MATH4994 —– Capstone Projects in Mathematics and Economics

Topic 4 – Proportional representation and Apportionment Schemes

4.1 General issues of apportionment of legislature seats

- Gerrymandering

4.2 Quota Method of the Greatest Remainder (Hamilton's method) and paradoxes

- Alabama paradox
- New State paradox
- Population monotonicity

4.3 Divisor methods

- Choices of divisors
- Huntington approach: Pairwise comparison of inequity
- Rank index
- US history of apportionment

4.4 Analysis of bias

- Probabilistic approach
- Majorization ordering
- "Near the quota" property

4.5 Proportionality in matrix apportionment

- Proportionality in districts and parties
- Greatest remainder biproportional rounding method

4.1 General issues of apportionment of legislature seats

To apportion is to distribute by right measure, to set off in just parts, to assign in due and proper proportion.

- Distributing available personnel or other resources in "integral parts" (integer programming):
  - distributing seats in a legislature based on populations or votes
  - distributing minister posts among political parties in a coalition government

- Some obvious process for *rounding fractions* or some optimal schemes for minimizing certain natural *measure of inequality* would fail. Each scheme may possess certain "flaws" or embarrassing "paradoxes" (反論, opposite to common sense or the truth).

Apportionment of US house seats based on states' populations

- $a_i =$ number of Representatives apportioned to the i^{th} state,

  $p_i =$ population in the i^{th} state, $i = 1, 2, \cdots, S$.

  The Constitution requires $a_i \geq 1$ and $p_i/a_i > 30,000$, where the current House size $= 435$* (fixed after New Mexico and Arizona became states in 1912).

$$\begin{aligned} &\text{Current number of constituents per Representative} \\ \approx\ & 300 \text{ million}/435 \gg 30,000 \end{aligned}$$

* In 1959, Alaska and Hawaii were admitted to the Union, each receiving one seat, thus temporarily raising the House to 437. The apportionment based on the census of 1960 reverted to a House size of 435.

Statement of the Problem of Apportionment of House Seats

$h =$ number of congressional seats; $P =$ total US population $= \sum_{i=1}^{S} p_i$;

the i^{th} state is entitled to $q_i = h\left(\frac{p_i}{P}\right)$ representatives.

Difficulty: the eligible quota $q_i = \frac{hp_i}{P}$ is in general not an integer. In simple terms, a_i is some form of *integer rounding* to q_i. Define $\overline{\lambda} = P/h =$ average number of constituents per Representative, then $q_i = p_i/\overline{\lambda}$. The (almost) continuous population weight p_i/P is approximated by the rational proportion a_i/h.

An apportionment solution is a function f, which assigns an apportionment vector $\boldsymbol{a}$ to any population vector $\boldsymbol{p}$ and fixed house size h. One usually talks about an apportionment method $M = M(\boldsymbol{p}, h)$, which is a non-empty set of apportionment solutions. Ties may occur, though unlikely, so the solution to $\boldsymbol{a}$ may not be unique.

Number of seats for the geographical constituency areas

District	*Number of seats*	*Estimated population (as on 30 June 2012)*	*% of deviation* $\frac{a_i - q_i}{q_i}$
Hong Kong Island	7	$1,295,800$	$+9.77\%$
Kowloon West	5	$1,081,700$	-5.45%
Kowloon East	5	$1,062,800$	-3.61%
New Territories West	9	$2,045,500$	-10.78%
New Territories East	9	$1,694,900$	$+8.21\%$

Changes made in 2016 election: one seat was moved from Hong Kong Island to Kowloon West.

Any justification to explain why not to do a similar swap of one seat from New Territories East to New Territories West?

Related problem

Apportionment of legislature seats to political parties is based on the votes received by the parties.

Inconsistencies in apportionment based on either the *district* or *state-wide* criterion.

2004 Connecticut congressional elections – District criterion

District	1st	2nd	3rd	4th	5th	Total	Seats
Republican	$73,273$	$\mathbf{165,558}$	$68,810$	$\mathbf{149,891}$	$\mathbf{165,440}$	$622,972$	3
Democratic	$\mathbf{197,964}$	$139,987$	$\mathbf{199,652}$	$136,481$	$105,505$	$779,589$	2

We pick the winner in each district. The Democratic Party receives only 2 seats though the Party receives more votes ($779,589$ versus $622,972$) state-wide. This is a real life contemporary example where 田忌賽馬 is put into practice.

State-wide criterion versus district criterion

If the state-wide criterion is used, then the Republican Party with only $\frac{622,972}{779,589 + 622,972} \times 100\% = 44.42\%$ of votes should receive only 2 seats.

Each Republican seat requires $622,973/3 \approx 207,657$ votes while each Democratic seat requires $779,589/2 \approx 389,795$ votes, which is 1.877 times that of the Republican party.

This appears to be contradicting the principle: parties should share the seats according to their total votes in each state. How can we resolve the inconsistencies?

Gerrymandering

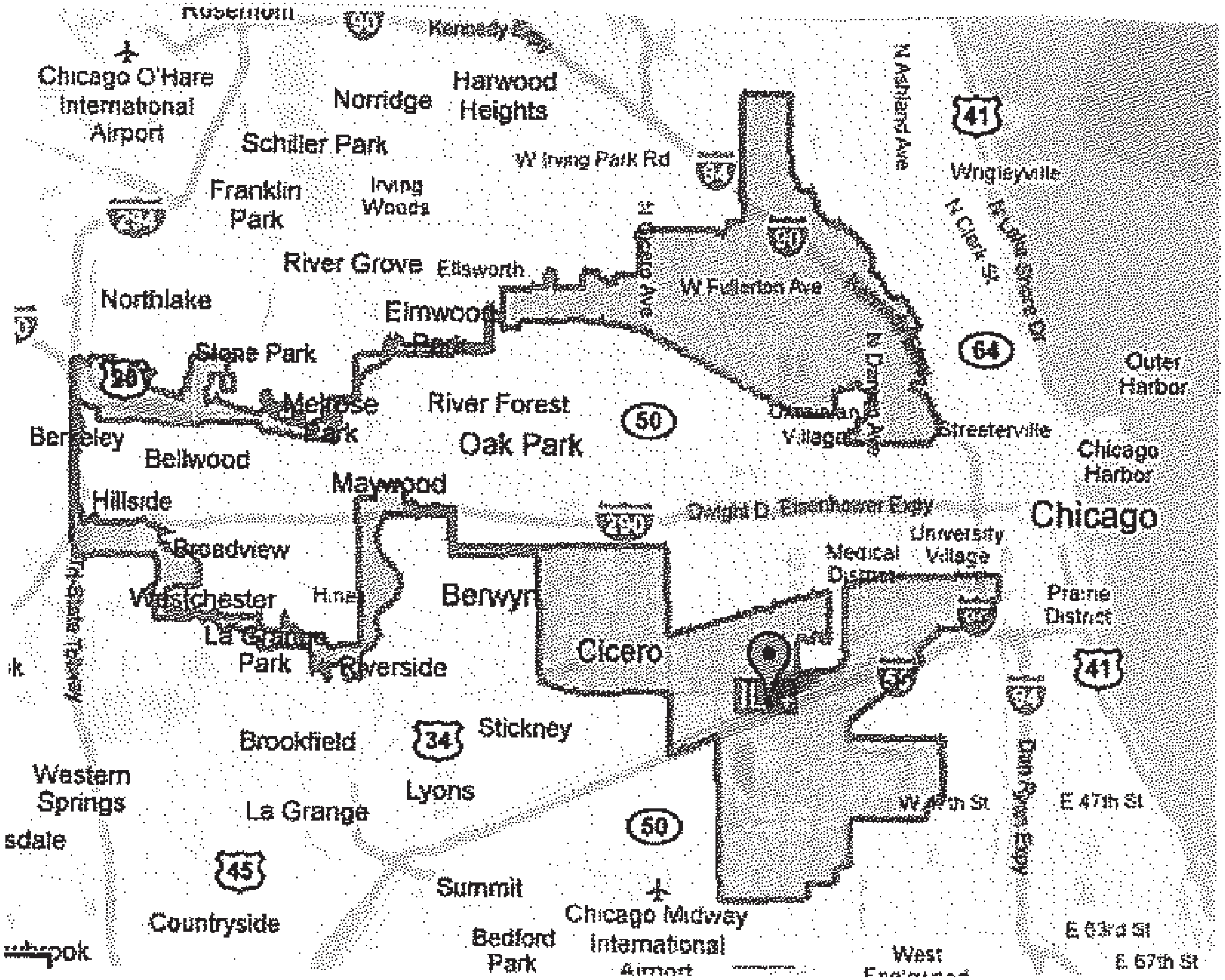
The practice of dividing a geographic area into electoral districts, often of highly irregular shape, to give one political party an unfair advantage by diluting the opposition's strength.

For example, Texas had redistributed following the census of 2000, but in the state elections of 2002, the Republicans took control of the state government and decided to redistribute once again. Both parties determine districts to maximize their advantage whenever they have the power to do so.

