MATH4994 — Capstone Projects in Mathematics and Economics

Homework Three

Course instructor: *Prof. Y.K. Kwok*

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1. Suppose there are four alternatives, named A, B, C and D. There are three voters who have the following individual rankings:

$$\begin{pmatrix} B \\ C \\ D \\ A \end{pmatrix}, \begin{pmatrix} C \\ D \\ A \\ B \end{pmatrix}, \begin{pmatrix} D \\ A \\ B \\ C \end{pmatrix}.$$

   You are asked to design an agenda in pairwise contests via majority vote in which A wins.

2. If there are only three alternatives, is the Hare elimination procedure equivalent to the plurality voting with a runoff?

3. Recall that we allow ties in the social choice but not the input. Prove or disprove each of the following statements:

   (a) Plurality voting always yields a unique social choice.
   (b) The Borda count always yields a unique social choice.
   (c) The Hare system always yields a unique social choice.
   (d) Sequential pairwise voting with a fixed agenda always yields a unique social choice.
   (e) A dictatorship always yields a unique social choice.

4. If we have a sequence of individual preference lists, and r and s are two of the alternatives, then "Net$(r > s)$" is defined to be the number of voters who prefer r to s minus the number of voters who prefer s to r. Suppose we change the way that we assign points in computing the Borda score of an alternative so that these scores are symmetric about zero. That is, for three alternatives, the first place will be worth 2 points, the second place will be worth 0 points, and the third place will be worth -2 points. We let "$B(r)$" denote the Borda score of the alternative r computed using these points. Consider the following preference lists:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} x \\ z \\ y \end{pmatrix} \begin{pmatrix} y \\ z \\ x \end{pmatrix}$$

   (a) Evaluate $\text{Net}(x > y)$, $\text{Net}(x > z), B(x)$, and $B(z)$.
   (b) Prove each of the following results for this example:

$$\text{Net}(x > y) + \text{Net}(x > z) = B(x).$$
$$\text{Net}(y > z) + \text{Net}(y > x) = B(y).$$
$$\text{Net}(z > x) + \text{Net}(z > y) = B(z).$$

5. Suppose we have a social choice procedure that satisfies monotonicity. Suppose that for the four alternatives a, b, c, d, we have a sequence of individual preference lists that yields d as the social choice. Suppose person one changes his list:

$$\text{from: } \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad \text{to: } \begin{pmatrix} d \\ a \\ b \\ c \end{pmatrix}$$

   Show that d is still the social choice, or at least tied for such.

6. Suppose we have three voters and four alternatives, and the individual preference lists are as follows:

$$\begin{pmatrix} a \\ b \\ d \\ c \end{pmatrix} \begin{pmatrix} c \\ a \\ b \\ d \end{pmatrix} \begin{pmatrix} b \\ d \\ c \\ a \end{pmatrix}.$$

   Show that if the social choice procedure being used is sequential pairwise voting with a fixed agenda, and suppose you have agenda setting power (i.e., you get to choose the order), then you can arrange for whichever alternative you want to be the social choice.

7. Prove that for a given social choice procedure and a given sequence of individual preference lists, a Condorcet winner, if it exists, must be unique.

8. An alternative is said to be a *Condorcet loser* if it would be defeated by every other alternative in one-on-one contest that takes place in sequential pairwise voting with a fixed agenda. Further, say that a social choice procedure satisfies the *Condorcet loser criterion* provided that a Condorcet loser is never among the social choices. Does the Condorcet loser criterion hold for:

   (a) plurality voting?
   (b) Borda count?
   (c) Hare system?
   (d) sequential pairwise voting with a fixed agenda?
   (e) dictatorship?

9. Prove that for three alternatives and an arbitrary sequence of individual preference lists, there is no Condorcet loser if and only if for each alternative there is an agenda under which that an alternative wins in sequential pairwise voting.

10. Consider the following social choice procedure. If there is a Condorcet winner, then it is the social choice. Otherwise, the alternative on top of the first person's list is the social choice. That is, if there is no Condorcet winner, then person one acts as the dictator. Give an example with three people and three alternatives showing that this procedure does not satisfy independence of irrelevant alternatives. (Hint: Start with the same sequence of lists that produces the voting paradox, and then move one alternative that should be irrelevant to the social choice). Prove that this social choice procedure satisfies the Pareto condition.

11. A social choice procedure is said to satisfy the "top condition" provided that an alternative is never among the social choices (ties are allowed) unless it occurs on top of at least one individual preference list. Prove or disprove each of the following:

    (a) Plurality voting satisfies the top condition.
    (b) The Borda count satisfies the top condition.
    (c) The Hare system satisfies the top condition.
    (d) Sequential pairwise voting satisfies the top condition.
    (e) A dictatorship satisfies the top condition.
    (f) If a procedure satisfies the top condition, then it satisfies the Pareto condition.

12. Under the social choice procedure of anti-plurality voting, the social choice (winner) is the alternative with the fewest last-place ranking. Check whether this anti-plurality voting satisfies (i) Pareto criterion, (ii) monotonicity criterion, (iii) independent of irrelevant alternatives criterion.

13. Consider the following sequence of individual preference lists:

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \begin{pmatrix} a \\ d \\ b \\ e \\ c \end{pmatrix} \begin{pmatrix} a \\ d \\ b \\ e \\ c \end{pmatrix} \begin{pmatrix} c \\ b \\ d \\ e \\ a \end{pmatrix} \begin{pmatrix} c \\ d \\ b \\ a \\ e \end{pmatrix} \begin{pmatrix} b \\ c \\ d \\ a \\ e \end{pmatrix} \begin{pmatrix} e \\ c \\ d \\ b \\ a \end{pmatrix}.$$

    (a) Consider the social welfare function obtained by the plurality procedure. Write down the social preference list that results from applying this function to the above sequence of individual preference lists.
    (b) Do the same for the social welfare function obtained by the Borda count.
    (c) Do the same for the social welfare function obtained by the Hare procedure.
    (d) Do the same for the social welfare function obtained by sequential pairwise voting with a fixed agenda.

14. Suppose $A = \{a, b, c\}$ and a given social welfare function produces output $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when confronted with input $\begin{pmatrix} c \\ b \\ a \end{pmatrix} \begin{pmatrix} a \\ c \\ b \end{pmatrix} \begin{pmatrix} b \\ c \\ a \end{pmatrix}$.

    (a) If neutrality is satisfied, what is the output when confronted with input:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} c \\ a \\ b \end{pmatrix} \begin{pmatrix} b \\ a \\ c \end{pmatrix}?$$

    (b) What input would definitely yield c over a over b as output?