MATH4994 — Capstone Projects in Mathematics and Economics

Homework Two

Course instructor: *Prof. Y.K. Kwok*

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1. Consider the following marriage problem where all men prefer the same woman as their first choice and all women prefer the same man as their first choice

$$\begin{array}{ll} P(m_1) = w_1, w_2, w_3 & P(w_1) = m_1, m_2, m_3 \\ P(m_2) = w_1, w_2, w_3 & P(w_2) = m_1, m_3, m_2 \\ P(m_3) = w_1, w_3, w_2 & P(w_3) = m_1, m_2, m_3 \end{array}$$

   Find the corresponding stable matchings for man-oriented μ_M and woman-oriented μ_W.

   Hint: Explain why any matching that does not pair m_1 with w_1 is unstable.

2. *Man-woman-child matching*
   There are three men, three women and three children. A matching is a division of the people into 3-person groups, each group containing one man, one woman, and one child. Each person has preferences over the sets of pairs he or she might possibly be matched with. A man, woman, and child (m, w, c) *block* a matching μ if m prefers (w, c) to $\mu(m)$; w prefers (m, c) to $\mu(w)$, and c prefers (m, w) to $\mu(c)$. A matching is stable only if it is not blocked by any such three agents.

   Consider three men, three women, and three children, with the following preferences:

$$\begin{array}{rcl} P(m_1) & = & (w_1, c_3), (w_2, c_3), (w_1, c_1), \ldots \text{(arbitrary)} \\ P(m_2) & = & (w_2, c_3), (w_2, c_2), (w_3, c_3), \ldots \text{(arbitrary)} \\ P(m_3) & = & (w_3, c_3), \ldots \text{(arbitrary)} \\ P(w_1) & = & (m_1, c_1), \ldots \text{(arbitrary)} \\ P(w_2) & = & (m_2, c_3), (m_1, c_3), (m_2, c_2), \ldots \text{(arbitrary)} \\ P(w_3) & = & (m_2, c_3), (m_3, c_3), \ldots \text{(arbitrary)} \\ P(c_1) & = & (m_1, w_1), \ldots \text{(arbitrary)} \\ P(c_2) & = & (m_2, w_2), \ldots \text{(arbitrary)} \\ P(c_3) & = & (m_1, w_3), (m_2, w_3), (m_1, w_2), (m_3, w_3), \ldots \text{(arbitrary)}. \end{array}$$

   Show that there is no stable matching in this example.

   Remark
   Observe that the preferences in this problem are "separable" into preferences over men, women, and children; that is, there are no preferences like (m, w, c) is preferred by m to (m, w, c'), but (m, w', c') is preferred to (m, w', c).

3. *Many-to-one matching*
   There are 2 firms and 3 workers. Each worker can work for at most one firm and has preferences over those firms he is willing to work for. Each firm can hire as many workers as it wishes and has preferences over those subsets of workers it is willing to employ. A

firm F and a subset of workers C *block* a matching μ if F prefers C to the set of workers assigned to it at μ, and every worker in C who is not assigned to F prefers F to the firm he is assigned by μ. Consider two firms and three workers with the following preferences:

$$\begin{aligned}
P(F_1) &= \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\} \\
P(F_2) &= \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2\}, \{w_3\}, \{w_1\}, \{w_2\} \\
P(w_1) &= F_2, F_1 \\
P(w_2) &= F_2, F_1 \\
P(w_3) &= F_1, F_2.
\end{aligned}$$

Find the five individually rational matchings without unemployment. Show that each of these matchings can be blocked by some matching pair. Check that any matching that leaves w_1 unmatched is blocked either by (F_1, w_1) or by (F_2, w_1); any matching that leaves w_2 unmatched is blocked either by (F_1, w_2), (F_2, w_2), or $(F_2, \{w_2, w_3\})$. Finally, any matching that leaves w_3 unmatched is blocked by $(F_2, \{w_1, w_3\})$.