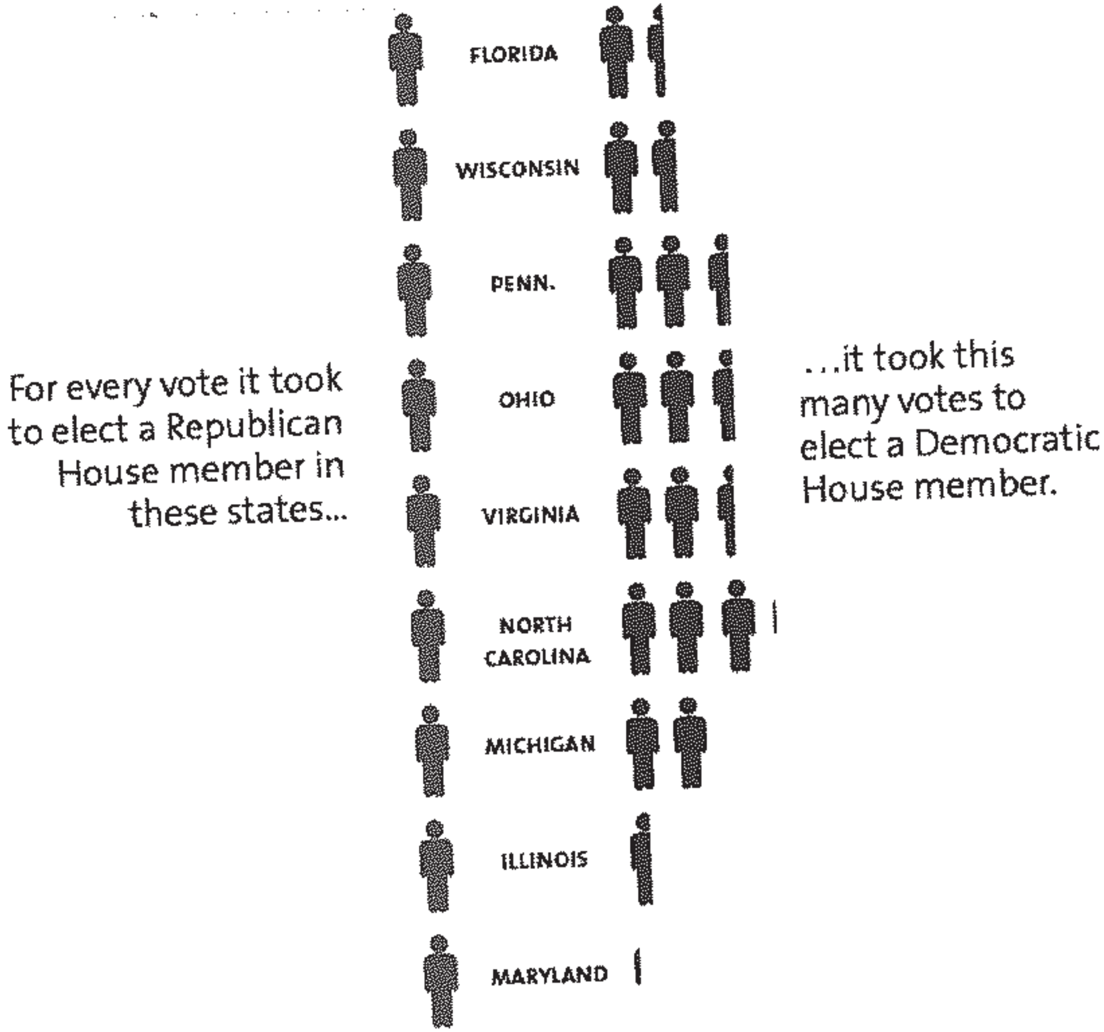
In 2012, the 234-201 House seats majority goes to the Republicans though the Democrats have a slight edge in the popular vote for House seats, 48.8%-47.6%.

Measure to resolve gerrymandering Allocation to district winners is assigned such that it also depends on the state wide popularity vote.

Illinois Congressional District 4: Worst Example of Gerrymandering



Republicans *Democrats*



- In Florida, Democrats won nearly half the popular votes but filled about a third of the state's congressional seats.

Issues addressed in apportionment schemes

1. Find an operational method for interpreting the mandate of proportional representation (with reference to population counts or votes).

2. Identify the desirable properties that any fair method ought to observe. Not to produce paradoxes.

- The "best" method is unresolvable since there is no one method that satisfies all reasonable criteria and produce no paradoxes – *Balinski-Young Impossibility Theorem*.

- Intense debate is surrounding the basis of population counts: How to count Federal employees living outside the US? Should we count illegal immigrants and permanent residents?

4.2 Quota Method of the Greatest Remainder (Hamilton's method) and paradoxes

After assigning at least one seat to each state, every state is then assigned its lower quota. This is possible provided that

$$h \geq \sum_{i=1}^{S} \max(1, \lfloor q_i \rfloor), \qquad (i)$$

a condition which holds in general. Next, we order the remainders $q_i - \lfloor q_i \rfloor$, and allocate seats to the states having the largest fractional remainders in sequential order.

- By its construction, the Hamilton method satisfies the quota property: $\lfloor q_i \rfloor \leq a_i \leq \lfloor q_i \rfloor + 1$.
- Recall that $h = \sum_{i=1}^{S} q_i$, thus $h \geq \sum_{i=1}^{S} \lfloor q_i \rfloor$; so condition (i) is not satisfied only when there are too many states with very small population that are rounded up to one seat based on the minimum requirement.

Constrained integer programming problem

We minimize $\sum_{i=1}^{S} (a_i - q_i)^2$

subject to $\sum_{i=1}^{S} a_i = h \quad$ and $\quad a_i \geq 1, \quad i = 1, \cdots, S.$

It seeks for integer allocations a_i that are never less than unity and staying as close as possible (in some measure) to the fair shares q_i. The "inequity" is measured by the totality of $(a_i - q_i)^2$ summed among all states.

- Actually, in a more generalized setting, Hamilton's method minimizes

$$\sum_{i=1}^{S} |a_i - q_i|^p, \quad p \geq 1.$$

  This amounts to a norm-minimizing approach.

- Provided $h \geq \sum_{i=1}^{S} \max(1, \lfloor q_i \rfloor)$, each state would receive at least $\max(1, \lfloor q_i \rfloor)$ seats. Due to the minimum requirement that $a_i \geq 1$, it may be possible that not all states are assigned seats with number that is guaranteed to be at least the lower quota.

- Any state which has been assigned the lower quota $\lfloor q_i \rfloor$ already will not be assigned a new seat until all other states have been assigned the lower quota. This is because the states that have been assigned the lower quota would have value of $q_i - a_i$ smaller than those states that have not.

- The Hamilton apportionment procedure minimizes the sum of inequity as measured by $\sum_{i=1}^{S} |a_i - q_i|^p$. The assignment of the remaining seats coinciding with the ranking of the largest remainders minimizes the contribution to the chosen criterion of sum of inequity. The worst discrepancy between a_i and q_i among all states is measured by $\max_i |a_i - q_i|$. Among all apportionment methods, Hamilton's method minimizes $\max_i |a_i - q_i|$.

Loss of House Monotone Property

State	Population	25 seats exact quota	26 seats exact quota	27 seats exact quota
A	9061	8.713 [9]	9.061 [9]	9.410 [9]
B	7179	6.903 [7]	7.179 [7]	7.455*[8]
C	5259	5.057 [5]	5.259 [5]	5.461*[6]
D	3319	3.191 [3]	**3.319***[4]	**3.447** [3]
E	1182	1.137 [1]	1.182 [1]	1.227 [1]
	26000	25	26	27

- The integers inside [ ] show the apportionments.
- When $h = 26$, State D is assigned an additional seat beyond the lower quota of 3. However, when $h = 27$, the extra seat is taken away since States B and C take the two additional seats beyond their lower quotas. State D suffers a drop from 4 seats to 3 seats when the total number of seats increases from 26 to 27.

Alabama Paradox (1882)

In 1882, the US Census Bureau supplied Congress with a table showing the apportionment produced by Hamilton's method for all sizes of the House between 275 and 350 seats. Using Hamilton's method, the state of Alabama would be entitled to 8 representatives in a House having 299 members, but in a House having 300 members it would only receive 7 representatives – loss of *house monotone property*.

- Alabama had an exact quota of 7.646 at 299 seats and 7.671 at 300 seats, while Texas and Illinois increased their quotas from 9.640 and 18.640 to 9.682 and 18.702, respectively.
- At $h = 300$, Hamilton's method gave Texas and Illinois each an additional representative. Since only one new seat was added, Alabama was forced to lose one seat. Apparently, the more populous state has the larger increase in the remainder part. The less populous states may fall victim in this Alabama paradox.

House monotone property (Property H)

An apportionment method M is said to be house monotone if for every apportionment solution $f \in M$

$$f(\boldsymbol{p}, h) \leq f(\boldsymbol{p}, h+1).$$

That is, if the House increases its size, then no state will lose a former seat using the same method M.

A method observes house monotone property if the method awards extra seats to states when h increases, rather than computing a general redistribution of the seats.

