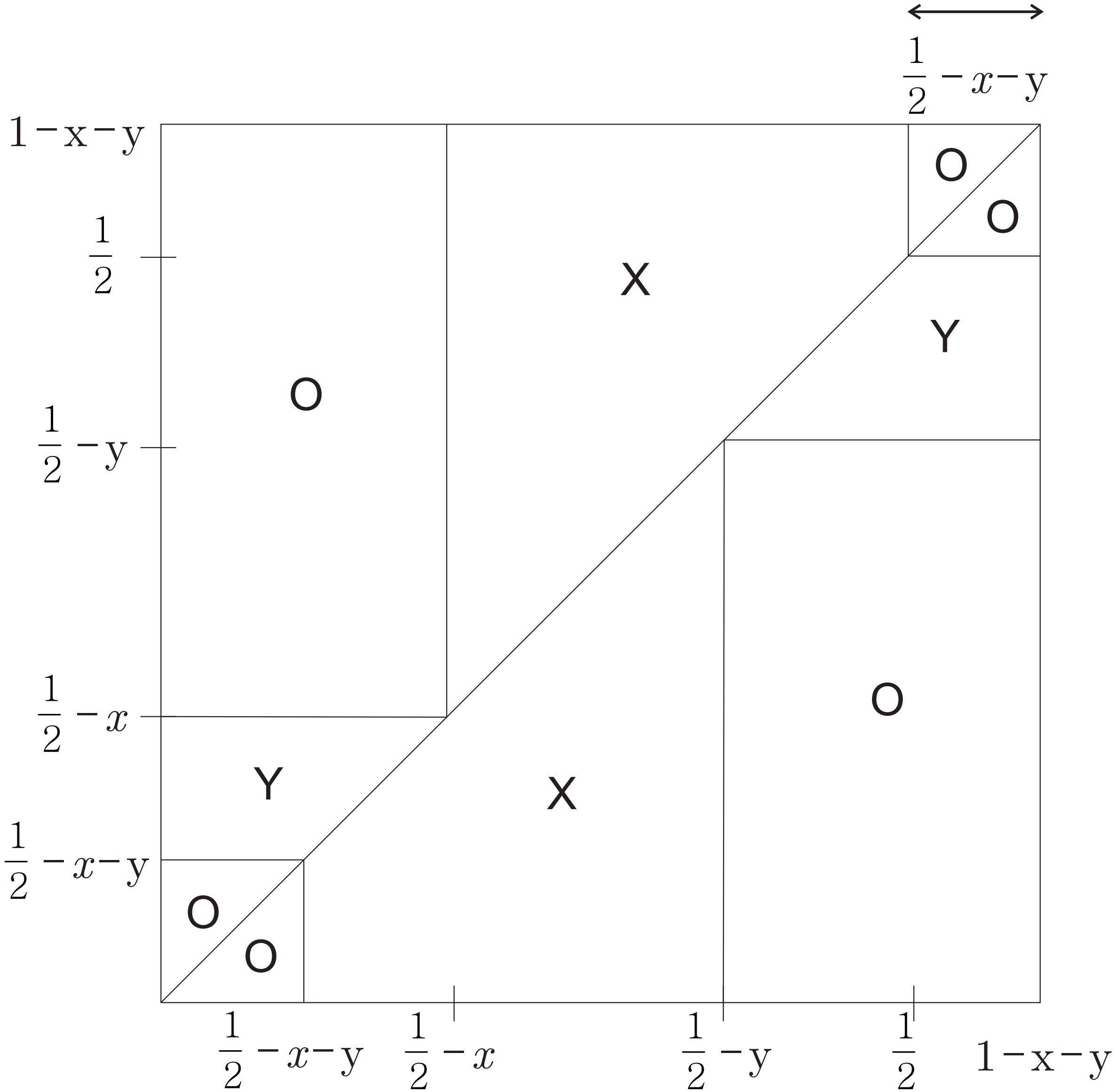
$$\phi_X(x, y) = \frac{\left(\frac{1}{2} - y\right)^2}{(1 - x - y)^2} \quad \text{and} \quad \phi_Y(x, y) = \frac{\left(\frac{1}{2} - x\right)^2}{(1 - x - y)^2}.$$

Since the power of any player holding x votes is the same across all players, so we expect

$$\phi_X(x, y) = \phi_Y(y, x).$$

Lastly, $\phi_X(x, y) + \phi_Y(x, y) + \phi_O(x, y) = 1$.

(ii) $x + y \leq \frac{1}{2}$

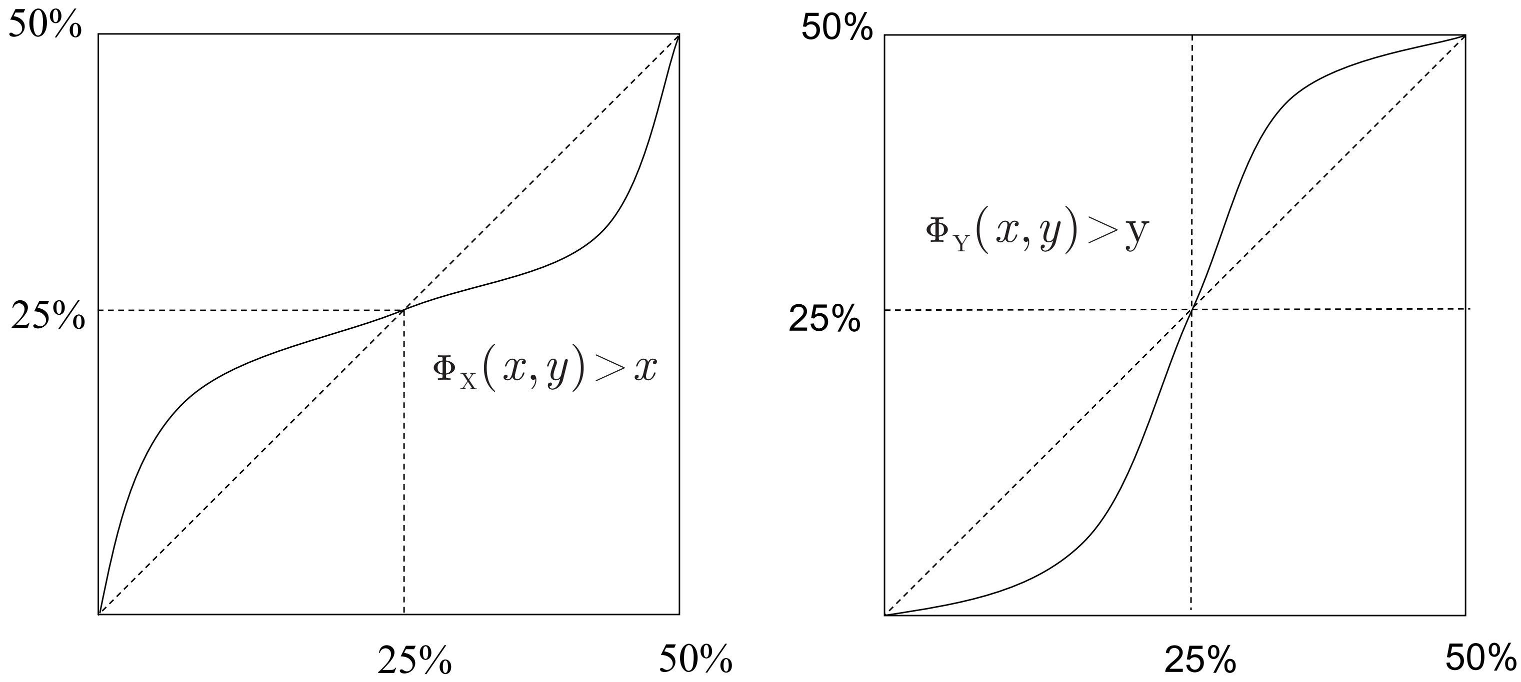


Assuming $a < b$, a slight modification is required when $x + y \leq \frac{1}{2}$ since X pivots if and only if $\frac{1}{2} - x \leq a \leq \frac{1}{2}$. For $a > \frac{1}{2}$, O pivots. Similarly, when $a < \frac{1}{2} - x - y$ and $b < \frac{1}{2} - x - y$, O pivots. We obtain

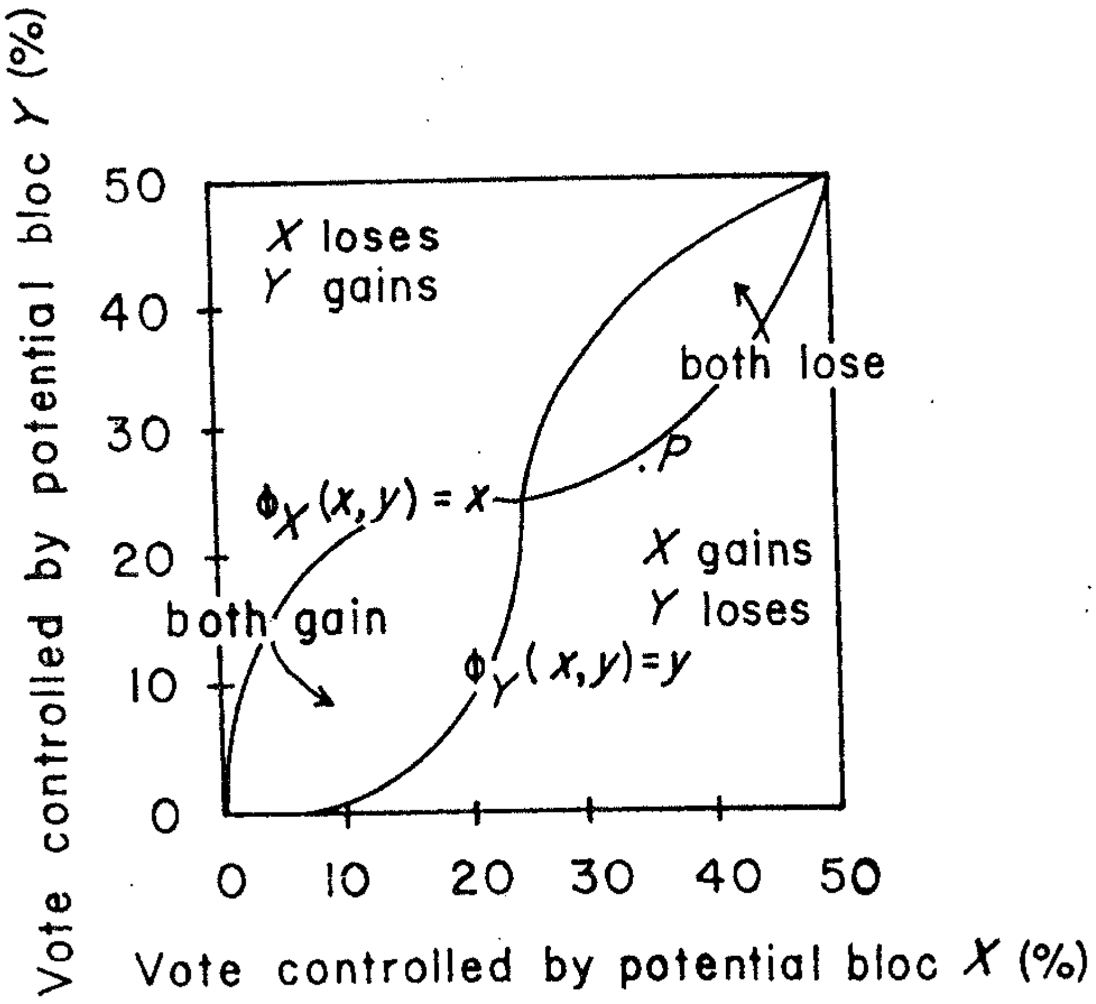
$$\begin{aligned}
\phi_X(x,y) &= \frac{\left(\frac{1}{2} - y\right)^2 - \left(\frac{1}{2} - x - y\right)^2}{(1-x-y)^2} = \frac{x(1-x-2y)}{(1-x-y)^2};\\
\phi_Y(x,y) &= \frac{y(1-y-2x)}{(1-x-y)^2};\\
\phi_U &= 1 - \phi_X - \phi_Y.
\end{aligned}$$

By symmetry, $\phi_X(x,y) = \phi_Y(y,x)$. This is because $\phi_X(x,y)$ is the power of X when he holds x votes and the other holds y votes; while $\phi_Y(y,x)$ gives the power of Y when he holds x votes and the other holds y votes.

We compare the power of x when both blocs organize or both do not organize by examining the relative magnitude of $\phi_X(x,y)$ and x.



Note that $0 \leq x \leq \frac{1}{2}$ and $0 \leq y \leq \frac{1}{2}$. It is seen that $\phi_X(x,y) = x$ when $x = y = \frac{1}{4}$, $x = y = 0$ or $x = y = \frac{1}{2}$. When both x and y are less than $\frac{1}{4}$, the organization of bloc X is beneficial when bloc Y is not substantially larger than bloc X.



The members of X will be better off if both X and Y organize precisely when $\phi_X(x, y) > x$ [the region below the curve $\phi_X(x, y) = x$]. If X represents Town J and Y represents Town B, then point P corresponds to the above example.

Quarreling paradoxes

- What happens if the two players quarrel, and refuse to enter into a coalition together?
- We normally think that we maximize our power by keeping as many options open as possible, and that restricting our freedom to act lessens our influence.
- Quarreling, of course, restricts our freedom to act.

Example

Consider the weighted voting game

$$\begin{array}{c}[5;\, 3,\, 2,\, 2]\\ \quad A\ \ B\ \ C\end{array}$$

For this game, $\phi = (2/3, 1/6, 1/6)$ and $\beta = (3/5, 1/5, 1/5)$, as seen by writing out the orderings with the pivots underlined:

$$\begin{array}{ccc} A\underline{B}C & B\underline{A}C & C\underline{A}B \\ A\underline{C}B & *\,BC\underline{A} & *\,CB\underline{A} \end{array}$$

and the winning coalitions with the critical defectors underlined:

$$\underline{AB} \quad \underline{AC} \quad *\,\underline{A}BC.$$

Suppose members B and C quarrel. What is the effect on the Shapley-Shubik index?