4. Consider the following matching where not all players have strict preferences. Here, $P(m_1)$ represents the preference list of man m_1; and similar notation for others.

$$\begin{aligned}
P(m_1) &= [w_2, w_3], w_1 & \qquad P(w_1) &= m_1, m_2, m_3 \\
P(m_2) &= w_2, w_1 & \qquad P(w_2) &= m_1, m_2 \\
P(m_3) &= w_3, w_1 & \qquad P(w_3) &= m_1, m_3
\end{aligned}$$

Note that m_1 is indifferent to w_2 and w_3. There are **only two** stable matching solutions μ_1 and μ_2 that can be found, namely,

$$\mu_1 = \begin{matrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \end{matrix} \quad \text{and} \quad \mu_2 = \begin{matrix} w_1 & w_2 & w_3 \\ m_3 & m_2 & m_1 \end{matrix}.$$

(a) Verify that the matching solution μ_1 is stable.

Hint: A man-woman pair (not one of the pairs in the matching scheme) blocks a matching scheme if they prefer each other **strictly** to their spouses under the matching scheme.

(b) Show that matching solution μ_2 does not achieve woman-optimality.

Hint: Since all stable matching solutions have been given, find the set of achievable men for each woman. Apply the property of woman-optimality once the set of achievable men is known.

5. Given the following preferences for both colleges (c_1, c_2 and c_3) and students (s_1, s_2 and s_3):

colleges' preferences		*students' preferences*	
c_1 :	$s_1 - s_3 - s_2$	s_1 :	$c_2 - c_1 - c_3$
c_2 :	$s_2 - s_3 - s_1$	s_2 :	$c_1 - c_3 - c_2$
c_3 :	$s_2 - s_1 - s_3$	s_3 :	$c_1 - c_2 - c_3$

(a) The Boston school choice mechanism chooses students according to the students' first choice as the first priority, then the second choice, and so forth. Find the Boston school choice solution for the given preference lists.

(b) Can a student game around to get into a better school by misrepresenting his preference list? If yes, find that student and his gaming strategy.

6. There are three agents i_1, i_2, i_3, and three houses h_1, h_2, h_3. Agent i_1 is a current tenant and occupies house h_1. Agents i_2, i_3 are new applicants and houses h_2, h_3 are vacant houses. The following matrix gives the utilities of agents over houses:

	h_1	h_2	h_3
i_1	3	4	1
i_2	4	3	1
i_3	3	4	1

Agent i_1 has two options:

1. keep house h_1 or
2. give it up and enter the lottery.

If i_1 enters the lottery then there are 6 possibilities depending on the chosen ordering, summarized in the following table:

ordering	assignment of i_1	assignment of i_2	assignment of i_3
i_1–i_2–i_3	h_2	h_1	h_3
i_1–i_3–i_2	h_2	h_3	h_1
i_2–i_1–i_3	h_2	h_1	h_3
i_2–i_3–i_1	h_3	h_1	h_2
i_3–i_1–i_2	h_1	h_3	h_2
i_3–i_2–i_1	h_3	h_1	h_2

(a) Assuming that agent i_1 is an expected-utility maximizer, show that the optimal strategy of i, is keeping house h_1.

(b) Suppose i_1 chooses keeping h_1. Note that both agent i_2 and agent i_3 prefer house h_2 to house h_3, find the eventual outcomes and their associated probabilities of occurrence. Check whether both outcomes are Pareto optimal.

(c) The following procedure guarantees the existing tenant a house that is at least as good as the one he is already holds.

   (i) Order the agents by means of a lottery.

   (ii) Assign the first agent his or her top choice, the second agent his or her top choice among the remaining houses, and so on, *until someone, demands the house the existing tenant holds.*

(iii) If the existing tenant is already assigned a house, then do not disturb the procedure. If the existing tenant is not assigned a house, then modify the remainder of the ordering by inserting him or her at the top, and proceed with the procedure.

For each of the 6 possible orderings of choices of houses based on lottery, find the modified orderings based on the new mechanism. Find the possible outcomes and their associated probabilities of occurrences. Check whether they are Pareto efficient.