Why does Hamilton's method not observe the House monotone property?

The rule of assignment of the additional seat may alter the existing allocations. With an increase of one extra seat, the quota $q_i = h\frac{p_i}{P}$ becomes $\widehat{q}_i = (h+1)\frac{p_i}{P}$. The increase in the quota is p_i/P, which differs across the different states (a larger increase for the more populous states). It is possible that a less populous state that is originally over-rounded becomes under-rounded.

- When the number of states is 2, Alabama paradox will not occur. When a state is favorable (rounded up) at h, it will not be rounded down to the floor value of the original quota at the new house size $h+1$. Though the new seat may be given to the other state, there is no third state that takes the seat by winning in the ranking of fractional remainders.

New States Paradox

If a new state enters, bringing in its complement of new seats, a given state may lose representation to another even though there is no change in either of their population.

Example

In 1907, Oklahoma was added as a new state with 5 new seats to house (386 to 391). Maine's apportionment went up (3 to 4) while New York's went down (38 to 37). This is due to the change in priority order of assigning the surplus seats based on the fractional remainders.

Consider an apportionment of h seats among 3 states, we ask "If the population $\boldsymbol{p} = (p_1 \quad p_2 \quad p_3)$ apportions h seats to $\boldsymbol{a} = (a_1 \quad a_2 \quad a_3)$, is it possible that the population $\boldsymbol{p}' = (p_1 \quad p_2)$ apportions $h - a_3$ seats to $\boldsymbol{a}' = (a_1 + 1 \quad a_2 - 1)$?

Example

Consider the Hamilton apportionment of 4 seats to 2 states whose populations are 623 and 377. Now suppose a new state with population 200 joins the union and the house size is increased to 5.

- Earlier case, $\boldsymbol{q} = (2.49 \quad 1.51)$ so states 1 and 2 each receives 2 seats.

- After the addition of a new state, $\boldsymbol{q} = (2.60 \quad 1.57 \quad 0.83)$. State 2 loses a seat to state 1 since the new apportionment is $(3 \quad 1 \quad 1)$.

Consistency (uniformity)

Let $\boldsymbol{a} = (\boldsymbol{a}^{S_1}, \boldsymbol{a}^{S_2}) = M(\boldsymbol{p}, h)$, where S_1 and S_2 are two subsets of S that partition S. An apportionment method is said to be uniform if $(\boldsymbol{a}^{S_1}, \boldsymbol{a}^{S_2}) = M(\boldsymbol{p}, h)$ would imply $\boldsymbol{a}^{S_1} = M(\boldsymbol{p}^{S_1}, \Sigma_{S_1} a_i)$.

This would mean if a method apportions $\boldsymbol{a}^{S_1}$ to the states in S_1 in the entire problem, then the same method applied to apportioning $h_{S_1} = \Sigma_{S_1} a_i$ seats among the states in S_1 with the same data in the subproblem would give the same result.

Example

Consider the Hamilton apportionment of 100 seats based on the following population data among 5 states.

State	Population	Quota	Number of seats
1	7368	29.578	30
2	1123	4.508	4
3	7532	30.236	30
4	3456	13.873	14
5	5431	21.802	22
total	24910	100	100

Consider the subproblem of assigning 64 seats among the first 3 states.

State	Population	Quota	Number of seats
1	7368	29.429	**29**
2	1123	4.485	**5**
3	7532	30.085	30
total	16023	64	64

Surprisingly, restricting the apportionment problem to a subset of all states does not yield the same seat assignment for the states involved in the subproblem: state 1 loses one seat to state 2.

- The New State Paradox occurs since the apportionment solution changes with the addition of 2 new states: state 4 and state 5.
- A consistent apportionment scheme would not admit the “New States” Paradox.

Population monotonicity

Suppose the population (quota) of a state changes due to redrawing of state boundaries or actual migration of population. Given the fixed values of h and S, if a state's quota increases, then its apportionment does not decrease.

Failure of the population monotone property in Hamilton's method

Suppose a state R_ℓ decreases in population and the excess population is distributed to one state called "lucky" in class D (rounding down) with a larger share of the excess population and another state called "misfortune" in class U (rounding up) with a smaller share. After the redistribution, it is possible that R_ℓ remains in class U, while state "lucky" moves up to class U but state "misfortune" goes down to class D.

Example $h = 32, \quad \boldsymbol{q} = (2.34 \quad 4.88 \quad 8.12 \quad 7.30 \quad 9.36)$

with $\quad \boldsymbol{a} = (2 \quad 5 \quad 8 \quad 7 \quad 10)$.

Population migration from State B to State A and State E lead to

$$\begin{aligned} \boldsymbol{q}_{new} &= (2.42 \quad 4.78 \quad 8.12 \quad 7.30 \quad 9.38) \\ \boldsymbol{a}_{new} &= (3 \quad 5 \quad 8 \quad 7 \quad 9). \end{aligned}$$

State A has a larger share of the migrated population compared to State E, where

$$\begin{aligned} q_A &: \quad 2.34 \to 2.42 \\ q_E &: \quad 9.36 \to 9.38 \\ q_B &: \quad 4.88 \to 4.78. \end{aligned}$$

What has happened to State E? The quota of State E increases but its apportionment decreases.

Quota property (Property Q)

An apportionment method M is said to satisfy the quota property if for every apportionment solution f in M, and any $\boldsymbol{p}$ and h, the resulting apportionment $\boldsymbol{a} = f(\boldsymbol{p}, h)$ satisfies

$$\lfloor q_i \rfloor \leq a_i \leq \lceil q_i \rceil \quad \text{for all } i.$$

Hamilton's method satisfies the Quota Property by its construction. By virtue of the Quota Property, it is impossible for any state to lose more than one seat when the house size is increased by one.

Balinski-Young Impossibility Theorem

Any apportionment method that does not violate the quota rule must produce paradoxes, and any apportionment method that does not produce paradoxes must violate the quota rule.

Lower quota property

M satisfies lower quota if for every $\boldsymbol{p}, h$ and $f \in M$,

$$\boldsymbol{a} \geq \lfloor \boldsymbol{q} \rfloor.$$

Upper quota property

M satisfies upper quota if for every $\boldsymbol{p}, h$ and $f \in M$,

$$\boldsymbol{a} \leq \lceil \boldsymbol{q} \rceil.$$

Relatively well-rounded

If $a_i > q_i + \frac{1}{2}$ (rounded up even when the fractional remainder is less than 0.5), State i is over-rounded. If $a_j < q_j - \frac{1}{2}$ (rounded down even when the fractional remainder is larger than 0.5), State j is under-rounded. If there exists no pair of States i and j with a_i over-rounded and a_j under-rounded, then $\boldsymbol{a}$ is *relatively well-rounded*.

Remarks

1. Hamilton's apportionment satisfies both the quota property and relatively well-rounded property.

2. For $\boldsymbol{q} = (13.3 \; 14.4 \; 17.6 \; 18.7)$, the apportionment $\boldsymbol{a} = (14 \; 14 \; 18 \; 18)$ satisfies the quota property but not the relatively well-rounded property. Note that State 1 is over-rounded while State 4 is under-rounded.

3. The example demonstrated on p.64 under the Webster apportionment reveals the case where the apportionment solution satisfies relatively well-rounded property but not the quota property.

Binary fairness (pairwise switching)

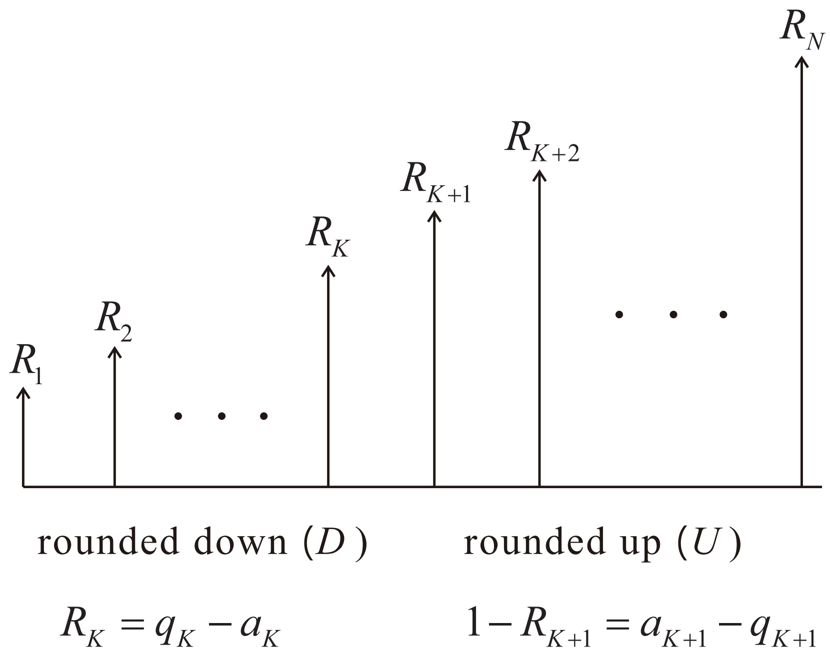
One cannot switch a seat from any state i to any other state j and reduce the sum: $|a_i - q_i| + |a_j - q_j|$.