In considering the orders in which the players might join a coalition in support of a proposal, we must now rule out those orderings in which B and C join together to help put the coalition over the top, i.e. those orderings in which both B and C join at or before the pivot.

There are two orderings in which this happens, marked by an $*$. In the four other orderings, the coalition becomes winning with the help of only one of B or C. By the original Shapley-Shubik assumption, these four orderings are equally likely. We obtain the *Shapley-Shubik index with quarreling* as

$$\phi^Q_{BC} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Banzhaf model for quarreling

We merely eliminate from consideration those winning coalitions containing both B and C (just ABC above) and compute proportions of critical defections in the remaining winning coalitions. We obtain

$$\beta^{Q}_{BC} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

with the same qualitative effect of an increase in B and C's share of the power at the expense of A.

What are the possibilities?

1. If two members quarrel, they may both gain power (as measured by ϕ or β). See the above example.

2. If two members quarrel, they may both lose power.

   This, of course, seems much more natural than (1).

3. If two members quarrel, one may gain power while the other loses power. (A quarrel might hurt you while helping your opponent, or vice versa $\cdots$.)

   Example: A and D quarrel in $[5; 3, 2, 2, 1]$

$$\phi = \beta = \left(\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}\right)$$

$$\phi^Q_{AD} = \beta^Q_{AD} = \left(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right).$$

4. A quarrel may not affect the power of the quarrelers at all, but change the power of innocent bystanders. (神仙打架，禍及凡人)

Example: B and C quarrel in the game of the last example.

$$\phi_{BC}^{Q} = \beta_{BC}^{Q} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right)$$

Poor D, whose only chance to become part of a minimal winning coalition was with BC, has become a dummy.

5. The presence of dummies does not change the relative proportion of pivotal orderings among the players. However, quarreling with a dummy always hurts you. This is because the quarreler with the dummy would lose more on the number of pivotal orderings that involve the dummy compared to other players. It is still possible for a bystander to lose in power (the loss is less than that of the quarreler) if this bystander derives most of her power via "joining with the quarreler".

   Example: A quarrels with D in $[4; 2, 2, 2, 1]$

$$\phi = \beta = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$$

$$\phi^Q_{AD} = \beta^Q_{AD} = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, 0\right)$$

   It may be worthwhile staying on friendly terms even with those who have no power.

One-way quarrel (hostility)

A one-way quarreler can help or hurt his victim, and he may either help or hurt himself.

For example, in $[5; 3, 2, 2, 1]$, if B or C is hostile to D, D loses the chance of forming a minimal winning coalition, making D to be a dummy. In this case, the victim D is hurt.

Suppose that, in the weighted voting game $\underset{A\ \ B\ \ C\ \ D}{[7;\ 4,\ 3,\ 2,\ 1]}$ player B hates player C and refuses to join any coalition in support of a proposal that C has already joined. Player C has no such hostile feeling about B. What is the effect upon the power of B and C?

Orderings and pivots are

$A\underline{B}CD$	$B\underline{A}CD$	$^*CA\underline{B}D$	$DA\underline{B}C$
$A\underline{B}DC$	$B\underline{A}DC$	$CA\underline{D}B$	$DA\underline{C}B$
$^*AC\underline{B}D$	$^+BC\underline{A}D$	$^*CB\underline{A}D$	$DB\underline{A}C$
$AC\underline{D}B$	$^+BCD\underline{A}$	$^*CBD\underline{A}$	$^+DBC\underline{A}$
$AD\underline{B}C$	$BD\underline{A}C$	$CD\underline{A}B$	$DC\underline{A}B$
$AD\underline{C}B$	$^+BDC\underline{A}$	$^*CDB\underline{A}$	$^*DCB\underline{A}$

1. Consider $^*AC\underline{B}D$, C joins earlier. Since B is hostile to C, B will not join later to form a winning coalition.

2. Consider $^+BC\underline{A}D$, B joins earlier. Since C is hostile to B, C will not join later to form a winning coalition.

Both orderings will be ruled out if B and C quarrel.

With no quarreling, we have

$$\phi = \left(\frac{14}{24}, \frac{6}{24}, \frac{2}{24}, \frac{2}{24}\right) \approx (0.58, 0.25, 0.08, 0.08).$$

B's hostility to C rules out orderings marked by $*$, giving

$$\phi^Q_{B\to C} = \left(\frac{10}{18}, \frac{4}{18}, \frac{2}{18}, \frac{2}{18}\right) \approx (0.56, 0.22, 0.11, 0.11).$$

B has hurt himself and helped his victim.

If we reversed the situation and had C hating B, the orderings marked by$^+$ would be ruled out, giving

$$\phi^Q_{C\to B} \left(\frac{10}{20}, \frac{6}{20}, \frac{2}{20}, \frac{2}{20}\right) = (0.50, 0.30, 0.10, 0.10)$$

C would help B, and also help herself!

Example

Weighted voting game: $[5;4,2,1,1,1]$; the first two voters quarrel. Note that the first two players can form a winning coalition together.

$$\begin{array}{ccccc} 4 & \underline{2} & 1 & 1 & 1 \\ 4 & \underline{1} & 2 & 1 & 1 \\ 4 & \underline{1} & 1 & 2 & 1 \\ 4 & \underline{1} & 1 & 1 & 2 \end{array} \qquad \begin{array}{ccccc} 2 & \underline{4} & 1 & 1 & 1 \\ 1 & \underline{4} & 2 & 1 & 1 \\ 1 & \underline{4} & 1 & 2 & 1 \\ 1 & \underline{4} & 1 & 1 & 2 \end{array} \qquad \begin{array}{ccccc} 2 & 1 & \underline{4} & 1 & 1 \\ 1 & 2 & \underline{4} & 1 & 1 \\ 1 & 1 & \underline{4} & 2 & 1 \\ 1 & 1 & \underline{4} & 1 & 2 \end{array}$$

$$\begin{array}{ccccc} 2 & 1 & 1 & \underline{4} & 1 \\ 1 & 2 & 1 & \underline{4} & 1 \\ 1 & 1 & 2 & \underline{4} & 1 \\ 1 & 1 & 1 & \underline{4} & 2 \end{array} \qquad \begin{array}{ccccc} 2 & 1 & 1 & \underline{1} & 4 \\ 1 & 2 & 1 & \underline{1} & 4 \\ 1 & 1 & 2 & \underline{1} & 4 \\ 1 & 1 & 1 & \underline{2} & 4 \end{array}$$

20 distinct orderings without consideration of quarrel

$$\phi = \left(\frac{6}{10}\ \frac{1}{10}\ \frac{1}{10}\ \frac{1}{10}\ \frac{1}{10}\right).$$

Rule out 7 orderings when the first two voters quarrel

4	$\underline{2}$	1	1	1
2	$\underline{4}$	1	1	1
2	1	1	$\underline{4}$	1
2	1	$\underline{4}$	1	1

1	2	$\underline{4}$	1	1
1	2	1	$\underline{4}$	1
1	1	2	$\underline{4}$	1

These “illegal” orderings must lie in the set of pivotal orderings held by the two quarreling players. In other words, none of these “illegal” orderings are pivotal orderings of the other players.

After ruling out 7 “illegal” orderings, the remaining 13 coalitions are:

$$
\begin{array}{ccccc}
4 & \underline{1} & 2 & 1 & 1 \\
4 & \underline{1} & 1 & 2 & 1 \\
4 & \underline{1} & 1 & 1 & 2 \\
1 & \underline{4} & 2 & 1 & 1 \\
1 & \underline{4} & 1 & 2 & 1 \\
1 & \underline{4} & 1 & 1 & 2
\end{array}
\qquad\qquad
\begin{array}{ccccc}
1 & 1 & \underline{4} & 2 & 1 \\
1 & 1 & \underline{4} & 1 & 2 \\
1 & 1 & 1 & \underline{4} & 2 \\
2 & 1 & 1 & \underline{1} & 4 \\
1 & 2 & 1 & \underline{1} & 4 \\
1 & 1 & 2 & \underline{1} & 4 \\
1 & 1 & 1 & \underline{2} & 4
\end{array}
$$

Out of these 13 orderings, 6 orderings of which “4”-voter pivots, only one ordering of which “2”-voter pivots, 6 orderings of which either one of the three “1”-voter pivots. The new power indexes with quarreling is

$$\phi = \left(\frac{6}{13} \quad \frac{1}{13} \quad \frac{2}{13} \quad \frac{2}{13} \quad \frac{2}{13}\right).$$

Lemma

Assume that the two players A and B in a N-person voting game can always form a winning coalition just with themselves. Suppose A and B now quarrel, show that the other players in the game enjoy an increase in power.

Proof

We let n_i and n_i' denote the number of orderings that player i is pivotal without and with A and B quarreling, respectively. The Shapley-Shubik power index of player i without and with quarreling are

$$\phi_i = \frac{n_i}{\sum_{i=1}^{N} n_i} \quad \text{and} \quad \phi_i' = \frac{n_i'}{\sum_{i=1}^{N} n_i'},$$

respectively. It is easily seen that $n_A' < n_A$ and $n_B' < n_B$ due to the constraint imposed by quarreling.

However, $n_i = n'_i$, $i \neq A, B$, since quarrel between A and B does not change the number of pivotal orderings of player i. This is because A and B together can form a winning coalition, so the ruling out of orderings with both A and B together would not reduce the pivotal orderings held by player i. We then have

$$\begin{aligned}\phi'_i = \frac{n'_i}{\sum_{i=1}^N n'_i} \quad &> \quad \frac{n'_i}{n_A + n_B + \sum_{\substack{i=1 \\ i \neq A,B}}^N n'_i} \\ &= \quad \frac{n_i}{n_A + n_B + \sum_{\substack{i=1 \\ i \neq A,B}}^N n_i} = \phi_i, \ \ i \neq A, B.\end{aligned}$$

Therefore, all other players other than the quarrelers in the game enjoy an increase in power.

Warning on the results

1. The conclusion is true, a subtlety of political situations that precise analysis has thrown light upon.
2. The conclusion is a peculiarity of the power indices, showing that they have strange properties that should make us wary of where and how we use them.
3. The conclusion is a peculiarity not of the indices but of the model of quarreling we made using the indices. The model does not adequately reflect properties of real world quarrels.

Bandwagon effect (跟紅頂白)

- Two opposing blocs compete for the support of uncommitted voters in an attempt to achieve winning sizes. Uncommitted voters *suddenly* find it *advantageous* to begin committing themselves to the larger bloc, quickly enlarging it to winning size.

  "be (jump) on the bandwagon" – join in what seems likely to be a successful enterprise.

Two opposing blocs, X and Y, and a collection of uncommitted voters U.

- If X and Y are opposing, we rule out orderings in which they join together to win. We consider only orderings in which exactly one of X or Y is present when the coalition first becomes winning. Also, we rule out the possibility of uncommitted voters uniting to win without either X or Y.

Example

Consider $[5; 3, 2, 1, 1, 1, 1]$

- Of the 30 orderings, 9 of them are ruled out by the restriction that exactly one of the "3" or "2" should appear at or before the pivot:

$3\ \underline{2}\ 1\ 1\ 1\ 1$	$2\ \underline{3}\ 1\ 1\ 1\ 1$	$2\ 1\ \underline{3}\ 1\ 1\ 1$	$2\ 1\ 1\ \underline{3}\ 1\ 1$	
$3\ 1\ \underline{2}\ 1\ 1\ 1$	$1\ 3\ \underline{2}\ 1\ 1\ 1$	$1\ 2\ \underline{3}\ 1\ 1\ 1$	$1\ 2\ 1\ \underline{3}\ 1\ 1$	$1\ 1\ 2\ \underline{3}\ 1\ 1$

- The use of the remaining 21 orderings leads to a modified Shapley-Shubik index

$$\left(\frac{6}{21}, \frac{3}{21}, \frac{3}{21}, \frac{3}{21}, \frac{3}{21}, \frac{3}{21}\right) \approx (0.286, 0.143, 0.143, 0.143, 0.143, 0.143).$$

- An uncommitted voter should commit to X if the increment of power he will add to X is larger than the power he would have if he remains uncommitted (毀滅自我，成就他人霸業).

- If an uncommitted voter commits to X, then the new game is

$$\begin{array}{ccccccccccccccc} [\,5; & 3, & 2, & 1, & 1, & 1, & 1\,] & \longrightarrow & [\,5; & 4, & 2, & 1, & 1, & 1\,] \\ & X & Y & & U & & & & & & & & & \end{array}$$

with only 3 uncommitted voters.

To calculate the modified Shapley-Shubik index by writing down the 20 possible orderings for this game, crossing out the 7 "illegal" orderings since X and Y are not supposed to appear together in a winning coalition. We obtain

$$\left(\frac{6}{13}, \frac{1}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13}\right) \approx (0.462, 0.077, 0.154, 0.154, 0.154).$$

- The uncommitted voter raises the power of X from 0.286 to 0.462, an increment of 0.176, which is more than 0.143 he would have by remaining uncommitted.

Example - 1956 Democratic Vice-Presidential Nomination

Jack Kennedy	618	45%
Estes Kefauver	551.5	40%
Albert Gore	110.5	8%
Others	92	7%
Total	1372	(687 is needed to be nominated)

As a model, we consider the weighted voting game

$$[50;\ \underset{J}{45},\ \underset{E}{40},\ \underset{G}{8},\ \underbrace{1,1,1,1,1,1,1}_{\text{uncommitted}}]$$

JE should be ruled out. We also rule out JG and allow EG.

Background

- Gore was friendly with Kennedy, but Gore and Kefauver were both senators from Tennessee.

- Gore's supporters would not follow Gore to Kennedy. Gore was under constant pressure to withdraw in favor of Kefauver. In fact, Kefauver went on to win the nomination.

We would like to examine the degree that Kennedy is disadvantaged by not being able to form an alliance with Gore.

Under the above alliance conditions, 408 of the $10!/7! = 720$ possible orderings were ruled out. For example,

$$40\ 1\ 1\ \underline{45}\ 1\ 1\ 8\ 1\ 1\ 1;\ \ 1\ 8\ 40\ \underline{45}\ 1\ 1\ 1\ 1\ 1\ 1$$

are ruled out since J and E should not appear together to form a winning coalition.

