MATH4994 — Capstone Projects in Mathematics and Economics

Solution to Homework Two

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1. In this problem, woman w_1 is the first choice of every man, and man m_1 is the first choice of every woman. No two men would agree on what is the best matching, since each man's favorite matching is one at which he is married to woman w_1. Similarly, no two women agree on what is the best matching. But let us now turn our attention to the set of *stable* matching. Any matching that does not pair m_1 with w_1 is unstable, since m_1 and w_1 are each other's first choice, and so form a blocking pair for any matching at which they are not mates. Consequently there are only two stable matchings. These are

$$\mu_M = \begin{matrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{matrix} \quad \text{and} \quad \mu_W = \begin{matrix} w_1 & w_3 & w_2 \\ m_1 & m_2 & m_3 \end{matrix}.$$

2. We consider the following 3 cases:

   (i) All matchings that give m_1 (respectively m_2 and w_2) a better family than (m_1, w_1, c_1) [respectively, (m_2, w_2, c_2)] are unstable.

   To see this, note that any matching containing either (m_1, w_1, c_3) or (m_2, w_2, c_3) is blocked by (m_3, w_3, c_3), and any matching containing (m_1, w_2, c_3) is blocked by (m_2, w_3, c_3).

   (ii) Any matching that does not contain (m_1, w_1, c_1) [respectively, (m_2, w_2, c_2)] is either blocked by (m_1, w_1, c_1) [respectively, (m_2, w_2, c_2)] or is unstable as already shown in part 1 above.

   (iii) Finally, (m_1, w_2, c_3) blocks any matching that contains (m_1, w_1, c_1) and (m_2, w_2, c_2). So all matchings are unstable.

3. We observe that

$$\begin{aligned}
\mu_1 &= \begin{matrix} F_1 & F_2 \\ \{w_1, w_3\} & \{w_2\} \end{matrix}, && \text{which is blocked by } (F_2, w_1) \\
\mu_2 &= \begin{matrix} F_1 & F_2 \\ \{w_1, w_2\} & \{w_3\} \end{matrix}, && \text{which is blocked by } (F_2, \{w_1, w_3\}) \\
\mu_3 &= \begin{matrix} F_1 & F_2 \\ \{w_2, w_3\} & \{w_1\} \end{matrix}, && \text{which is blocked by } (F_2, \{w_1, w_2\}) \\
\mu_4 &= \begin{matrix} F_1 & F_2 \\ \{w_2\} & \{w_1, w_3\} \end{matrix}, && \text{which is blocked by } (F_1, \{w_2, w_3\}) \\
\mu_5 &= \begin{matrix} F_1 & F_2 \\ \{w_1\} & \{w_2, w_3\} \end{matrix}, && \text{which is blocked by } (F_2, \{w_1, w_3\}).
\end{aligned}$$

4. (a) Consider μ_1, we check whether a blocking pair can be found for m_1, m_2 and m_3.

(i) m_1 is matched to one of his best choice w_2 under μ_1;

(ii) m_2 may want to match with w_2, but w_2 has been matched with her best choice under μ_1;

(iii) m_3 is matched to his best choice w_3 under μ_1.

(b) The sets of achievable men for w_1, w_2 and w_3 are

$$A(w_1) = \{m_2, m_3\}, \quad A(w_2) = \{m_1, m_2\}, \quad A(w_3) = \{m_1, m3\}.$$

Consider μ_2, w_1 is matched to m_3, who is the worst choice within $A(w_1)$; w_2 is matched to m_2, who is the worst choice within $A(w_2)$; w_3 is matched to m_1, who happens to be the best choice within $A(w_3)$. Note that *not all* women achieve their best achievable partner. Therefore, woman-optimality is not achieved under μ_2.

5. (a) S_1 comes to c_2 first; S_2 and S_3 compete for c_1, but S_3 wins; lastly, S_2 settles with c_3. The outcome is

$$\begin{array}{ccc} S_1 & S_2 & S_3 \\ c_2 & c_3 & c_1. \end{array}$$

(b) Both S_1 and S_3 receive their top choice already, so they do not game around. Can S_2 gain by gaming? His top choice is c_1 but c_1 places S_2 the lowest priority. Therefore, S_2 cannot gain by gaming to improve his assignment of c_3, which is his second choice already.

6. (a) The expected utility of i_1 under the lottery is given by

$$\frac{1}{6}u(h_1) + \frac{3}{6}u(h_2) + \frac{2}{6}u(h_3) = \frac{3}{6} + \frac{12}{6} + \frac{2}{6} = \frac{17}{6}.$$

The utility of i_1 if he chooses keeping his house h_1 is 3, which is higher than $\frac{17}{6}$. Therefore, i_1 should choose keeping h_1.

(b) The two possible outcomes are

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_2 & h_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_3 & h_2 \end{pmatrix},$$

both with 1/2 probability. Among these two matchings the first is Pareto dominated by

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_1 & h_3 \end{pmatrix}.$$

Therefore, this mechanism may lead to Pareto inefficient outcomes.

(c) The lottery can result in six orderings. If the ordering is one of $i_1 - i_2 - i_3$, $i_1 - i_3 - i_2$, or $i_3 - i_1 - i_2$, then agent i_1 leaves before anyone demands house h_1 and therefore the resulting allocation is not affected:

initial ordering	modified ordering	assignment of i_1	assignment of i_2	assignment of i_3
i_1–i_2–i_3	i_1–i_2–i_3	h_2	h_1	h_3
i_1–i_3–i_2	i_1–i_3–i_2	h_2	h_3	h_1
i_3–i_1–i_2	i_3–i_1–i_2	h_1	h_3	h_2

If the ordering is $i_2 - i_1 - i_3$ or $i_2 - i_3 - i_1$, then in the first step agent i_2 demands house h_1. In both cases the ordering is changed to $i_1 - i_2 - i_3$ and the resulting outcome is as follows:

initial ordering	modified ordering	assignment of i_1	assignment of i_2	assignment of i_3
i_2–i_1–i_3	i_1–i_2–i_3	h_2	h_1	h_3
i_2–i_3–i_1	i_1–i_2–i_3	h_2	h_1	h_3

Finally if the ordering is $i_3 - i_2 - i_1$, then in the first step agent i_3 is assigned house h_2 and in the next step agent i_2 demands house h_1. The remainder of the ordering is changed to $i_1 - i_2$ and this results in the following outcome:

initial ordering	modified ordering	assignment of i_1	assignment of i_2	assignment of i_3
i_3–i_2–i_1	i_3–i_1–i_2	h_1	h_3	h_2

Therefore, the modified mechanism selects one of

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_1 & h_3 \end{pmatrix}, \begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_3 & h_1 \end{pmatrix}, \quad \text{or} \begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_3 & h_2 \end{pmatrix},$$

with probabilities of 1/2, 1/6, and 1/3 respectively. Note that all these matchings are Pareto efficient.