Hamilton's method, which minimizes $\sum_{i=1}^{S} |a_i - q_i|^p$, $p \geq 1$, does satisfy "binary fairness".

Proof

Among N states, we rank the fractional remainders $R_1, R_2, \ldots, R_N$, K of them are rounded down (set D) and $N-K$ of them are rounded up (Set U). We than have

$$\max_i |a_i - q_i| = \max(R_K, 1 - R_{K+1}).$$



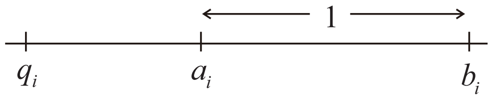
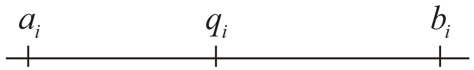
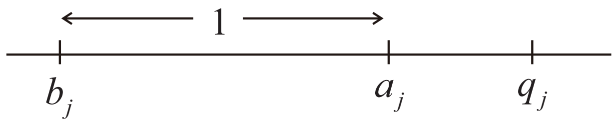
State K is worst off: $|a_K - q_K| = R_K$ while state $K+1$ is best off: $|a_{K+1} - q_{K+1}| = 1 - R_{K+1}$.

Consider another apportionment for the pair of states i and j such that

$$b_i > a_i \quad \text{and} \quad b_j < a_j.$$

The most complicated case corresponds to: state $i \in D$ so $b_i \geq a_i$ and state $j \in U$ so $a_j > b_j$.

State i

<table>
<tr><td></td><td></td><td>U</td><td>D</td></tr>
<tr><td>State j</td><td>U</td><td>$|b_i - q_i| > 1 > \max_i |a_i - q_i|$
</td><td>
$|b_i - q_i| = 1 - R_i > 1 - R_K > 1 - R_{K+1}$
$|b_i - q_i| = R_j > R_{K+1} > R_K$
so that
$\max(|b_i - q_i|, |b_j - q_j|) > \max(R_K, 1 - R_{K+1}) = \max_i |a_i - q_i|$</td></tr>
<tr><td></td><td>D</td><td>$|b_i - q_i| > 1$ and $|b_j - q_j| > 1$</td><td>$|b_j - q_j| > 1 > \max_i |a_i - q_i|$
</td></tr>
</table>

Summary of Hamilton's method

Assuming no minimum requirement:

- Every state is assigned at least its lower quota. Order the fractional remainders. Assign the extra seats to those states with larger values of fractional remainder.

- Minimize $\sum_{i=1}^{S} |a_i - q_i|^p$ subject to $\sum_{i=1}^{S} a_i = h$.

- Satisfying the quota property: the quota q_i of each state is either rounded up or rounded down to give a_i.

- Binary fairness

- $\min_a \max_i |a_i - q_i|$

Paradoxes House Monotone; New State Paradox; Population Monotone

4.3 Divisor methods

Based on the idea of an *ideal district size* or common divisor, a divisor λ is specified, where λ is an approximation to the theoretical population size per seat $\overline{\lambda} = P/h$. Some rounding of the numbers p_i/λ are used to determine a_i, whose sum equals h. This class of methods are called the divisor methods.

Jefferson's method (used by the US Congress from 1794 through 1832)

Let $\lfloor\lfloor x \rfloor\rfloor$ be the greatest integer less than x if x is non-integer, and otherwise be equal to x or $x-1$. For example, $\lfloor\lfloor 4 \rfloor\rfloor$ can be equal to 4 or 3.

For a given $h, \overline{\lambda} =$ average size $= \sum_{i=1}^{S} p_i/h$ and $q_i = \dfrac{p_i}{\overline{\lambda}}$. We choose $\lambda\,(\leq \overline{\lambda})$ such that $\displaystyle\sum_{i=1}^{S} \left\lfloor\left\lfloor \frac{p_i}{\lambda} \right\rfloor\right\rfloor = h$ has a solution.

To meet the requirement of giving at least one representative to each state, we take $a_i = \max\left(1, \left\lfloor\left\lfloor \frac{p_i}{\lambda} \right\rfloor\right\rfloor\right)$, where λ is a positive number chosen so that $\sum_{i=1}^{S} a_i = h$. Here, λ is a quantity that is close to $\overline{\lambda} =$ average population represented by a single representative.

The generalized floor $\lfloor\lfloor\ \rfloor\rfloor$ provides flexibility to avoid potential "nonsolvability". This occurs when the sum may jump from $h-1$ to $h+1$ without hitting h as λ decreases continuously.

- The more populous state is favored over the less populous state in Jefferson's apportionment. For example, in 1794 apportionment in which $h = 105$, Virginia with $q = 18.310$ was rewarded with 19 seats while Delaware with $q = 1.613$ was given only one seat. This demonstrates the failure of the relatively well-rounded property. The method was challenged due to its violation of the quota property, which was then replaced by another divisor method (Webster's method) in 1842.

Adams Method

Alternatively, one might consider finding apportionment by *rounding up*. Let $\lceil\lceil x \rceil\rceil$ be the smallest integer greater than x if x is not an integer, and otherwise equal to x or $x+1$. Choose $\lambda\,(\geq \overline{\lambda})$ such that

$$\sum_{i=1}^{S} \lceil\lceil p_i/\lambda \rceil\rceil = h$$

can be obtained, then apportionment for h can be found by taking

$$a_i = \lceil\lceil p_i/\lambda \rceil\rceil$$

satisfying $\sum_{i=1}^{S} a_i = h$. This is called the Adams method.

Since all quota values are rounded up, the Adams method guarantees at least one seat for every state. The Adams method favors the smaller state (just opposite to that of the Jefferson method).

Lemma on the Jefferson apportionment

Given $\boldsymbol{p}$ and $h, \boldsymbol{a}(a_1 \cdots a_S)$ is a Jefferson apportionment for h if and only if

$$\max_i \frac{p_i}{a_i + 1} \leq \min_i \frac{p_i}{a_i}. \qquad (A)$$

Proof

By definition, $a_i = \lfloor\lfloor p_i/\lambda \rfloor\rfloor$. When $\frac{p_i}{\lambda}$ is not an integer, then $a_i + 1 > \frac{p_i}{\lambda} > a_i$. When $\frac{p_i}{\lambda}$ is an integer, then a_i equals either $\frac{p_i}{\lambda}$ or $\frac{p_i}{\lambda} - 1$. Combining the results, we have

$$a_i + 1 \geq \frac{p_i}{\lambda} \geq a_i \Leftrightarrow \frac{p_i}{a_i + 1} \leq \lambda \leq \frac{p_i}{a_i} \text{ for all } i,$$

(if $a_i = 0, p_i/a_i = \infty$). The inequality remains valid when we take minimum among p_i/a_i and maximum among $p_i/(a_i+1)$ for all states. We then deduce that

$$\max_i \frac{p_i}{a_i + 1} \leq \min_i \frac{p_i}{a_i}.$$

Interpretation of the Lemma

Recall that the smaller value of p_i/a_i (district size = population size represented by each seat) the better for that state. Alternatively, state i is better off than another state j if $\dfrac{p_i}{a_i} < \dfrac{p_j}{a_j}$.

- To any state k, assignment of an additional seat would make it to become the best off state among all states since

$$\frac{p_k}{a_k^{new}} = \frac{p_k}{a_k + 1} \le \max_i \frac{p_i}{a_i + 1} \le \min_i \frac{p_i}{a_i} \le \min_{i \neq k} \frac{p_i}{a_i}.$$

- Though there may be inequity among states as measured by their shares of p_i/a_i, the "unfairness" is limited to less than one seat (the assignment of one extra seat makes that state to become the best off).

Jefferson's lower quota property and Adams' upper quota property

Jefferson apportionment satisfies the lower quota property. Suppose not, there exists $\boldsymbol{a}$ for h such that $a_i < \lfloor q_i \rfloor$ or $a_i \leq q_i - 1$. For some state $j \neq i$, we have $a_j > q_j$. Recall $q_i = p_i/\overline{\lambda}$ and $q_j = p_j/\overline{\lambda}$ so that

$$\frac{p_j}{a_j} < \overline{\lambda} \leq \frac{p_i}{a_i + 1},$$

a contradiction to the Lemma. However, it does not satisfy the upper quota property (historical apportionment in 1832, where New York State was awarded 40 seats with quota of 38.59 only).

In a similar manner, the Adams method satisfies

$$\max_i \frac{p_i}{a_i} \leq \min_i \frac{p_i}{a_i - 1} \text{ for } a_i \geq 1.$$

Based on this inequality, it can be shown that it satisfies the upper quota property. Similarly, the Adams method does not satisfy the lower quota property.