Of the remaining orderings, J has 20 pivots, E and G both have 98, and uncommitted delegates have 96, giving the following modified SS index

(0.064 0.314 0.314 0.043 0.043 0.043 0.043 0.043 0.043 0.043).

Interestingly, E and G have equal power under the above alliance conditions. This is because E cannot form a winning coalition even with the inclusion of all uncommitted voters. Poor J, since he quarrels with both E and G, his power is much undermined.

Consider the following possible changes:

1. G joins E, giving $[50; 45, 48, 1, 1, 1, 1, 1, 1, 1]$. The new power indexes are now

   (0.111 0.389 0.071 0.071 0.071 0.071 0.071 0.071 0.071).

   G can contribute only 0.075 to E, much less than he has by not joining E.

2. An uncommitted delegate joins J, giving

$$[50; 46, 40, 8, 1, 1, 1, 1, 1, 1]$$

with power indices

$$(0.088\ 0.285\ 0.285\ 0.057\ 0.057\ 0.057\ 0.057\ 0.057\ 0.057\ ).$$

She has contributed only 0.024 to J, less than she had while uncommitted.

3. An uncommitted delegate joins E, giving

$$[50; 45, 41, 8, 1, 1, 1, 1, 1, 1]$$

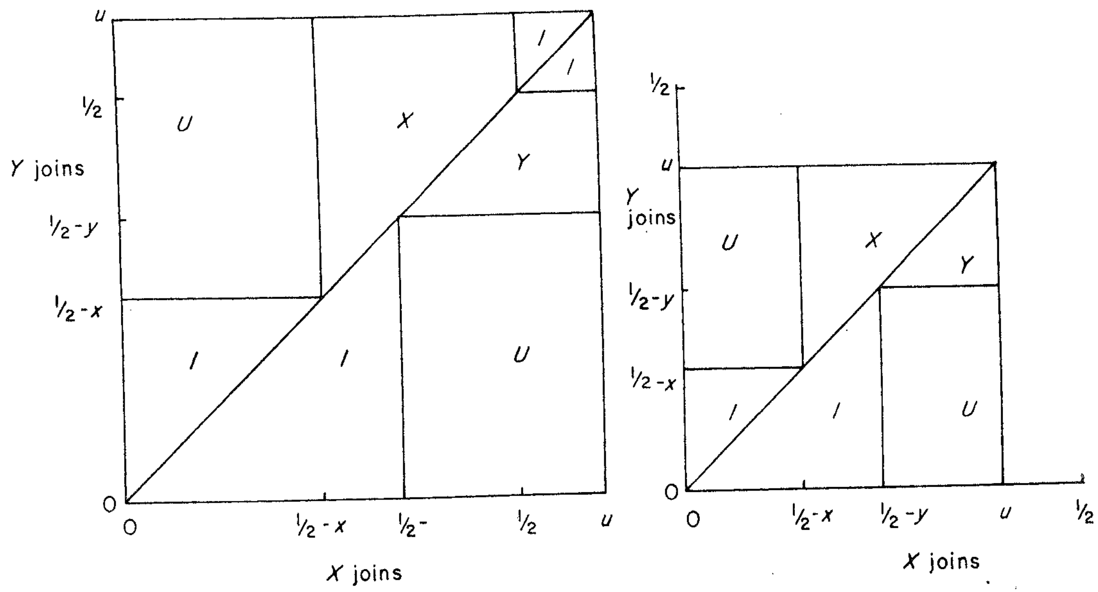
with power indexes

$$(0.039\ 0.377\ 0.377\ 0.034\ 0.034\ 0.034\ 0.034\ 0.034\ 0.034\ ).$$

She has committed 0.063 to E, more than 0.043 she had while uncommitted. A bandwagon effect for E occurs.

Example

Let x be the fraction of votes controlled by bloc X, y the fraction controlled by bloc Y, and $u = 1 - x - y$ the fraction held by an ocean of uncommitted voters. Assume majority rule and $x < \frac{1}{2}, y < \frac{1}{2}$.



(a) $x + y \leq 1/2$ $(u \geq 1/2)$. (b) $x + y \geq 1/2$ $(u \leq 1/2)$.

The label Y in the figure means player Y pivots; U means an uncommitted voter pivots; I means an illegal ordering.

- Ordering represented by points in the lower left-hand corners and the upper right-hand corner are ruled out since we require exactly one of X or Y to appear at or before the pivot. For example, considering the case $x + y \geq 1/2$, the left bottom triangle above the diagonal corresponds to Y being pivotal if X and Y do not oppose each other. Since this triangle corresponds to the scenario where X joins before Y and Y pivots, this should be ruled out as "illegal". Similarly, for the case $x + y \leq 1/2$, the box at the top right corner corresponds to the scenario where U pivots without the participation of either X or Y. The power index of X is found to be

$$\phi_X(x,y) = \begin{cases} \frac{x(1-x-2y)}{1-x-y-x^2-y^2}, & \text{if } x+y \leq \frac{1}{2} \\ \frac{\left(\frac{1}{2}-y\right)^2}{(1-x-y)^2+2\left(\frac{1}{2}-x\right)\left(\frac{1}{2}-y\right)}, & \text{if } x+y \geq \frac{1}{2} \end{cases}.$$

The formulas for ϕ_Y are symmetric to this.

The total power of the uncommitted voters:

$$\phi_U(x,y) = \begin{cases} \frac{(1-2x)(1-2y)}{1-x-y-x^2-y^2}, & \text{if } x+y \le \frac{1}{2}, \\ \\ \frac{(1-2x)(1-2y)}{(1-x-y)^2+2\left(\frac{1}{2}-x\right)\left(\frac{1}{2}-y\right)}, & \text{if } x+y \ge \frac{1}{2}. \end{cases}$$

- Consider a small bloc of uncommitted voters, comprising a fraction Δx of the total vote, considering whether to join X.

- If they do, the power increment they will contribute is $\phi_X(x+\Delta x, y) - \phi_X(x,y)$. If they remain uncommitted, as they are independent and sharing equal power, their total power will be $(\Delta x/u)\phi_U(x,y)$.

- They should join X if

$$\phi_X(x+\Delta x, y) - \phi_X(x,y) > \frac{\Delta x}{u}\phi_U(x,y).$$

Taking the limit $\Delta x \to 0$, we obtain

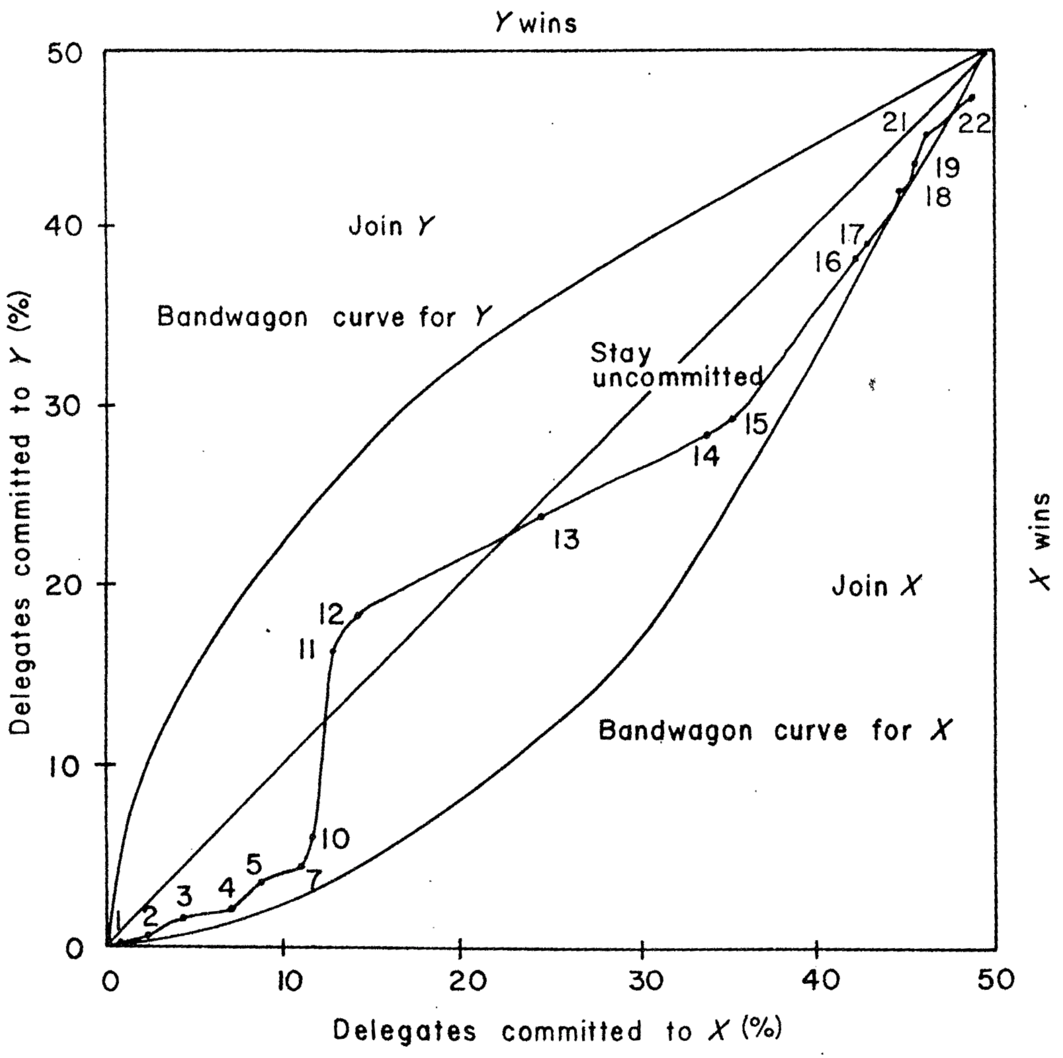
$$\frac{\partial \phi_X}{\partial x} > \frac{\phi_U}{u}.$$

- When $x + y \geq \frac{1}{2}$, the curve where $\frac{\partial \phi_X}{\partial x} = \frac{\phi_U}{u}$ is a straight line

$$\frac{1}{2} - y = a\left(\frac{1}{2} - x\right), \quad a \approx 1.78.$$

- When $x + y \leq \frac{1}{2}$, the bandwagon curves for X and Y are plotted in the figure. It is advantageous to join X when (x,y) lies below the bandwagon curve for X. When x and y are both small, x has to be several times larger than y in order that the uncommitted voters join X (see the bandwagon curve at the lower left corner).

1976-Race between Ford (player X) and Reagan (player Y) for the Republican nomination. The small numbers along the curve represent the number of weeks lapsed since the race started. By week 22, the uncommitted voters joined Ford to take advantage of the bandwagon effect.



2.6 Power distribution in weighted voting systems

Meeting the target

- Suppose the players are the countries of a federal union, one may wish the power of a player to be proportional to its population.

- Given the "population" vector $\boldsymbol{p}$ representing the ideal influence of the players, find the distribution of weights $(w_1, \cdots, w_n)$ and the quota of such that the sum of the differences between the target and the power is minimized.

- Let $d(x, y)$ be the distance between the power index and the target:

$$d(x, y) = \sum_{i=1}^{n} (x_i - y_i)^2, \quad \boldsymbol{x} \in \mathbb{R}^n \quad \text{and} \quad \boldsymbol{y} \in \mathbb{R}^n.$$

We write $d^{SS} =$ distance between the Shapley-Shubik index and the target, and similar definition for d^B.

3-player case

Given $w_1 \geq w_2 \geq w_3$, the four different weighted quota games for $n = 3$, with examples, are presented below.

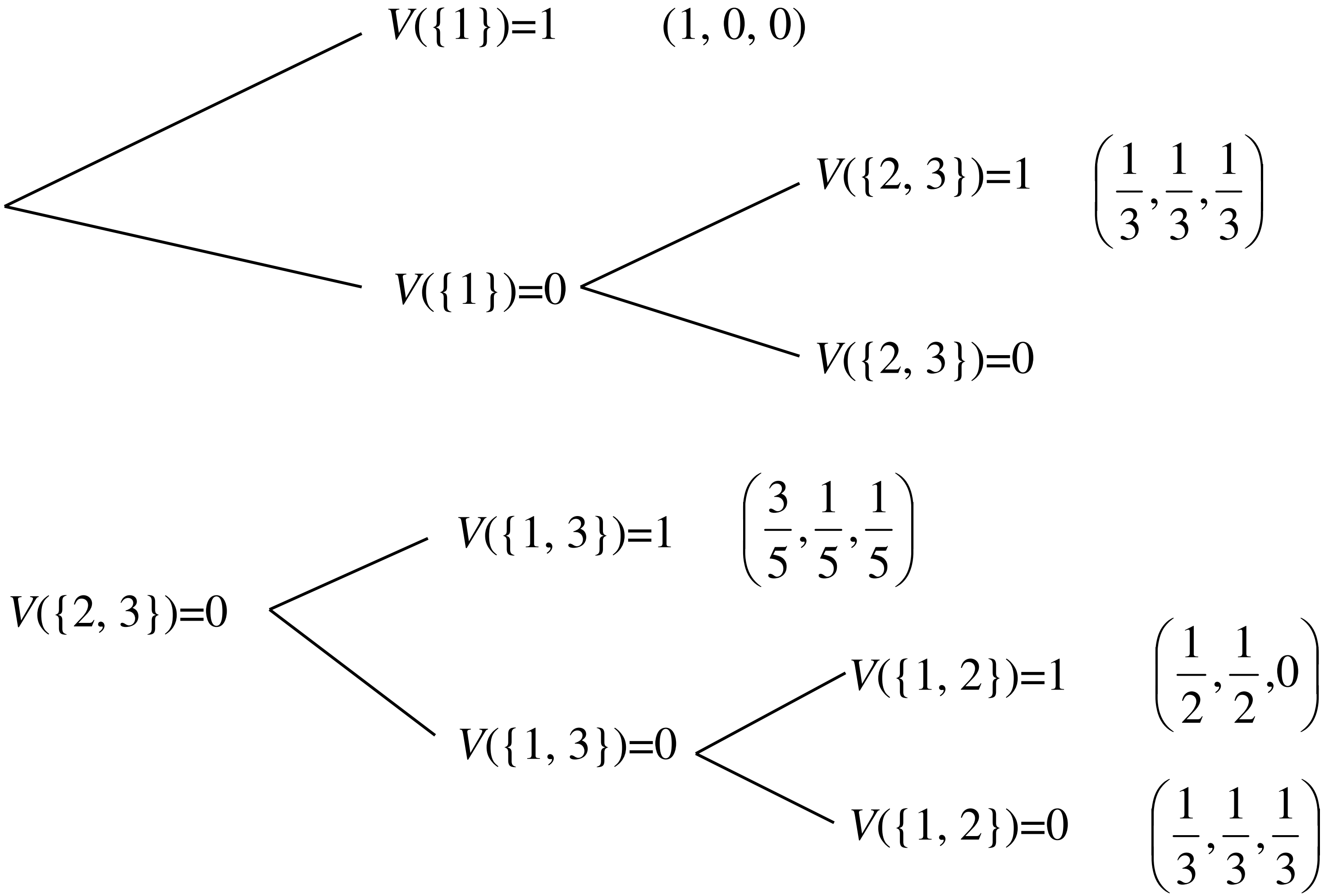
Conditions on w	Banzhaf	Shapley-Shubik	Example
$w_1 \geq q$	$\beta^1 = (1, 0, 0)$	$\phi^1 = (1, 0, 0)$	$G_1 = (2; 2, 0, 0)$
$w_1 + w_3 < q$ and $w_1 + w_2 \geq q$	$\beta^2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$\phi^2 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$G_2 = (2; 1, 1, 0)$
$w_1 < q$ and $w_2 + w_3 \geq q$	$\beta^3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\phi^3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$G_3 = (4; 2, 2, 2)$
$w_1 < q$ and $w_2 + w_3 < q$ and $w_1 + w_3 \geq q$	$\beta^4 = \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right)$	$\phi^4 = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$	$G_4 = (5; 4, 2, 2)$

For example, suppose $w_1 < q$ and $w_2 + w_3 \geq q$, then the winning coalitions are $\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$. All players then have equal power.

Notation

Let W be the set of all winning coalitions. A voting game is *proper* if $S \in W$ then $N \backslash S \notin W$, where N is the set of all players. If a coalition $S \in W$, then $V(S) = 1$; otherwise, $V(S) = 0$.

- It is impossible to have $V(\{2\}) = 1$ or $V(\{3\}) = 1$. If otherwise, $V(\{1\}) = 1$, then $V(\{2,3\}) = 0$. This leads to a contradiction.



Vector of power under Banzhaf index

1. $V(\{1\}) = 1$

   This means player 1 has all the power, so the vector of power is $(1, 0, 0)$.

2. $V(\{1\}) = 0$ and $V(\{2, 3\}) = 1$

   We have $V(\{1, 2, 3\}) = 1, V(\{1, 3\}) = 1$ and $V(\{1, 2\}) = 1$. There is no decisive player since $V(\{1\}) = V(\{2\}) = V(\{3\}) = 0$. However, in $\{1, 2\}, \{1, 3\}, \{2, 3\}$, every player is decisive. Therefore, the vector of power is $(1/3, 1/3, 1/3)$.

3. $V(\{1\}) = 0, V(\{2, 3\} = 0$ and $V(\{1, 3\}) = 1$.

   We have $V(\{1, 2, 3\}) = 1$ and $V(\{1, 2\}) = 1$.

   In $\{1, 2, 3\}$, only player 1 is decisive while in $\{1, 2\}, \{1, 3\}$, every player is decisive. Therefore, the vector of power is $(3/5, 1/5, 1/5)$.

4. $V(\{1\}) = 0, V(\{2,3\}) = 0, V(\{1,3\}) = 0, V(\{1,2\}) = 1$

   We have $V(\{1,2,3\}) = 1$. Only player 3 is not decisive while in $\{1,2\}$, every player is decisive. Therefore, the vector of power is $(1/2, 1/2, 0)$.

5. $V(\{1\}) = 0, V(\{2,3\}) = 0, V(\{1,3\}) = 0, V(\{1,2\}) = 0$

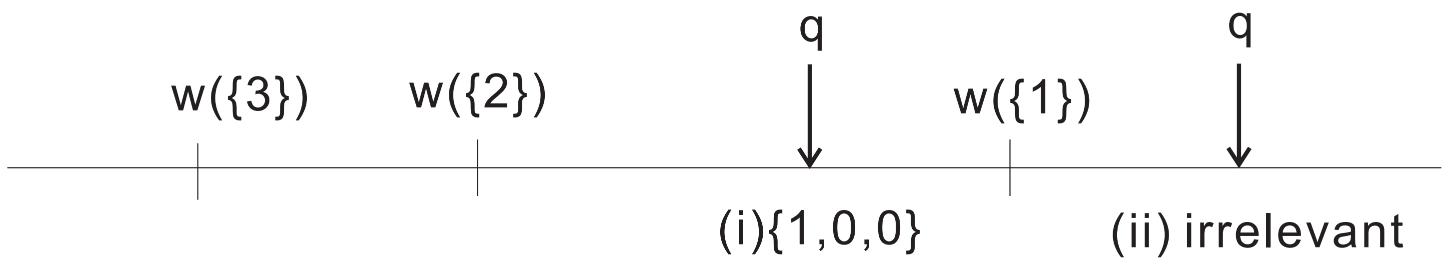
   We have $V(\{1,2,3\}) = 1$. In $\{1,2,3\}$, every player is decisive and the vector of power is $(1/3, 1/3, 1/3)$.

In summary, there are 5 possible cases but only 4 vectors of power. The Banzhaf indexes are:

$$(1/3, 1/3, 1/3), (3/5, 1/5, 1/5), (1/2, 1/2, 0) \text{ and } (1, 0, 0).$$

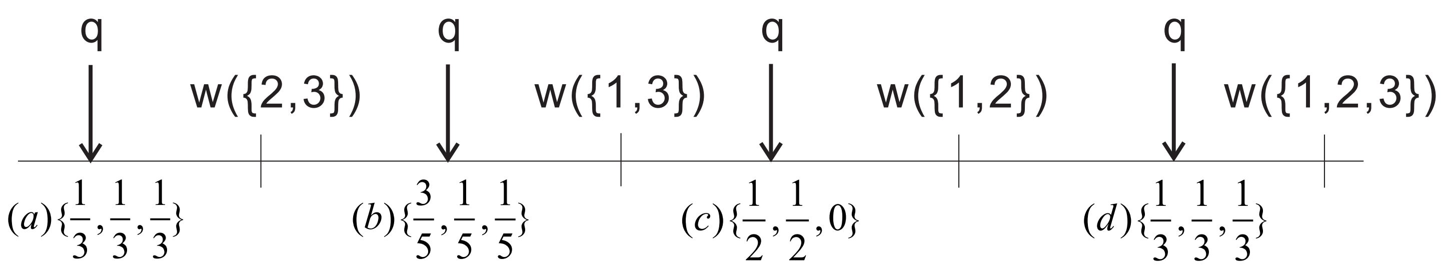
Let $w(S)$ denote the sum of weights in the coalition S.

First, we consider one-player coalitions



The quota q may fall (i) below $w(\{1\})$ and player 1 is the dictator; or (ii) above $w(\{1\})$, and as a result, there is no one-player winning coalition. We rule out the case where q falls below $w(\{2\})$. If otherwise, $\{2,3\}$ is also winning and its complement $\{1\}$ cannot be winning.

Next, we consider two-player and three-player coalitions



Remarks

It is necessary to refine the potential ordering of weights in coalitions according to $w(\{1\}) < w(\{2,3\})$ or otherwise. We avoid the discussion on the special case of $w(\{1\}) = w(\{2,3\})$.

1. Suppose $w(\{1\}) < w(\{2,3\})$, then $\{1\}$ can never be winning. Implicitly, we deduce that $w(\{1\}) < q$, and there will be no one-player winning coalition. We can perform the analysis solely based on the two-player and three-player coalitions.

2. Suppose $w(\{1\}) > w(\{2,3\})$, then $\{2,3\}$ can never be winning. We rule out the scenario where $q < w(\{2,3\})$ and need to include the new cases (i) $w(\{2,3\}) < q < w(\{1\})$ and (ii) $w(\{1\}) < q < w(\{2,3\})$.

   Case (i) gives the power distribution $\{1,0,0\}$ while case (ii) gives $\left\{\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right\}$.

Assuming $w(\{1\}) < w(\{2,3\})$, we examine the various outcomes when q increases gradually.

(a) $w(\{1\}) < q < w(\{2,3\})$
All two-player coalitions and three-player coalitions are winning, so the players have equal power.

(b) $w(\{2,3\}) < q < w(\{1,3\})$
Since $\{2,3\}$ is not winning, player 1 has higher power compared to players 2 and 3, so the power distribution is $\left\{\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right\}$.

(c) $w(\{1,3\}) < q < w(\{1,2\})$
Since $\{1,2\}$ is the only two-player winning coalitions, so player 3 is a dummy. This gives the power distribution $\left\{\frac{1}{2}, \frac{1}{2}, 0\right\}$.

(d) $w(\{1,2\}) < q < w(\{1,2,3\})$
All two-player coalitions are losing and only $\{1,2,3\}$ is the winning coalition. Therefore, the players have equal power.

What would happen when some of the voters have the same weight?

1. Suppose $w_1 = w_2 > w_3$, we rule out the case where player 1 is the dictator of the game. We consider the following cases:

   (a) $w_1 < q < w_1 + w_3$, we have the power distribution $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$. This is because player 3 is critical in $\{1,3\}$ and $\{2,3\}$, while player 1 is critical in $\{1,2\}$ and $\{1,3\}$ (similarly for player 2).

   (b) $w_1 + w_3 < q < 2w_1$, the power distribution is $\left\{\frac{1}{2}, \frac{1}{2}, 0\right\}$.

   (c) $2w_1 < q < 2w_1 + w_3$, the power distribution is $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$.

   Note that the power distribution $\left\{\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right\}$ is ruled out.

2. Suppose $w_1 > w_2 = w_3$, we rule out the power distribution $\left\{\frac{1}{2}, \frac{1}{2}, 0\right\}$. It is still possible for player 1 to be the dictator.

3. Suppose $w_1 = w_2 > w_3$, the 3 players are equal in power.

Shapely-Shubik index

There are 6 possible orders with 3 players

1. $V(\{1\}) = 1$

   Player 1 is the only pivotal, even if it arrives last in the coalition. The vector of power is $(1, 0, 0)$

2. $V(\{1\}) = 0$ and $V(\{2, 3\}) = 1$

   We have $V(\{1, 2, 3\}) = 1, V(\{1, 3\}) = 1$ and $V(\{1, 2\}) = 1$

   For each order, the player in the second position is pivotal, so the vector of power is $(1/3, 1/3, 1/3)$.

3. $V(\{1\}) = 0, V(\{2,3\}) = 0$ and $V(\{1,3\}) = 1$

   We have $V(\{1,2,3\}) = 1$ and $V(\{1,2\}) = 1$.

   When player 1 is not first in the order, it is always pivotal.

   When player 1 is first in the orders, the pivotal is the player which arrives second in the order. Therefore, the vector of power is $(2/3, 1/6, 1/6)$.

4. $V(\{1\}) = 0, V(\{2,3\}) = 0, V(\{1,3\}) = 0, V(\{1,2\}) = 1$

   We have $V(\{1,2,3\}) = 1$. When player 1 is first in the order, player 2 is pivotal. When player 2 is first in the order, player 1 is pivotal. In the orders 312 and 321, the player who arrives last is pivotal. Therefore, the vector of power is $(1/2, 1/2, 0)$.

5. $V(\{1\}) = 0, V(\{2,3\}) = 0, V(\{1,3\}) = 0, V(\{1,2\}) = 0$

   We have $V(\{1,2,3\}) = 1$. It is always the player who arrives last in the order the pivotal. The vector of power is $(1/3, 1/3, 1/3)$.

In summary, there are 4 vectors of power using the Shapley-Shubik index:

$$(1/3, 1/3, 1/3), (2/3, 1/6, 1/6), (1/2, 1/2, 0) \text{ and } (1, 0, 0).$$

Remark

There has not existed a general formula to determine all the possible vectors for a given n and a given power index.