Recursive scheme of Jefferson's apportionment

The set of Jefferson solutions is obtained recursively as follows:

(i) $\boldsymbol{f}(\boldsymbol{p}, 0) = \boldsymbol{0}$;

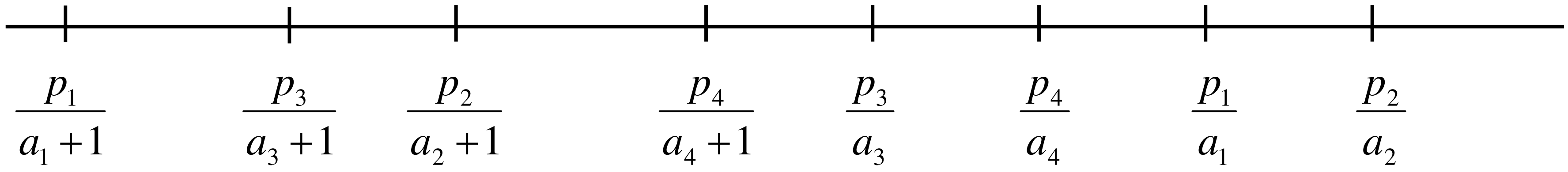
(ii) if $a_i = f_i(\boldsymbol{p}, h)$ is an apportionment to state i for house size h, let k be some state for which $\dfrac{p_k}{a_k + 1} = \max_i \dfrac{p_i}{a_i + 1}$, then

$$f_k(\boldsymbol{p}, h+1) = a_k + 1 \quad \text{and} \quad f_i(\boldsymbol{p}, h+1) = a_i \text{ for } i \neq k.$$

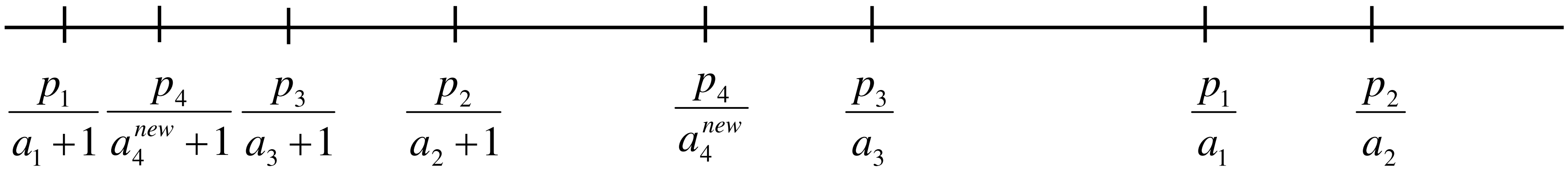
Remark

The above algorithm dictates how the additional seat is distributed while other allocations remain the same. Hence, house monotone property of the Jefferson apportionment is automatically observed.

Consider the case $S = 4$, we rank $\dfrac{p_i}{a_i + 1}$, $i = 1, 2, 3, 4$.



Since $\dfrac{p_i}{a_i + 1}$ is maximized at $i = 4$, we assign the extra seat to State 4. Now, $a_4^{new} = a_4^{old} + 1$.



After one seat has been assigned to State 4, $\frac{p_i}{a_i+1}$ is maximized at $i = 2$. Next, we assign the extra seat to State 2.

Note that State 4 is the most eligible state since $\frac{p_4}{a_4+1}$ is the largest among $\frac{p_i}{a_i+1}$, $i = 1, 2, 3, 4$. Once an additional seat is allocated to State 4, $\frac{p_4}{a_4^{\text{new}}}$ becomes the best off state since it is the smallest among $\frac{p_i}{a_i}$, $i = 1, 2, 3, 4$.

Imagine that if the additional seat is signed to State 2 even $\frac{p_2}{a_2+1} < \frac{p_4}{a_4+1}$ so that $a_2^{\text{new}} = a_2 + 1$. Now, we have

$$\frac{p_2}{a_2^{\text{new}}} < \frac{p_4}{a_4+1}.$$

This is a violation of the requirement that $\frac{p_2}{a_2^{\text{new}}}$ cannot be smaller than $\frac{p_4}{a_4+1}$ (inequality A).

Webster's method (first adopted in 1842, replacing Jefferson's method but later replaced by Hill's method in 1942)

For any real number z, whose fractional part is not $\frac{1}{2}$, let $[z]$ be the integer closest to z. If the fractional part of z is $\frac{1}{2}$, then $[z]$ has two possible values.

The Webster Method is

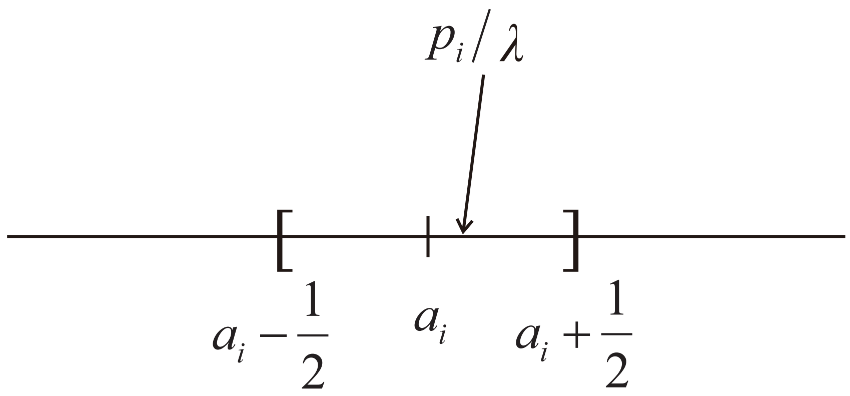
$$f(\boldsymbol{p}, h) = \{\boldsymbol{a} : a_i = [p_i/\lambda], \sum_{i=1}^{S} a_i = h \text{ for some positive } \lambda\}.$$

It can be shown that λ satisfies

$$\max_{a_i \geq 0} \frac{p_i}{a_i + \frac{1}{2}} \leq \lambda \leq \min_{a_i > 0} \frac{p_i}{a_i - \frac{1}{2}}.$$

This is obvious from the property that

$$a_i + \frac{1}{2} \geq \frac{p_i}{\lambda} \geq a_i - \frac{1}{2} \text{ for all } i.$$



The special case $a_i = 0$ has to be ruled out in the right side inequality since $a_i - \dfrac{1}{2}$ becomes negative when $a_i = 0$.

Violation of upper quota

1. Violation of the upper quota by both Jefferson's and Webster's Methods

State i	$p_i = 100q_i$	$\lfloor q_i \rfloor$	$\lceil q_i \rceil$	Ham	Jeff	Web
1	8785	87	88	88	90	90
2	126	1	2	2	1	1
3	125	1	2	2	1	1
4	124	1	2	1	1	1
5	123	1	2	1	1	1
6	122	1	2	1	1	1
7	121	1	2	1	1	1
8	120	1	2	1	1	1
9	119	1	2	1	1	1
10	118	1	2	1	1	1
11	117	1	2	1	1	1
$\sum$	$10,000$	97	108	100	100	100

Here, $h = 100$ and $P = 10,000$, so the average district size is 100.

Violation of lower quota

2. Violation of the lower quota by Webster's Method

State i	$p_i = 100q_i$	$\lfloor q_i \rfloor$	$\lceil q_i \rceil$	Ham	Jeff	Web
1	9215	92	93	92	95	90
2	159	1	2	2	1	2
3	158	1	2	2	1	2
4	157	1	2	2	1	2
5	156	1	2	1	1	2
6	155	1	2	1	1	2
$\sum$	$10,000$	97	103	100	100	100

The 100$^{\text{th}}$ seat is allocated to state 6 under Webster's apportionment since $101.82 = \frac{9215}{90+0.5} = \frac{9215}{90.5} < \frac{155}{1.5} = \frac{155}{1+0.5} = 103.3$.

Relatively well-rounded property *(see Qn 9 in HW 4)*

Webster's method can never produce an apportionment that rounds up for q_i for one state i with $q_i - \lfloor q_i \rfloor < 0.5$ while rounding down q_j for another state j with $q_j - \lfloor q_j \rfloor > 0.5$.

Integer programming formulation of Webster's Method

Recall that $\frac{a_i}{p_i}$ gives the per capital representation of state $i, i = 1, \cdots, S$; and the ideal per capital representation is h/P. Consider the sum of squared difference of $\frac{a_i}{p_i}$ to $\frac{h}{P}$ weighted by p_i

$$\overline{s} = \sum_{i=1}^{S} p_i \left(\frac{a_i}{p_i} - \frac{h}{P} \right)^2 = \sum_{i=1}^{S} \frac{a_i^2}{p_i} - \frac{h^2}{P}.$$

Webster's method: minimizes $\overline{s}$ subject to $\sum_{i=1}^{S} a_i = h$.

Proof

Suppose $\boldsymbol{a}$ is a Webster apportionment solution, then it satisfies the min-max property:

$$\max_{a_i \geq 0} \frac{p_i}{a_i + \frac{1}{2}} \leq \lambda \leq \min_{a_i > 0} \frac{p_i}{a_i - \frac{1}{2}}.$$

It suffices to show that if an apportionment has been made under the Webster scheme, then an interchange of a single seat between any 2 states r and s cannot reduce $\overline{s}$.