Target power distribution

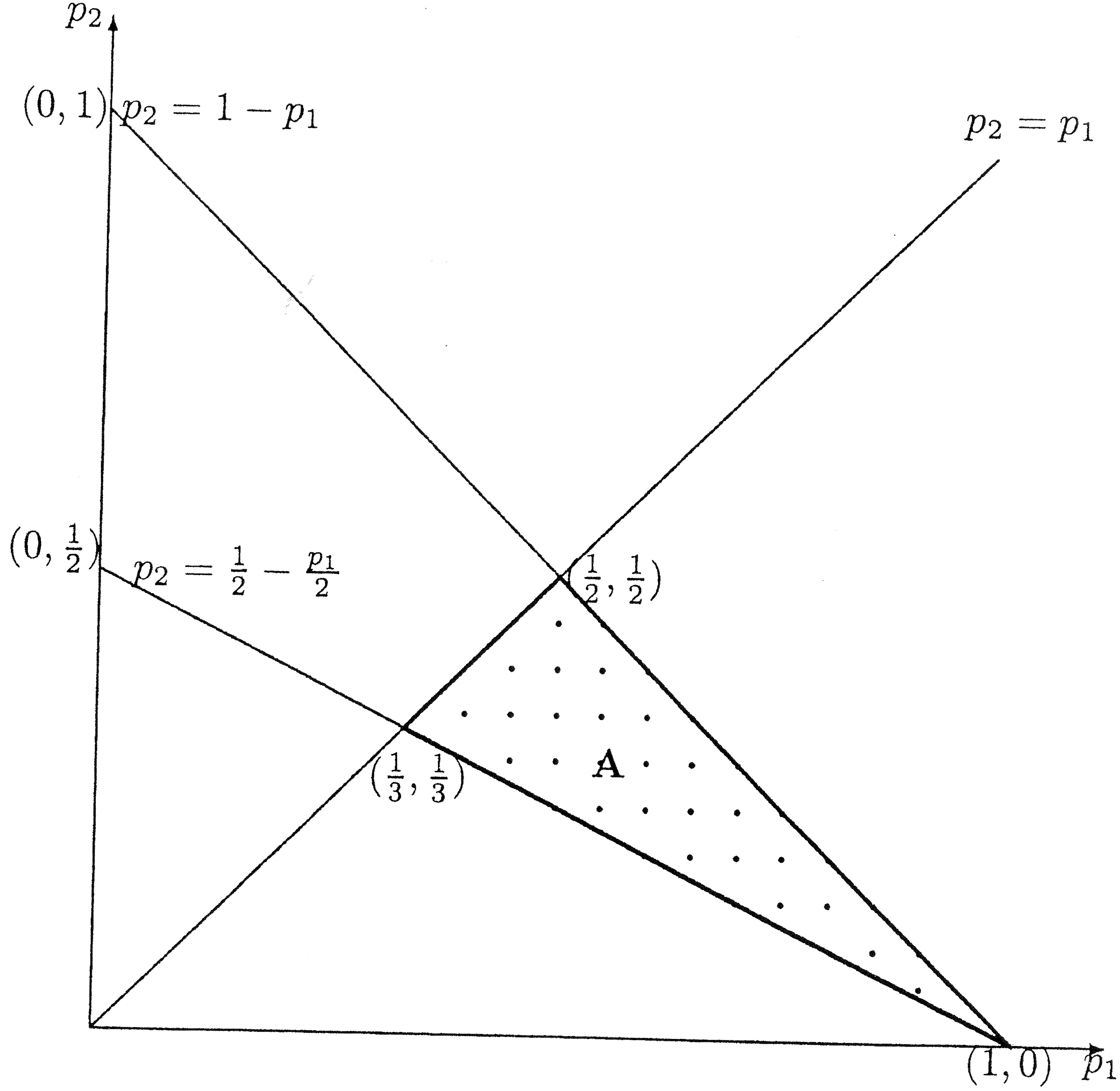
Let p_i denote the power index of player i, $i = 1,2,3$. Assume that $p_2 + p_2 + p_3 = 1$, we can represent all the possible targets in a simplex. Further, $p_1 \geq p_2 \geq p_3$ which implies $p_2 \geq \frac{1}{2} - \frac{p_1}{2}$ since $2p_2 + p_1 \geq 1$.

In summary, the simplex that contains feasible power distributions satisfies

$$\begin{aligned} p_1 &\geq p_2 \\ p_2 &\geq \frac{1}{2} - \frac{p_1}{2} \\ p_2 &\leq 1 - p_1. \end{aligned}$$

The area of the admissible region A is equal to

$$\frac{1}{2}\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{3} & 1 \\ 1 & 0 & 1 \end{vmatrix} = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{2} - \frac{1}{3} - \frac{1}{6}\right) = \frac{1}{12} \approx 0.0833.$$



The possible target vectors

If we calculate the distances d^B and d^{SS} between a vector of power and a target p, we obtain easily:

- $d^B = d^{SS} = (p_1 - \frac{1}{3})^2 + (p_2 - \frac{1}{3})^2 + (p_3 - \frac{1}{3})^2$
  $= \sum_{i=1}^{3} p_i^2 - \frac{1}{3}$ for the vector $(1/3, 1/3, 1/3)$.

- $d^B = \sum_{i=1}^{3} p_i^2 + \frac{1}{25} - \frac{4p_1}{5}$ for the vector $(3/5, 1/5, 1/5)$.

- $d^{SS} = \sum_{i=1}^{3} p_i^2 + \frac{1}{6} - p_1$ for the vector $(2/3, 1/6, 1/6)$.

- $d^B = d^{SS} = \sum_{i=1}^{3} p_i^2 + \frac{1}{2} - p_1 - p_2$ for the vector $(1/2, 1/2, 0)$.

- $d^B = d^{SS} = \sum_{i=1}^{3} p_i^2 + 1 - 2p_1$ for the vector $(1, 0, 0)$.

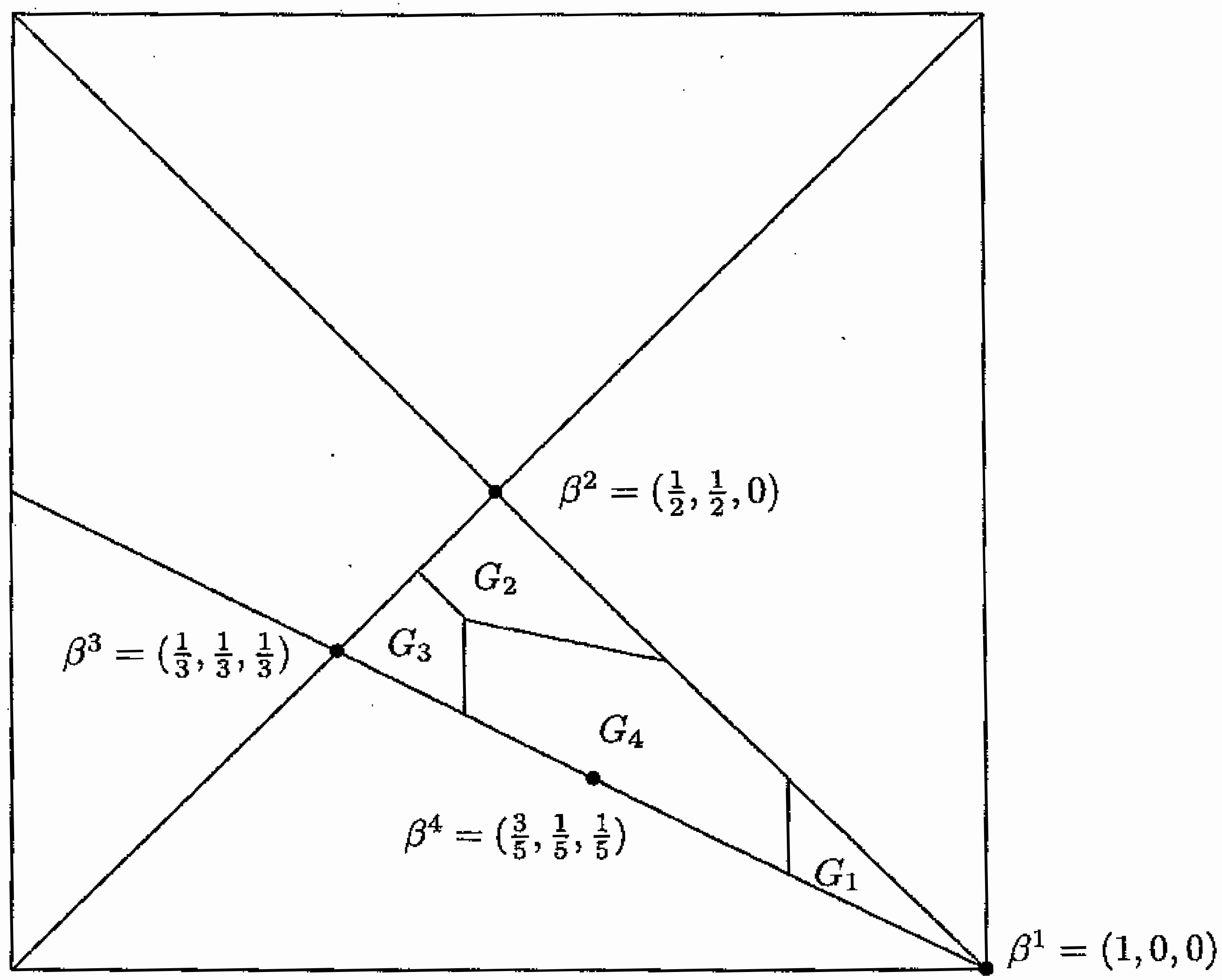
We can compare the distances and obtain for the Banzhaf index:

- The vector of power $\beta^1 = (1,0,0)$ and G_1 minimizes d^B if $p_1 > 5/6$. The result is obtained by comparing $d^B = \sum_{i=1}^{3} p_i^2 + \frac{1}{6} - p_1$ for $(3/5, 1/5, 1/5)$ and $d^B = \sum_{i=1}^{3} p_i^2 + 1 - 2p_1$ for $(1,0,0)$.

- The vector of power $\beta^2 = (1/2, 1/2, 0)$ and G_2 minimizes d^B if $p_2 > 1/3$ and $p_2 > 5/6 - p_1$.

- The vector of power $\beta^3 = (1/3, 1/3, 1/3)$ and G_3 minimizes d^B if $p_2 < 1/2$ and $p_2 < 5/6 - p_1$.

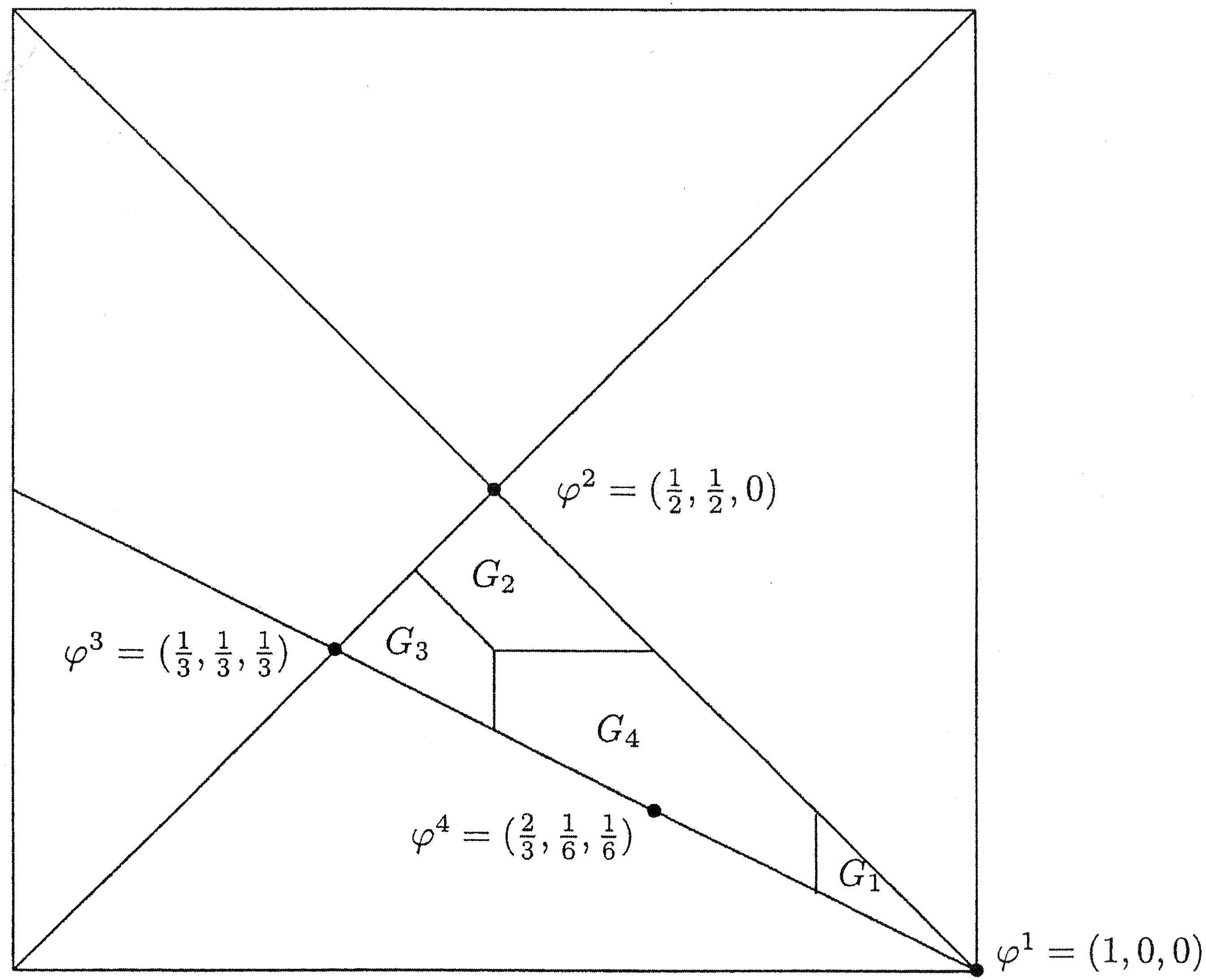
- The vector of power $\beta^4 = (3/5, 1/5, 1/5)$ and G_4 minimizes d^B otherwise. The 3 boundaries are given by: $1/25 - 4p_1/5 < -1/3$, $1/25 - 4p_1/5 < 1/2 - p_1 - p_2$ and $1/25 - 4p_1/5 < 1 - 2p_1$; that is if $7/15 < p_1 < 4/5$ and $p_1 + 5p_2 < 23/10$.