We prove by contradiction. Suppose such an interchange is possible in reducing $\overline{s}$, where $a_r > 0$ and $a_s \geq 0$, then this implies that (all other allocations are kept the same)

$$\frac{(a_r - 1)^2}{p_r} + \frac{(a_s + 1)^2}{p_s} < \frac{a_r^2}{p_r} + \frac{a_s^2}{p_s}$$

$$\Leftrightarrow \quad \frac{p_r}{a_r - \frac{1}{2}} < \frac{p_s}{a_s + \frac{1}{2}}.$$

This is a violation to the min-max property. Therefore, the Webster apportionment minimizes $\overline{s}$ subject to $\sum_{i=1}^{S} a_i = h$.

Generalized formulation of the divisor method

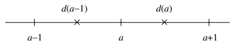
Any rounding procedure can be described by specifying a dividing point $d(a)$ in each interval $[a, a+1]$ for each non-negative integer a.

Any monotone increasing $d(a)$ defined for all integers $a \geq 0$ and satisfying

$$a \leq d(a) \leq a+1$$

is called a divisor criterion.

For any positive real number z, a d-rounding of z (denoted by $[z]_d$) is an integer a such that $d(a-1) \leq z \leq d(a)$. This is unique unless $z = d(a)$, in which case it takes on either a or $a+1$.



- For example, Webster's $d(a) = a + \frac{1}{2}$. Suppose z lies in $(2.5, 3.5)$, it is rounded to 3. When $z = 3.5$, it can be either rounded to 3 or 4.

- Also, Jefferson's $d(a) = a + 1$ (Greatest Divisor Method) while Adams' $d(a) = a$ (Smallest Divisor Method). For Jefferson's method, if $a < z < a + 1$, then $[z]_d = a$. When $z = a + 1$, then $[z]_d$ can be either a or $a + 1$. For example, when $z = 3.8$, then $d(2) \leq z \leq d(3) = 4$, so $[3.8]_d = 3$; when $z = 4 = 3 + 1$, then $a = 3$ and $[4]_d = 3$ or 4.

The divisor method based on d is

$$M(\boldsymbol{p}, h) = \left\{ \boldsymbol{a} : a_i = [p_i/\lambda]_d \quad \text{and} \quad \sum_{i=1}^{S} a_i = h \text{ for some positive } \lambda \right\}.$$

In terms of the min-max inequality:

$$M(\boldsymbol{p}, h) = \left\{ \boldsymbol{a} : \min_{a_i > 0} \frac{p_i}{d(a_i - 1)} \geq \max_{a_j \geq 0} \frac{p_j}{d(a_j)}, \quad \sum_{i=1}^{S} a_i = h \right\}.$$

This is a consequence of $d(a_i - 1) \leq \frac{p_i}{\lambda} \leq d(a_i)$ when $a_i = \left[\frac{p_i}{\lambda}\right]_d$. We exclude $a_i = 0$ in the left inequality since $d(a_i - 1)$ is in general negative when $a_i = 0$.

The divisor method M based on d may be defined recursively as:

(i) $M(\boldsymbol{p}, 0) = \boldsymbol{0}$,

(ii) if $\boldsymbol{a} \in M(\boldsymbol{p}, h)$ and k satisfies

$$\frac{p_k}{d(a_k)} = \max_i \frac{p_i}{d(a_i)},$$

then $\boldsymbol{b} \in M(\boldsymbol{p}, h+1)$, with $b_k = a_k + 1$ and $b_i = a_i$ for $i \neq k$.

Dean's method (Harmonic Mean Method)

The i^{th} state receives a_i seats where p_i/a_i is closer to the common divisor λ when compared to $\frac{p_i}{a_i+1}$ and $\frac{p_i}{a_i-1}$. For all i, we have

$$\frac{p_i}{a_i} - \lambda \leq \lambda - \frac{p_i}{a_i+1} \quad \text{and} \quad \lambda - \frac{p_i}{a_i} \leq \frac{p_i}{a_i-1} - \lambda$$

which simplifies to

$$\frac{a_i + \frac{1}{2}}{a_i(a_i+1)} p_i \leq \lambda \leq \frac{a_i - \frac{1}{2}}{a_i(a_i-1)} p_i \text{ for all } i.$$

Define $d(a) = \frac{a(a+1)}{a+\frac{1}{2}} = \frac{1}{\frac{1}{2}\left(\frac{1}{a} + \frac{1}{a+1}\right)}$ (harmonic mean of consecutive integers a and $a+1$), then

$$\max_i \frac{p_i}{d(a_i)} \leq \lambda \leq \min_j \frac{p_j}{d(a_j-1)}.$$

Hill's method (Equal Proportions Method)

- Besides the Harmonic Mean, where $\frac{1}{d(a)} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{a+1}\right)$ (Dean's method) and the Arithmetic Mean $d(a) = \frac{1}{2}(a+a+1)$ (Webster's method), the choice of the Geometric Mean $d(a) = \sqrt{a(a+1)}$ leads to the Equal Proportions method (also called Hill's method).

- For a population p_i and common divisor λ, suppose p_i/λ falls within $[a, a+1]$, then p_i/λ is rounded up to $a+1$ seats if $p_i/\lambda > d(a) = \sqrt{a(a+1)}$ and rounded down to a seats if $p_i/\lambda < d(a) = \sqrt{a(a+1)}$. If $p_i/\lambda = \sqrt{a(a+1)}$, the rounding is not unambiguously defined.

MF: Major Fractions (Webster's method)

EP: Equal Proportion (Hill's method)

HM: Harmonic Mean (Dean's method)

State	p_i	q_i	a_i for Method GR	SD	HM	EP	MF	GD
1	91,490	91.490	92	88	89	90	93	94
2	1,660	1.660	2	2	2	2	2	1
3	1,460	1.460	2	2	2	2	1	1
4	1,450	1.450	1	2	2	2	1	1
5	1,440	1.440	1	2	2	2	1	1
6	1,400	1.400	1	2	2	1	1	1
7	1,100	1.100	1	2	1	1	1	1
Totals	100,000	100	100	100	100	100	100	100
Min λ				1,040	1,023	1,011	979	964
Max λ				1,051	1,033	1,018	989	973

Allocations for the six divisor methods with $S = 100$. Note that $\overline{\lambda} = \frac{100{,}000}{100} = 1,000$. The minimum and maximum integer values of λ which yield these allocations are also shown.

Huntington approach: Pairwise comparison of inequity

- Consider the ratio $p_i/a_i =$ average number of constituents per seat (district size) in state i, the ideal case would be the same value of p_i/a_i for all states. Between any 2 states, there will always be certain inequity which gives one of the states a slight advantage over the other. For a population $\boldsymbol{p} = (p_1, p_2, \cdots, p_S)$ and an apportionment $(a_1, a_2, \cdots, a_S)$ for House size h, if $p_i/a_i > p_j/a_j$, then state j is "better off" than state i in terms of district size.

- How is the "amount of inequity" between 2 states measured? Some possible choices of measure of inequity are:

(i) $\left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right|$, (ii) $\left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right| \Big/ \min\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right)$, (iii) $\left|\frac{a_i}{p_i} - \frac{a_j}{p_j}\right|$,

(iv) $\left|a_i - a_j\frac{p_i}{p_j}\right|$, (v) $\left|a_i\frac{p_j}{p_i} - a_j\right|$.

Huntington's rule

A transfer is made from the more favored state to the less favored state if this reduces this measure of inequity.

- An apportionment is stable in the sense that no inequity, computed according to the chosen measure, can be reduced by transferring one seat from a better off state to a less well off state.

Huntington considered 64 cases involving the relative and absolute differences and ratios involving the 4 parameters p_i, a_i, p_j, a_j for a pair of states i and j. He arrived at 5 different apportionment methods.

- Some schemes are "unworkable" in the sense that the pairwise comparison approach would not in general converge to an overall minimum – successive pairwise improvements could lead to cycling.

Hill's method (Method of Equal Proportions) revisited

Hill's method has been used to apportion the House since 1942.

Let $T_{ij}\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right)$ be the relative difference between $\frac{p_i}{a_i}$ and $\frac{p_j}{a_j}$, defined by

$$T_{ij}\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right) = \left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right| \Big/ \min\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right).$$

The ideal situation is $T_{ij} = 0$ for all pairs of i and j.

Suppose $\frac{a_i}{p_i} > \frac{a_j}{p_j}$, state i is better off, then

$$T_{ij} = \left(\frac{p_j}{a_j} - \frac{p_i}{a_i}\right) \Big/ \frac{p_i}{a_i} = \frac{a_i/p_i}{a_j/p_j} - 1.$$

Lemma on Hill's method

Between two states i and j, we consider (i) $a_i + 1$ and a_j to be a better assignment than (ii) a_i and $a_j + 1$

if and only if $\frac{p_i}{\sqrt{a_i(a_i+1)}} > \frac{p_j}{\sqrt{a_j(a_j+1)}}$.

Remark

With an additional seat, should it be assigned to State i with a_i seats or State j with a_j seats? The decision factor is to compare

$$\frac{p_i}{\sqrt{a_i(a_i+1)}} \quad \text{and} \quad \frac{p_j}{\sqrt{a_j(a_j+1)}}.$$

The one with a higher rank index value $r(p,a) = \frac{p}{\sqrt{a(a+1)}}$ should receive the additional seat.