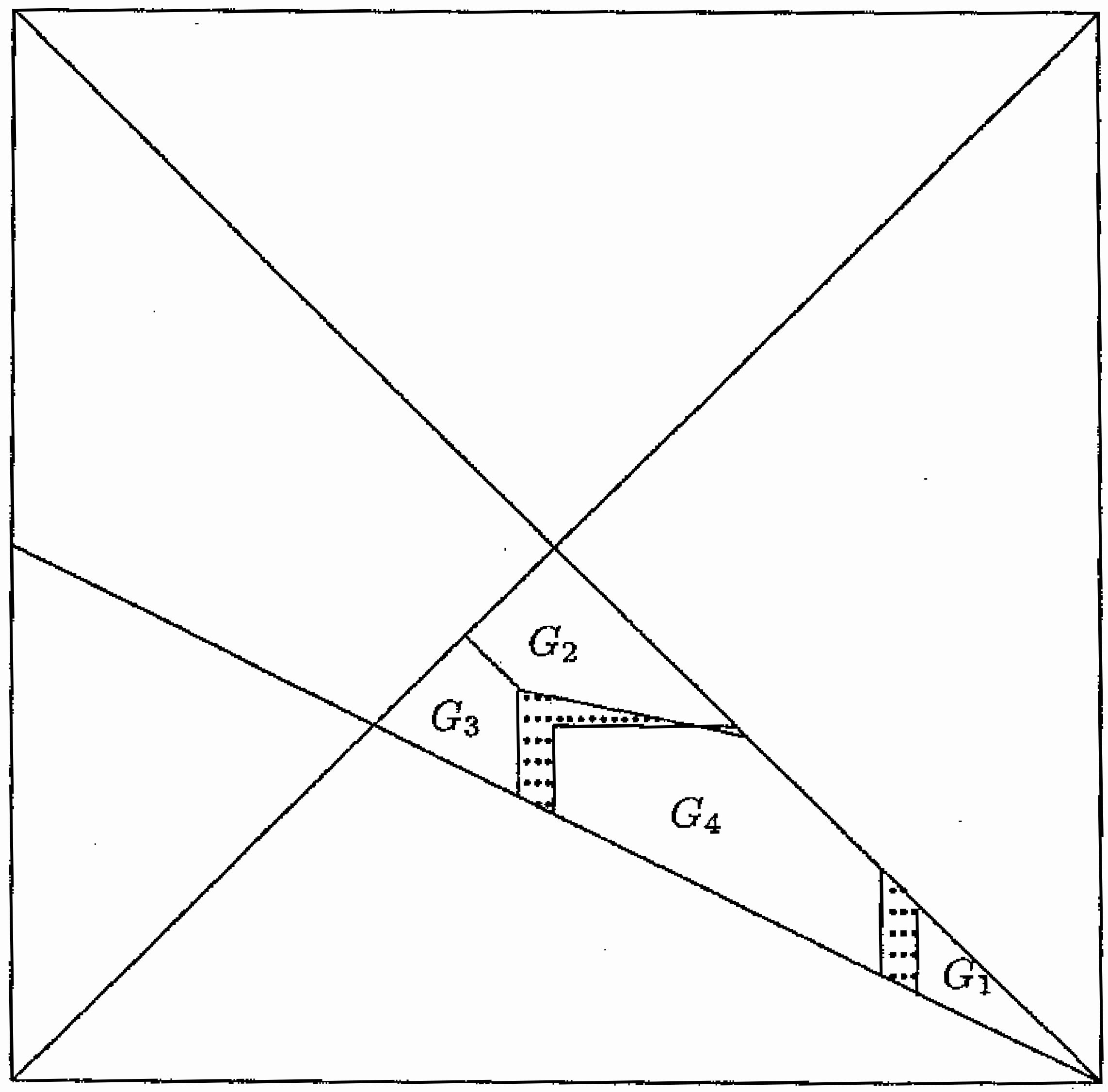
The different closest games for the Banzhaf index for $n = 3$

The same reasoning for the Shapley-Shubik index enables us to define the following domains:

- The vector of power $\phi^1 = (1, 0, 0)$ and G_1 minimizes d^{SS} if $p_1 > 4/5$.
- The vector of power $\phi^2 = (1/2, 1/2, 0)$ and G_2 minimizes d^{SS} if $p_2 > 23/50 - p_1/5$ and $p_2 > 5/6 - p_1$.
- The vector of power $\phi^3 = (1/3, 1/3, 1/3)$ and G_3 minimizes d^{SS} if $p_2 < 7/15$ and $p_2 > 5/6 - p_1$.
- The vector of power $\phi^4 = (2/3, 1/6, 1/6)$ and G_4 minimizes d^{SS} otherwise, that is if $7/5 < p_1 < 4/5$ and $p_1 + 5p_2 < 23/10$.



The different closest games for the Shapley-Shubik index for $n = 3$



Different Shapley-Shubik and Banzhaf inverse games for $n = 3$

Example

For a voting game with 3 players, suppose the target power distribution of the 3 players is

$$\boldsymbol{p}_{\text{tar}} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right).$$

Based on Shapley-Shubik indexes, find the weighted voting system whose power distribution is closest to the given target distribution.

Solution

Recall that there are only 4 possible power distributions in a 3-player voting game under the Shapley-Shubik indexes: $(1, 0, 0)$, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$.

$$\text{The distance between } \boldsymbol{p}_{\text{tar}} \text{ and } (1,0,0) = \left(\frac{1}{2}-1\right)^2+\left(\frac{1}{4}-0\right)^2+\left(\frac{1}{4}-0\right)^2 = \frac{3}{8}.$$

$$\text{The distance between } \boldsymbol{p}_{\text{tar}} \text{ and } \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) = \left(\frac{1}{2}-\frac{1}{3}\right)^2+\left(\frac{1}{4}-\frac{1}{3}\right)^2+\left(\frac{1}{4}-\frac{1}{3}\right)^2 = \frac{1}{24}.$$

$$\text{The distance between } \boldsymbol{p}_{\text{tar}} \text{ and } \left(\frac{1}{2},\frac{1}{2},0\right) = \left(\frac{1}{2}-\frac{1}{2}\right)^2+\left(\frac{1}{4}-\frac{1}{2}\right)^2+\left(\frac{1}{4}-0\right)^2 = \frac{1}{16}.$$

$$\text{The distance between } \boldsymbol{p}_{\text{tar}} \text{ and } \left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right) = \left(\frac{1}{2}-\frac{2}{3}\right)^2+\left(\frac{1}{4}-\frac{1}{6}\right)^2+\left(\frac{1}{4}-\frac{1}{6}\right)^2 = \frac{1}{24}.$$

The closest distance is $\frac{1}{24}$, achieved by either choosing $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ or $\left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right)$. The corresponding power distribution can be achieved by the respective weighted voting vector $[2;1,1,1]$ or $[5;3,2,2]$.

Four-player weighted voting systems

Editorial Committee

Editor-in-chief has 3 votes; Managing Editor has 2 votes; News Editor and Feature Editor, each has 1 vote. Total votes $= 7$, quota $= 4$.

- Given the weighted voting vector $[4; 3, 2, 1, 1]$, we aim to achieve the power distribution that matches with hierarchy of influence, where

$$P_{Chief} > P_{Man} > P_{News} = P_{Feat}.$$

- Take the 4 editors as A,B,C,D for convenience: $S_A^{(1)} = \{B, C\}$, $S_A^{(2)} = \{B, D\}$, $S_A^{(3)} = \{C, D\}$, $S_A^{(4)} = \{B\}$, $S_A^{(5)} = \{C\}$, $S_A^{(6)} = \{D\}$; $S_B^{(1)} = \{A\}$, $S_B^{(2)} = \{C, D\}$; $S_C^{(1)} = \{A\}$, $S_C^{(2)} = \{B, D\}$; etc.

  The corresponding Banzhaf indexes are

$$B_{Chief} = \frac{1}{2}, \quad B_{Man} = \frac{1}{6}, \quad B_{News} = \frac{1}{6}, \quad B_{Feat} = \frac{1}{6}.$$

- Can we design the weights so that the desired hierarchy of influence is achieved?

  Consider a weighted voting system of size 4 as represented by

$$[q; w_1, w_2, w_3, w_4]$$

  with $w_1 \geq w_2 \geq w_3 \geq w_4$, what are the possible power distributions?

  We assume

$$\frac{w_1 + w_2 + w_3 + w_4}{2} < q \leq w_1 + w_2 + w_3 + w_4.$$

  Furthermore, we assume

(a) $w_i < q$ for each i; that is, none of the players form a winning coalition on herself.

(b) $\sum_{j \neq i} w_j \geq q$; that is, none of the players has veto power.

Republic of PAIN (Pennsylvania and its Neighbors)

Total of 90 votes; quota $= 46$

Apportionment according to population:

New York	38
Pennsylvania	25
Ohio	23
West Virginia	4

Any two of New York, Pennsylvania and Ohio can pass a bill. West Virginia is a dummy. The Banzhaf indexes of the 4 states are found to be

$$B_{NY} = B_{Pen} = B_{Ohio} = \frac{1}{3} \quad \text{and} \quad B_{WV} = 0.$$

Any weighted voting system of the form

$$[2m; m, m, m, 1], \quad m \geq 2$$

would yield the same power distribution.

Theorem

In any 4-player weighted voting system with no veto power, there are only 5 possible power distributions:

(a) $B(P_i) = \frac{1}{4}$ for every i.

(b) $B(P_1) = B(P_2) = \frac{1}{3}, B(P_3) = B(P_4) = \frac{1}{6}$.

(c) $B(P_1) = \frac{5}{12}, B(P_2) = B(P_3) = \frac{1}{4}$ and $B(P_4) = \frac{1}{12}$.

(d) $B(P_1) = \frac{1}{2}$ and $B(P_2) = B(P_3) = B(P_4) = \frac{1}{6}$.

(e) $B(P_1) = B(P_2) = B(P_3) = \frac{1}{3}$ and $B(P_4) = 0$.

When we impose $w_3 = w_4$, then cases (c) and (e) are impossible.

Remarks

1. It is impossible to achieve

$$B_1 > B_2 > B_3 = B_4$$

   as required by the Editorial Committee.

2. The best weighted voting system for PAIN is case (c), which can be achieved by $[13; 8, 6, 6, 1]$.

3. For any 4-player weighted voting system with no player having veto power, there are at most one dummy player in the system. One can show easily that if there exist more than one dummy in the system, then at least one player will have veto power.

Proof of the Theorem

1. In the absence of veto power, any 4-player coalition is a winning coalition that does not yield any critical instances. Suppose any one of the players is critical, then that player would have veto power.

2. All the 3-player coalitions are winning coalitions. Otherwise, the missing player in that 3-player coalition has veto power. These 4 three-player coalitions are: $\{P_2, P_3, P_4\}$, $\{P_1, P_3, P_4\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_2, P_3\}$. Since all 3-player coalitions are winning, any single-player coalition (complement of the respective 3-player coalition) is losing.

3. In any 2-player winning coalition, both players must be critical since we do not permit single-player winning coalitions.

The possible 2-player winning coalitions are

$$\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_4\}.$$

However, if $\{P_1, P_2\}$ wins then $\{P_3, P_4\}$ must lose;
similarly, if $\{P_1, P_3\}$ wins then $\{P_2, P_4\}$ must lose;
lastly, if $\{P_1, P_4\}$ wins then $\{P_2, P_3\}$ must lose; or otherwise.

Therefore, there are two cases of having three 2-player winning coalitions:

$$\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\} \text{ or } \{P_1, P_2\}, \{P_1, P_3\}, \{P_2, P_3\}.$$

Various possible number of 2-player winning coalitions

The possible number of two-player winning coalitions can be 0,1,2,3. Each of the first 3 cases of having 0,1 or 2 two-player coalition generates one power distribution. The 2 remaining power distributions are generated when the number of two-player coalitions is 3.

Case (i) There is no 2-player winning coalition.

Every player is critical in each of the four 3-player winning coalitions. This is because dropping any one player turns a winning three-player winning coalition into losing (as there is no two-player winning coalition). Total number of critical instances = 12, and we have equal share of power among the 4 players.

$$B(P_1) = B(P_2) = B(P_3) = B(P_4) = \frac{1}{4}.$$

Case (ii) There is only one 2-player winning coalition, which must be $\{P_1, P_2\}$. Therefore, P_1 and P_2 are both critical in this coalition and also in the coalitions $\{P_1, P_2, P_3\}$ and $\{P_1, P_2, P_4\}$.

- On the other hand, P_3 and P_4 are not critical in $\{P_1, P_2, P_3\}$ and $\{P_1, P_2, P_4\}$ but are critical in $\{P_1, P_3, P_4\}$ and $\{P_2, P_3, P_4\}$.

- Also, P_1 is critical in $\{P_1, P_3, P_4\}$ and P_2 is critical in $\{P_2, P_3, P_4\}$.

- As a summary, we have

  4 critical instances for each of P_1 and P_2

  2 critical instances for each of P_3 and P_4.

  Therefore, $B(P_1) = B(P_2) = \frac{1}{3}, B(P_3) = B(P_4) = \frac{1}{6}$.

Case (iii) If there are two winning 2-player coalitions, then they must be $\{P_1, P_2\}$ and $\{P_1, P_3\}$. This can only occur if $w_3 > w_4$. Otherwise, if $w_3 = w_4$, then $\{P_1, P_3\}$ winning would imply $\{P_1, P_4\}$ winning as well.

- P_1 is critical in each of the three 3-player coalitions containing P_1, yielding 5 critical instances for P_1.
- P_2 is critical in $\{P_1, P_2, P_4\}, \{P_2, P_3, P_4\}$ and $\{P_1, P_2\}$.
- P_3 is critical in $\{P_1, P_3, P_4\}, \{P_2, P_3, P_4\}$ and $\{P_1, P_3\}$.
- P_4 is critical in $\{P_2, P_3, P_4\}$.

There are 12 critical instances in total. Therefore,

$$B(P_1) = \frac{5}{12}, B(P_2) = B(P_3) = \frac{1}{4} \quad \text{and} \quad B(P_4) = \frac{1}{12}.$$

Case (iv) There are two cases involving three winning 2-player coalitions.

(a) $\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}$ winning

- Each of P_2, P_3, P_4 yields one critical instance in these 2-player winning coalitions while P_1 yields three critical instances.
- P_1 is critical in the coalitions $\{P_1, P_2, P_3\}, \{P_1, P_2, P_4\}, \{P_1, P_3, P_4\}$ while P_2, P_3 and P_4 are critical in $\{P_2, P_3, P_4\}$. Therefore,

$$B(P_1) = \frac{1}{2}, B(P_2) = B(P_3) = B(P_4) = \frac{1}{6}.$$

(b) We may have $\{P_1, P_2\}, \{P_1, P_3\}$ and $\{P_2, P_3\}$ winning. This occurs only when $w_3 > w_4$.

- Now P_4 is never a critical player since removing P_4 from any 3-player coalition leaves one of these three winning 2-player coalitions. Therefore, $B(P_4) = 0$.
- Removing any player from $\{P_1, P_2, P_3\}$ still leaves a winning coalition, but in the other three 3-player coalitions containing P_4, the other two players are critical.
- Here, we have a total of 12 critical instance, with each of P_1, P_2, P_3 being critical 4 times. As a result, we obtain

$$B(P_1) = B(P_2) = B(P_3) = \frac{1}{3}.$$

Remarks

1. With a complete enumeration of the power distributions feasible for weighted voting systems of size n, can one efficiently generate a complete list of feasible power distributions for size $n+1$ weighted voting system?