Proof

Suppose that when State i has $a_i + 1$ seats and State j has a_j seats, State i is the more favored state i.e.

$$\frac{p_j}{a_j} - \frac{p_i}{a_i + 1} > 0;$$

while when State i has a_i seats and State j has $a_j + 1$ seats, State j is the more favored state i.e.

$$\frac{p_i}{a_i} - \frac{p_j}{a_j + 1} > 0.$$

Should we transfer one seat in assignment (ii) from State j to State i so that assignment (i) is resulted?

Based on the Huntington rule and the given choice of inequity measure for the Hill methods, Assignment (i) is a better assignment than (ii) if and only if

$$T_{ij}\left(\frac{p_i}{a_i+1},\frac{p_j}{a_j}\right) < T_{ij}\left(\frac{p_i}{a_i},\frac{p_j}{a_j+1}\right)$$

$$\begin{aligned}
\Leftrightarrow \quad & \frac{p_j/a_j - p_i/(a_i+1)}{p_i/(a_i+1)} < \frac{p_i/a_i - p_j/(a_j+1)}{p_j/(a_j+1)} \\
\Leftrightarrow \quad & \frac{p_j(a_i+1) - p_i a_j}{p_i a_j} < \frac{p_i(a_j+1) - p_j a_i}{p_j a_i} \\
\Leftrightarrow \quad & \frac{p_j^2}{a_j(a_j+1)} < \frac{p_i^2}{a_i(a_i+1)}.
\end{aligned}$$

That is, the measure of inequity as quantified by T_{ij} of the Hill method is reduced.

Algorithm for Hill's method based on rank indexes

Compute the rank indexes $\dfrac{p_i}{\sqrt{n(n+1)}}$ for all i starting with $n = 1$ and then assign the seats in turn to the largest such numbers.

Floodland	Galeland	Hailland	Snowland	Rainland
$\frac{9061}{\sqrt{1 \cdot 2}}$	$\frac{7179}{\sqrt{1 \cdot 2}}$	$\frac{5259}{\sqrt{1 \cdot 2}}$	$\mathbf{\frac{3319}{\sqrt{1 \cdot 2}}}$	$\frac{1182}{\sqrt{1 \cdot 2}}$
$\frac{9061}{\sqrt{2 \cdot 3}}$	$\frac{7179}{\sqrt{2 \cdot 3}}$	$\frac{5259}{\sqrt{2 \cdot 3}}$	$\frac{3319}{\sqrt{2 \cdot 3}}$	$\frac{1182}{\sqrt{2 \cdot 3}}$
$\mathbf{\frac{9061}{\sqrt{3 \cdot 4}}}$	$\frac{7179}{\sqrt{3 \cdot 4}}$	...	...	...
...	...	...	...	...

Five seats have already been allocated (one to each state)

Comparing (i) Floodland with 4 seats and Snowland with 1 seat, against (ii) Floodland with 3 seats and Snowland with 2 seats, since $9061/\sqrt{3 \cdot 4} = 2616 > 3319/\sqrt{1 \cdot 2} = 2347$, so assignment (i) is better than assignment (ii).

Floodland	Galeland	Hailland	Snowland	Rainland
6407 - 6	5076 - 7	3719 - 8	**2347 – 12**	836
3699 - 9	2931 - 10	2147 - 13	1355 - 20	483
2616 – 11	2072 - 14	1518 - 18	958 - 27	...
2026 - 15	1605 - 17	1176 - 23	742	...
1658 - 16	1311 - 21	960 - 26	...	...
1401 - 19	1108 - 24	811	...	...
1211 - 22	959	...	...	...
1070 - 25	846	...	...	...

Remarks on the rank index

- Since the ranking function $\frac{p}{\sqrt{n(n+1)}}$ equal ∞ for $n = 0$, this method automatically gives each state at least one seat if $h \geq S$, so the minimum requirement of at least one seat for each state is always satisfied.

- If a tie occurs between states with unequal populations (extremely unlikely), Huntington suggests that it be broken in favor of the larger state.

- It does not satisfy the quota property. Actually, it can violate both lower and upper quota.

- The Hungtinton approach to the apportionment makes use of "local" measures of inequity.

Pairwise comparison using $\left|\frac{a_i}{p_i} - \frac{a_j}{p_j}\right|$, *Webster's method revisited*

Give to each state a number of seats so that no transfer of any seat can reduce the difference in per capita representation between those states.

That is, supposing that State i is favored over State j, $\frac{a_i}{p_i} > \frac{a_j}{p_j}$, no transfer of one seat from the more favored state to the less favored state will be made if

$$\frac{a_i}{p_i} - \frac{a_j}{p_j} \leq \frac{a_j + 1}{p_j} - \frac{a_i - 1}{p_i}$$

for all i and j.

This simplifies to

$$\begin{aligned} a_i p_j - p_i a_j &\le p_i(a_j + 1) - p_j(a_i - 1) \\ \frac{p_j}{a_j + \frac{1}{2}} &\le \frac{p_i}{a_i - \frac{1}{2}}. \end{aligned}$$

The above inequality is satisfied by any pair of states when the apportionment has been settled under the assumed pairwise inequity comparison scheme.

We can deduce the following min-max inequality:

$$\max_{all\ a_j} \frac{p_j}{a_j + \frac{1}{2}} \le \min_{a_i > 0} \frac{p_i}{a_i - \frac{1}{2}},$$

which takes the same form as that of the Webster method.

Five traditional divisor methods

Method	*Alternative name*	Divisor $d(a)$	Pairwise comparison $\left(\frac{a_i}{p_i} > \frac{a_j}{p_j}\right)$	Adoption by US Congress
Adams	Smallest divisors	a	$a_i - a_j \frac{p_i}{p_j}$	–
Dean	Harmonic means	$\frac{a(a+1)}{a+\frac{1}{2}}$	$\frac{p_j}{a_j} - \frac{p_i}{a_i}$	–
Hill	Equal propor-tions	$\sqrt{a(a+1)}$	$\frac{a_i/p_i}{a_j/p_j} - 1$	1942 to now
Webster	Major Fractions	$a + \frac{1}{2}$	$\frac{a_i}{p_i} - \frac{a_j}{p_j}$	1842; 1912; 1932*
Jefferson	Largest divisors	$a + 1$	$a_i \frac{p_j}{p_i} - a_j$	1794 to 1832

Resolution on the five divisor methods

A National Academy of Sciences Committee issued a report in 1929. The report considered the 5 divisor methods and focused on the pairwise comparison tests. The Committee adopted Huntington's reasoning that the Equal Proportions Method is preferred (the Method occupies mathematically a neutral position with respect to emphasis on larger and smaller states.)

Key result

The divisor method based on $d(a)$ is equivalent to the Hungtington method based on the rank index $r(p, a) = p/d(a)$.

$p_i/a_i =$ average district size of state i;
$a_i/p_i =$ per capita share of a representative of state i

- **Dean's Method** – absolute difference in average district sizes: $\left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right|$
- **Webster's Method** – absolute difference in per capita shares of a representative: $\left|\frac{a_i}{p_i} - \frac{a_j}{p_j}\right|$
- **Hill's Method** – relative differences in both district sizes and shares of a representative: $\left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right| \Big/ \min\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right)$
- **Adams' Method** – absolute representative surplus: $a_i - \frac{p_i}{p_j}a_j$ is the amount by which the allocation for state i exceeds the number of seats it would have if its allocation was directly proportional to the actual allocation for state j
- **Jefferson's Method** – absolute representation deficiency: $\frac{p_j}{p_i}a_i - a_j$

Minimum and maximum apportionment requirements

In order that every state receives at least one representative, it is necessary to have $d(0) = 0$ (assuming $p_i/0 > p_j/0$ for $p_i > p_j$ so that a state with a larger population will be assigned the first seat earlier). While the Adams, Hill and Dean methods all satisfy this property, we need to modify the Webster $\left[d(a) = a + \frac{1}{2}\right]$ and Jefferson Method $[d(a) = a + 1]$ by setting $d(0) = 0$ in the special case $a = 0$.

A divisor method M based on d for problems with both minimum requirement $\boldsymbol{r}^{min}$ and maximum requirement $\boldsymbol{r}^{max}, \boldsymbol{r}^{min} \leq \boldsymbol{r}^{max}$, can be formulated as

$$M(\boldsymbol{p}, h) = \left\{\boldsymbol{a} : a_i = mid\left(r_i^{\text{min}}, r_i^{max}, [p_i/\lambda]_d\right) \right.$$
$$\left. \text{and } \sum_{i=1}^{S} a_i = h \text{ for some positive } \lambda \right\}.$$

Here, $mid(u, v, w)$ is the middle in value of the three numbers u, v and w.

Balinski-Young Impossibility Theorem

- Divisor methods automatically satisfy the House Monotone Property.

- An apportionment method is uniform and population monotone if and only if it is a divisor method.