2. If a certain power distribution is desired, can one efficiently construct a weighted voting system that comes closest to the ideals?

Strict hierarchy of power $B(P_1) > B(P_2) > \cdots > B(P_n)$.

1. In the case of 4-player weighted voting system, there always exist at least two players with the same Banzhaf power index.

2. When $n = 5$, we have $[9; 5, 4, 3, 2, 1]$ which yields Banzhaf power distribution:$\left(\frac{9}{25}, \frac{7}{25}, \frac{5}{25}, \frac{3}{25}, \frac{1}{25}\right)$. Note that $[8; 5, 4, 3, 2, 1]$ does not induce the above power distribution. This is the only power distribution, of the 35 possibilities in the 5-player case, with strict hierarchy of power.

3. When $n = 6, [11; 6, 5, 4, 3, 2, 1]$ yields $\left(\frac{9}{28}, \frac{7}{28}, \frac{5}{28}, \frac{3}{28}, \frac{3}{28}, \frac{1}{28}\right)$. However, $[15; 9, 7, 4, 3, 2, 1]$ yields the power distribution with strict hierarchy

$$\left(\frac{5}{12}, \frac{3}{16}, \frac{1}{6}, \frac{1}{8}, \frac{1}{16}, \frac{1}{24}\right)$$

as desired.

2.7 Incomparability and desirability

We always consider *monotone* yes-no voting system — winning coalitions remain winning if new voters join them.

Let x and y be two players, how to formalize the following intuitive notions:

> "x and y have equal power"
> "x and y have the same amount of influence"
> "x and y are equally desirable in terms of the formation of a winning coalition"

The bottom line is to see whether x and y are both critical to turn the coalition Z from losing to winning upon joining the coalition.

Definition

Suppose x and y are two voters in a yes–no voting system. Then we shall say that x and y are equally desirable (or, the desirability of x and y is equal, or the same), denoted $x \approx y$, if and only if the following holds:

For every coalition Z containing neither x nor y,
the result of x joining Z is a winning coalition
if and only if
the result of y joining Z is a winning coalition,

We say: "x and y are equivalent" when $x \approx y$. In other words, x and y are equally desirable with reference to the formation of a winning coalition.

Example

Consider $[51; 1, 49, 50]$ with 3 players A, B and C. The winning coalitions are

$$\{A, C\}, \{B, C\} \quad \text{and} \quad \{A, B, C\}.$$

1. $A \approx B$

   Test for all coalitions that do not contain A and B. The only coalitions containing neither A nor B are the empty coalition (Z_1) and the coalition consisting of C alone (Z_2). The result of A joining Z_1 is the same as the result of B joining Z_1 (a losing coalition). The result of A joining Z_2 is the same as the result of B joining Z_2 (a winning coalition).

2. A and C are not equivalent

   Neither belongs to $Z = \{B\}$, but A joining Z yields $\{A, B\}$ which is losing while C joining Z yields $\{B, C\}$ which is winning with 51 votes.

Definition ‘

For two voters x and y in a yes–no voting system, we say that the *desirability of x and y is incomparable* (不可比較), denoted by

$$x|y,$$

if and only if there are coalitions Z and Z', neither one of which contains x or y, such that the following hold:

1. the result of x joining Z is a winning coalition, but the result of y joining Z is a losing coalition, and
2. the result of y joining Z' is a winning coalition, but the result of x joining Z' is a losing coalition.

x is critical but not y for coalition Z while y is critical but not x for another coalition Z'.

Example

In the US federal system, a House Representative and a Senator are incomparable. The Vice President and a Senator are not incomparable. Since the Vice President is the tie breaker, it is not possible that a coalition with the Vice President is critical while with a senator is not.

Question Which yes–no voting systems will have incomparable voters?

Proposition

For any yes–no voting system, the following are equivalent:

1. There exist voters x and y whose desirability is incomparable.

2. The system fails to be swap robust.

Proof

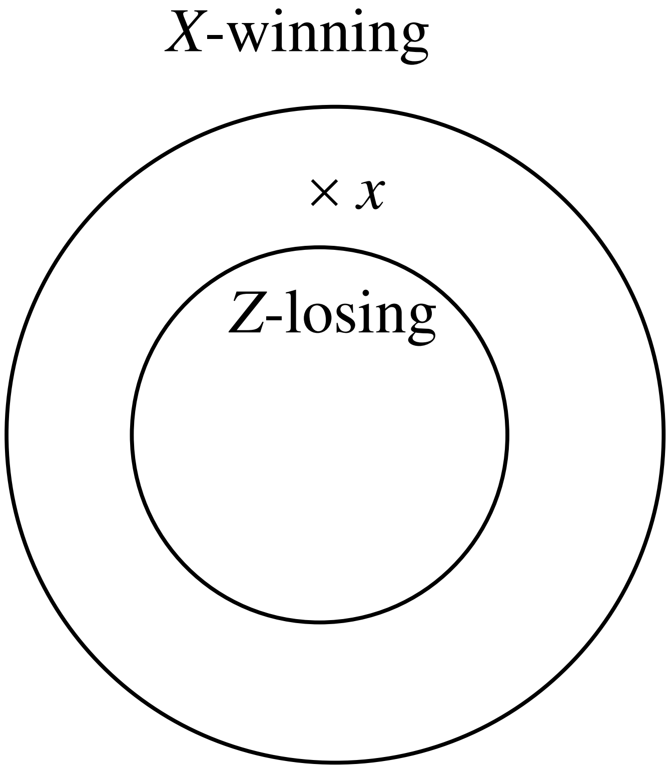
$1 \Rightarrow 2$

- Assume that the desirability of x and y is incomparable, and let Z and Z' be coalitions such that:

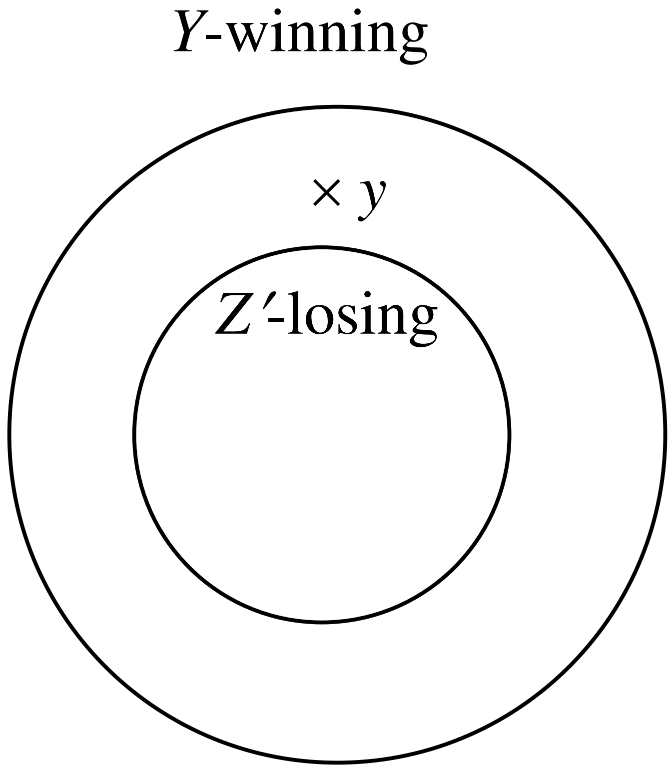
Z with x added is winning;
Z with y added is losing;
Z' with y added is winning; and
Z' with x added is losing.

- To see that the system is not swap robust, let X be the result of adding x to the coalition Z, and let Y be the result of adding y to the coalition Z'. Both X and Y are winning, but the one-for-one swap of x for y renders both coalitions losing.

Both Z and Z' do not contain x and y.



$Z \cup \{y\}$ remains to be losing



$Z' \cup \{x\}$ remains to be losing.

$2 \Rightarrow 1$

Assume that the system is not swap robust. Then we can choose winning coalitions X and Y with x in X but not in Y, and y in Y but not in X, such that both coalitions become losing if x is swapped for y. Let Z be the result of deleting x from the coalition X, and let Z' be the result of deleting y from the coalition Y. Then

Z with x added is X, and this is winning;
Z with y added is losing;
Z' with y added is Y, and this is winning; and
Z' with x added is losing.

This shows that the desirability of x and y is incomparable and completes the proof.

Corollary In a weighted voting system, we do not have voters whose desirability is incomparable. Say, suppose x's weight is heavier than y, while it may be possible that in one coalition x turns a coalition into winning but y does not, but it is not possible that the situation reverses in another coalition.

Proposition

Suppose the two players A and B are equally desirable in a yes-no voting system, then their Shapley-Shubik index are the same.

Proof

We compare the number of pivotal orderings of A and B. For any ordering that A pivots, we have the following two possibilities:

(i) B enters later than the pivotal position held by A

Let Z be the collection of players that have entered before A. Obviously, Z is losing and it does not contain A and B. Since $Z \cup \{A\}$ is winning, A and B are equally desirable, so $Z \cup \{B\}$ is also winning. Since Z is a losing coalition, so B is pivotal in the same ordering with A being swapped by B.

(ii) B has entered prior to the pivotal position held by A

Let Z' be the collection of players that have entered before A but excluding B in a given A-pivoted ordering. Note that $Z' \cup \{B\}$ is losing and so does $Z' \cup \{A\}$ since A and B are equally desirable. However, $Z' \cup \{B\} \cup \{A\}$ is winning, so B is pivotal in the ordering obtained by swapping A and B in this A-pivoted ordering.

In both cases, we observe that all A-pivoted orderings become B-pivoted orderings once A and B are swapped in position. Hence, A and B have the same number of pivotal orderings. As a result, $\phi_A = \phi_B$.

Remarks

1. Similar argument can be used to show that the Banzhaf indexes are the same for two players that are equally desirable in a yes-no voting game.

2. The converse statement is not true in general. Consider a voting system with 4 players A_1, A_2, A_3 and A_4, where the passage of a bill requires the support of (i) at least one of A_1 and A_2, (ii) at least one of A_3 and A_4. It is easy to show that A_1 and A_3 are incomparable since $\{A_2\} \cup \{A_1\}$ loses but $\{A_2\} \cup \{A_3\}$ wins. However, the 4 players share equal power, where

$$\phi_{A_1} = \phi_{A_2} = \phi_{A_3} = \phi_{A_4} = \frac{1}{4}.$$

Proposition

The relation of equal desirability is an equivalence relation on the set of voters in a yes–no voting system. That is, the following statements hold:

1. The relation is reflexive: if $x = y$ (that is, if x and y are literally the same voter), then x and y are equally desirable.

2. The relation is symmetric: if x and y are equally desirable, then y and x are equally desirable.

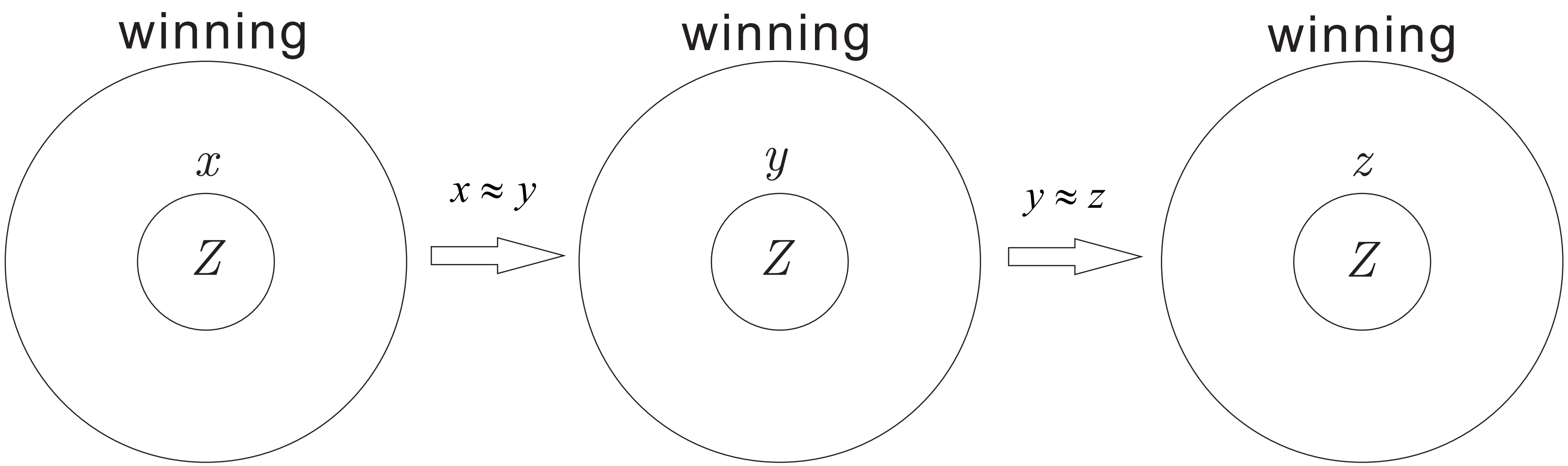
3. The relation is transitive: if x and y are equally desirable and y and z are equally desirable, then x and z are equally desirable.

Proof (transitivity)

Assume that Z is an arbitrary coalition containing neither x nor z. We must show that the result of x joining Z is a winning coalition if and only if the result of z joining Z is a winning coalition.

(i) $y \notin Z$

- Since $x \approx y$ and neither x nor y belongs to Z, the result of x joining Z is a winning coalition if and only if the result of y joining Z is a winning coalition.
- Since $y \approx z$ and neither y nor z belongs to Z, the result of y joining Z is a winning coalition if and only if the result of z joining Z is a winning coalition.
- The result of x joining Z is a winning coalition if and only if the result of z joining Z is a winning coalition, as desired.



(ii) $z \in Z$

- Let A denote the coalition resulting from y leaving Z, so $Z = A \cup \{y\}$. Assume that $Z \cup \{x\}$ is a winning coalition. We want to show that $Z \cup \{z\}$ is also a winning coalition. Now,

$$Z \cup \{x\} = A \cup \{x\} \cup \{y\}.$$

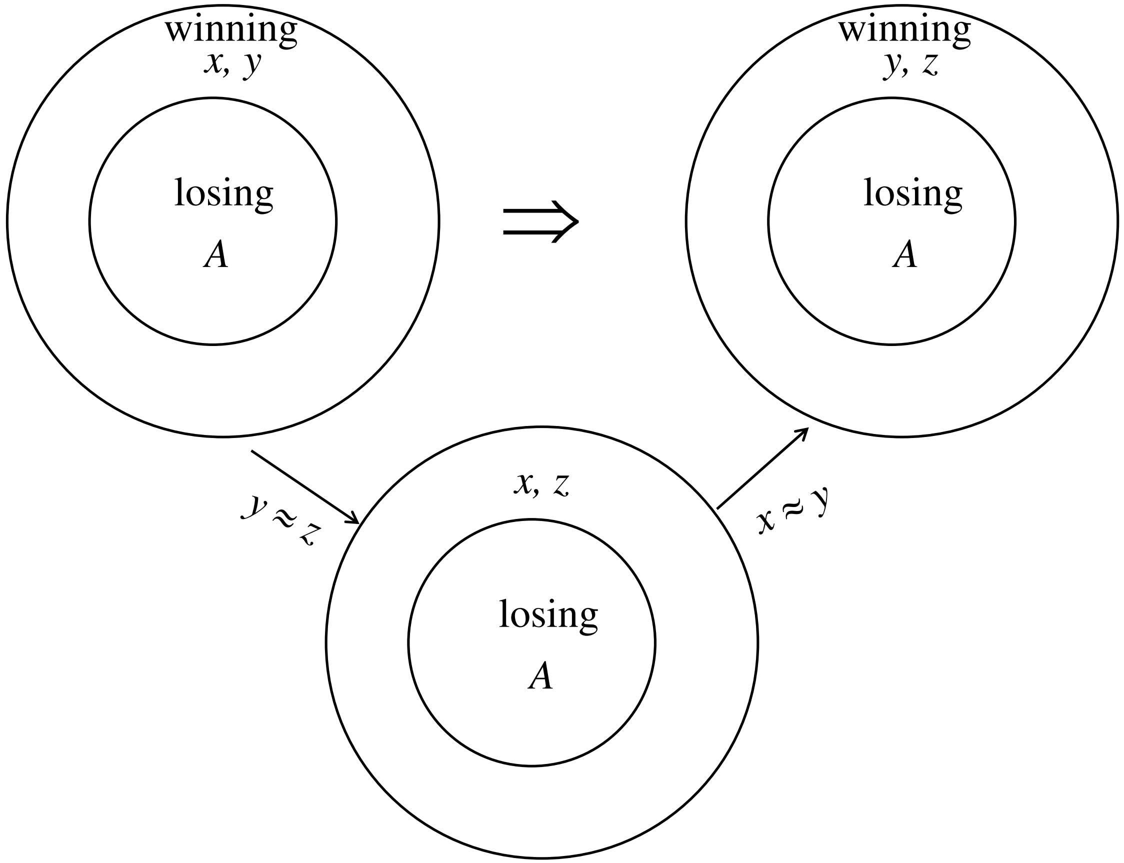
- Let $Z' = A \cup \{x\}$. Thus $Z' \cup \{y\}$ is a winning coalition. Since $y \approx z$ and neither y nor z belongs to Z', we know that $Z' \cup \{z\}$ is also a winning coalition. But $Z' \cup \{z\} = A \cup \{z\} \cup \{x\}$.
- Let $Z'' = A \cup \{z\}$. Thus $Z'' \cup \{x\}$ is a winning coalition. Since $x \approx y$ and neither x nor y belongs to Z'', we know that $Z'' \cup \{y\}$ is also a winning coalition. But $Z'' \cup \{y\} = A \cup \{z\} \cup \{y\} = Z \cup \{z\}$. Thus, $Z \cup \{z\}$ is a winning coalition as desired.
- A completely analogous argument would show that if $Z \cup \{z\}$ is a winning coalition, then so is $Z \cup \{x\}$. This completes the proof.

A does not contain x, y and z

$$Z = A \cup \{y\}, \quad Z' = A \cup \{x\}, \quad Z'' = A \cup \{z\}.$$

Assume that $Z \cup \{x\} = A \cup \{x, y\}$ is winning, we want to show that

$$Z \cup \{z\} = A \cup \{y, z\} \text{ is also winning.}$$



Proposition

For any two voters x and y in a weighted voting system, the following are equivalent:

1. x and y are equally desirable.

2. There exists an assignment of weights to the voters and a quota that realize the system and that give x and y the same weight.

3. There are two different ways to assign weights to the voters and two (perhaps equal) quotas such that both realize the system, but in one of the two weightings, x has more weight than y and, in the other weighting, y has more weight than x.

Proof

1. $\boxed{1 \Rightarrow 2}$

   Assume that x and y are equally desirable and there is a weighting and quota that realize the system. Let $w(x), w(y)$ and q denote the weight of x, weight of y and quota, respectively. [$w(Z) =$ total weight of the coalition Z].

   We try to construct a new weighting w_n such that with the same quota

$$w_n(x) = w_n(y).$$

   - The new weighting is obtained by keeping the weight of every voter except x and y the same and setting both $w_n(x)$ and $w_n(y)$ equal to the average of $w(x)$ and $w(y)$.

Assuming Z is a coalition, to show that this new weighting still realizes the same system, it is necessary to show that Z is winning in the new weighting if and only if Z is winning in the old weighting.

Case 1 Neither x nor y belong to Z

$w(Z) = w_n(Z)$ and so $w_n(Z) \geq q$ if and only if Z is winning.

Case 2 Both x and y belong to Z

$$\begin{aligned} w_n(Z) &= w_n(Z - \{x\} - \{y\}) + w_n(x) + w_n(y) \\ &= w(Z - \{x\} - \{y\}) + \frac{w(x) + w(y)}{2} \times 2 \\ &= w(Z). \end{aligned}$$

Case 3 x belongs to Z but y does not belong to Z

Let $Z' = Z - \{x\}$ so that $w_n(Z') = w(Z')$. Consider

$$\begin{aligned} w_n(Z) &= w_n(Z') + w_n(x) = w_n(Z') + \frac{w(x) + w(y)}{2} \\ &= \frac{w(Z') + w(x) + w(Z') + w(y)}{2} = \frac{w(Z) + w(Z - \{x\} \cup \{y\})}{2}. \end{aligned}$$

Since x and y are equally desirable, either both $Z = Z' \cup \{x\}$ and $Z - \{x\} \cup \{y\} = Z' \cup \{y\}$ are winning or both are losing.

If both are winning, then $w_n(Z) \geq \dfrac{q+q}{2} = q$.

If both are losing, then $w_n(Z) < \dfrac{q+q}{2} = q$.

2. $\boxed{2 \Rightarrow 3}$

We start with a weighting and quota where x and y have the same weight.

$$\begin{aligned} L_H &= \text{weight of the heaviest losing coalition} \\ W_L &= \text{weight of the lightest winning coalition} \end{aligned}$$

$$L_H < q \leq W_L.$$

Let q' be the average of L_H and q. It is seen that q' still works as a quota since

$$L_H < q' < W_L.$$

Let ϵ be any positive number such that

$$L_H + \epsilon < q' < W_L - \epsilon.$$

The system is unchanged if we either increase the weight of x by ϵ or decrease the weight of x by ϵ. There are 2 weightings that realize the system, one of which makes x heavier than y and the other of which makes y heavier than x.

3. $\boxed{3 \Rightarrow 1}$

Assume two weightings w and w', two quotas q and q', such that

(i) A coalition Z is winning if and only if $w(Z) \geq q$.

(ii) A coalition Z is winning if and only if $w'(Z) \geq q'$.

(iii) $w(x) > w(y)$. (iv) $w'(y) > w'(x)$.

We start with an arbitrary coalition Z containing neither x nor y. Suppose $Z \cup \{y\}$ is winning so that $w(Z \cup \{y\}) \geq q$. Since $w(x) > w(y)$, so $w(Z \cup \{x\}) \geq q$, thus $Z \cup \{x\}$ is winning. Similarly, $Z \cup \{x\}$ is winning $\Rightarrow Z \cup \{y\}$ is winning.

Example

Consider the yes–no voting system with 3 players whose winning coalitions are $\{A, C\}, \{B, C\}$ and $\{A, B, C\}$. The weighted voting system:

$$[51; 1, 49, 50] \quad \text{and} \quad [51; 49, 1, 50]$$

both realize the system. By the above result, we deduce that $A \approx B$.

How about two voters x and y whose desirability is neither equal nor incomparable? For any two voters x and y in a yes–no voting system, we say that x is more desirable than y, denoted by

$$x > y,$$

if and only if the following hold:

1. for every coalition Z containing neither x nor y, if $Z \cup \{y\}$ is winning then so is $Z \cup \{x\}$, and

2. there exists a coalition Z' containing neither x nor y such that $Z' \cup \{x\}$ is winning, but $Z' \cup \{y\}$ is losing.

Example

Consider the yes-no voting system with minority veto where the 9 voters are classified into the majority group of 6 voters $\{M_1, M_2, \ldots, M_6\}$ and the minority group of 3 voters $\{m_1, m_2, m_3\}$. The passage of a bill requires at least 5 votes from all voters and at least 1 vote from the minority voters.

(a) Any two members in the majority group are equally desirable (same result is applicable to the minority group).

Let M_1 and M_2 be two members in the majority group. Consider any coalition Z without M_1 and M_2, we have

$$\begin{aligned} & Z \cup \{M_1\} \text{ is winning} \\ \Rightarrow\ & Z \text{ must contain at least 4 votes from } \{M_3, \ldots, M_6, m_1, m_2, m_3\} \\ & \text{and at least one minority vote} \\ \Rightarrow\ & Z \cup \{M_2\} \text{ contains at least 5 votes and at least one minority vote} \\ \Rightarrow\ & Z \cup \{M_2\} \text{ is winning} \end{aligned}$$

Following similar steps, we have $Z \cup \{M_2\}$ is winning $\Rightarrow Z \cup \{M_1\}$ is winning. Combining the results, we obtain

$$Z \cup \{M_1\} \text{ is winning } \iff Z \cup \{M_2\} \text{ is winning}$$

(b) For every Z that does not contain x and y, it is easy to establish: $Z \cup \{M_1\}$ is winning $\Rightarrow Z \cup \{m_1\}$ is winning.

This is because the head count requirement and minority veto requirement are satisfied.

However, it is straightforward to choose a coalition Z' that does not contain M_1 and m_1, where $Z' \cup \{m_1\}$ is winnning and $Z' \cup \{M_1\}$ is losing. One such example is

$$Z' = \{M_2, M_3, M_4, M_5\}.$$

Example

In the US federal system, x is a senator and y is the Vice President, then $x > y$. *Hint*: Consider coalition Z where the number of senators is 50 or above (with the President and one-half majority of Representatives) and coalition Z' where the number of senators is 66 (without the President and two-thirds majority of Representatives).

We write $x \geq y$ to mean either $x > y$ or $x \approx y$. The relation $\geq$ is called the *desirability relation on individuals*.

- The binary relation $\geq$ is called a *preordering* because it is transitive and reflexive.

- A preordering is said to be *linear* if for every pair x and y, one has either $x \geq y$ or $y \geq x$.

Definition A yes–no voting system is said to be *linear* if and only if there are no incomparable voters (equivalently, if the desirability relation on individuals is a linear preordering).

Propositions

1. A yes–no voting system is linear if and only if it is swap robust. This is a direct consequence of the property that non swap robust is equivalent to the existence of a pair of voters whose desirability is incomparable.

   Corollary Every weighted voting system is linear.

2. In a weighted voting system we have $x > y$ if and only if x has strictly more weight than y in every weighting that realizes the system.

- When $x > y$, it is not possible that x has strictly more weight than y in one weighting and reverse in the other weighting. Also, the possibility that x and y have equal weight in any weighting is ruled out.

Example

Consider the yes-no voting system of 4 players with the following winning coalitions:

$\{A, C\}, \{B, C\}, \{A, B, C\}, \{C, D\}, \{A, C, D\}, \{B, C, D\}, \{A, B, D\}, \{A, B, C, D\}$.

(a) A and B are "equally desirable".

Test for all coalitions that do not contain A and B:

$$Z_1 = \phi, Z_2 = \{C\}, Z_3 = \{D\}, Z_4 = \{C, D\}.$$

The result of A joining any of these $Z_i, i = 1, 2, 3, 4$, is the same as that of B joining Z_i.

(b) C is more desirable than D, denoted by $C > D$, since the following coalitions that do not contain C and D:

$$Z'_1 = \phi, Z'_2 = \{A\}, Z'_3 = \{B\}, Z'_4 = \{A, B\},$$

we have

$Z'_1 \bigcup \{C\}$ is losing and $Z'_1 \bigcup \{D\}$ is losing
$Z'_4 \bigcup \{C\}$ is winning and $Z'_4 \bigcup \{D\}$ is winning

while

$Z'_i \bigcup \{C\}$ is winning but $Z'_i \bigcup \{D\}$ is losing, $i = 2, 3$.

(c) The given yes-no voting system is a weighted voting system. Possible assignments of voters' weights and quota that realize the system are:

$$\{4; 1, 1, 3, 2\}, \{9; 1, 3, 8, 5\}, \{9; 3, 1, 8, 5\}, \{9; 2, 2, 8, 5\} \text{ etc.}$$

Remarks

1. The first and the fourth weighted voted assignments give the same weight to A and B.

2. In the second system, B has more weight than A; while in the third system, A has more weight than B. The fourth system is obtained by taking the weight of A and B to be equal to the average of the weights of A and B in the second or the third system.

3. Since C is more desirable than D, the weight of C in any assignment is always greater than that of D.

4. The given yes-no voting system is swap robust (since it is a weighted voting system), so there are no incomparable voters. Indeed

$$A \approx B < D < C.$$