  The proof is highly technical.

- Divisor methods are known to produce violation of the quota property.

Conclusion It is impossible for an apportionment method that always satisfies quota and be incapable of producing paradoxes.

Let $r(p,a)$ be any real valued function of two real variables called a *rank-index*, satisfying $r(p,a) > r(p,a+1) \geq 0$, and $r(p,a)$ can be plus infinity. Given a rank-index, a Huntington Method M of apportionment is the set of solutions obtained recursively as follows:

(i) $f_i(\boldsymbol{p},0) = 0, \quad 1 \leq i \leq S$;

(ii) If $a_i = f_i(\boldsymbol{p},h)$ is an apportionment for h of M, and k is some state for which

$$r(p_k,a_k) \geq r(p_i,a_i) \quad \text{for} \quad 1 \leq i \leq S,$$

then

$$f_k(\boldsymbol{p},h+1) = a_k + 1 \quad \text{and} \quad f_i(\boldsymbol{p},h+1) = a_i \quad \text{for} \quad i \neq k.$$

The Huntington method based on $r(p,a)$ is

$$M(\boldsymbol{p},h) \;=\; \left\{\boldsymbol{a} \geq 0 : \sum_{i=1}^{S} a_i = h, \max_i r(p_i,a_i) \leq \min_{a_j>0} r(p_j,a_j-1)\right\}.$$

Theorem – Quota properties of Huntington family of methods

There exists no Huntington method satisfying quota. Of these five "known workable" method, only the Smallest Divisors Method satisfies upper quota and only the Jefferson Method satisfies lower quota.

			Apportionment for 36				
Party	Votes received	Exact quota	**SD**	**HM**	**EP**	**W**	**J**
A	$27,744$	9.988	10	10	10	10	11
B	$25,178$	9.064	9	9	9	9	9
C	$19,947$	7.181	7	7	7	8	7
D	$14,614$	5.261	5	5	6	5	5
E	$9,225$	3.321	3	4	3	3	3
F	$3,292$	1.185	2	1	1	1	1
	$100,000$	$36,000$	36	36	36	36	36

Quota Method

Uses the same rule as in the Jefferson method to determine which state receives the next seat, but rules this state ineligible if it will violate the upper quota. Recall that the Jefferson method satisfies the lower quota property.

Definition of eligibility

If f is an apportionment solution and $f_i(\boldsymbol{p}, h) = a_i$ and $q_i(\boldsymbol{p}, h)$ denotes the quota of the i^{th} state, then state i is eligible at $h+1$ for its $(a_i + 1)^{\text{st}}$ seat if $a_i < q_i(\boldsymbol{p}, h+1) = (h+1)p_i/P$. Write

$$E(\boldsymbol{a}, h+1) = \{i \in N_s : i \text{ is eligible for } a_i + 1 \text{ at } h+1\}.$$

Algorithm

The quota method consists of all apportionment solutions $f(\boldsymbol{p}, h)$ such that

$$f(\boldsymbol{p}, 0) = 0 \quad \text{for all } i$$

and if $k \in E(\boldsymbol{a}, h+1)$ and

$$\frac{p_k}{a_k + 1} \geq \frac{p_j}{a_j + 1} \text{ for all } j \in E(\boldsymbol{a}, h+1),$$

then

$$\begin{aligned} f_k(\boldsymbol{p}, h+1) &= a_k + 1 \quad \text{for one such } k \text{ and} \\ f_i(\boldsymbol{p}, h+1) &= a_i \quad \text{for all } i \neq k. \end{aligned}$$

As a result, no state will be assigned seats that are above the upper quota.

History of the US House apportionment

- The first apportionment occurred in 1794, based on the population figures* from the first national census in 1790. Congress needed to allocate exactly 105 seats in the House of Representatives to the 15 states.

- Hamilton's method was approved by Congress in 1791, but the bill was vetoed by President George Washington (first use of presidential veto).

- Washington's home state, Virginia, was one of the losers in the method, receiving 18 seats despite a standard quota of 18.310.

- The Jefferson apportionment method was eventually adopted and gave Virginia 19 seats.

*The population figures did not fully include the number of slaves and native Americans who lived in the U.S. in 1790.

- Jefferson's method is a divisor method, which may not satisfy the quota property. The year 1832 was the last use of Jefferson's method. If Jefferson's method has continued to be used, every apportionment of the House since 1852 would have violated quota. In 1832, Jefferson's method gave New York 40 seats in the House even though its standard quota was only 38.59.

- Websters' method, another but improved divisor method (regarded as the best apportionment method by modern day experts), was used for the apportionment of 1842. The method may violate quota, but the chance is very slim. If Webster's method has been used consistently from the first apportionment of the House in 1794 to the most recent reapportionment in 2012, it would still have yet to produce a quota violation.

- The very possibility of violating quota lead Congress leery of Webster's method. In 1850, Congressman Samuel Vinton proposed what be thought was a brand new method (actually identical to Hamilton's method). In 1852, Congress passed a law adopting Vinton's method.

- *Compromise adopted in 1852*

  In 1852, and future years, Congress would increase the total number of seats in the House to a number for which Hamilton's and Webster's method would yield identical apportionment.

- A major deficiency in Hamilton's method is the loss of *House Monotone* property. Such paradox occurred in 1882 and 1902. In 1882, US Congress opted to go with a House size of 325 seats to avoid the Alabama paradox. Another similar case occurred in 1902 (final death blow to Hamilton's method) lead Congress to adopt Webster's method with a total House size of 386 seats.

Debate between Webster's and Hill's methods

- In 1922 apportionment, the two methods produced significantly different outcomes. By this time, the number of seats in the House had been fixed by law. Consequently, the 1912 seat totals were held over without any reapportionment whatsover.

- In 1932 apportionment, Webster's and Hill's methods produced identical apportionment.

- For the 1942 apportionment, Webster's and Hill's method came very close except that Hill's method gave an extra seat to Arkansas at the expense of Michigan. Democrats favored Hill's since Arkansas tended to vote for Democrats. As the Democrat-s had the majority, it was Hill's method that passed through Congress. President Franklin Roosevelt (Democrat) signed the method into "permanent" law and it has been used ever since.

Court challenges

- In 1991, for the first time in US history, the constitutionality of an apportionment method was challenged in court, by Montana and Massachusetts in separate cases.

  - Montana proposed two methods as alternatives to EP (current method). Both HM and SD give Montana 2 seats instead of the single seat allocated by EP, but would not have increased Massachusetts' EP allocation of 10 seats. [Favoring small states.]
  - Massachusetts proposed MF, which would have allocated 11 seats to Massachusetts, and 1 to Montana. [Favoring medium states.]

"Apportionment Methods for the House of Representatives and the Court Challenges", by Lawrence R. Ernst, *Management Science*, vol. 40(10), p.1207-1227 (1994). Ernst is the author of the declarations on the mathematical and statistical issues used by the defense in these cases.

Supreme court case No. 91–860

US Department of Commerce versus Montana

1990 census	**Montana**	**Washington**
population	$803,655$	$4,887,941$
quota	1.40 seats	8.53 seats
Based on Hill's method	one seat	nine seats
district size	$803,655$	4,887,941/9=543,104.55

$$\begin{aligned} \text{absolute difference} &= 260,550.44 = 803,655 - 543,104.55 \\ \text{relative difference} &= 0.480 = \frac{260,550.44}{543,104.55}. \end{aligned}$$

How about the transfer of one seat from Washington to Montana?

New district size $401,827.5$ $610,992.625$

new absolute difference $= 209,165.125 = 610,992.625 - 401,827.5$

new relative difference $= 0.521 = \dfrac{209,165.125}{401,827.5}$.

A transfer of one seat from Washington to Montana results in a decrease of the absolute difference of the district sizes. According to Dean's method, this transfer should then happen.

The same transfer leads to an increase in the relative difference of the district sizes, and hence violates the stipulation of Hill's method.

The Supreme Court rejected the argument that Hill's method violates the Constitution and Montana did not gain a second seat. However, it ruled that apportionment methods are justiciable ,
opening the door to future cases.

US Presidential elections and Electoral College

- 538–member Electoral College (EC)

  435 (same apportionment as the House Representatives)

  + 3 from the District of Columbia (same number as the smallest state)

  + 2×50 states

- Presidential elections

  – The winner of the plurality vote in a state is entitled to all the electors from that state (except Maine and Nebraska).

- Maine and Nebraska give an elector to the winner of the plurality of votes in each congressional district and give additional two electors corresponding to Senate seats to the winner of the plurality of the statewide vote.

- Most states are small and benefit from having their proportional share in representation augmented by those two electoral votes corresponding to Senate seats (favoring small states over large states).

- In the 2000 election, the 22 smallest states had a total of 98 votes in the Electoral College (EC) while their combined population was roughly equal to that of the state of California, which had only 54 votes in the EC. Of those 98 EC votes, 37 went for Gore while 61 went for Bush.

- Gore would win for large House sizes and Bush would win for small House sizes as he did with the House size at 435. This is because Bush won many of the smaller states, where these small states have higher proportional share due to the additional two electoral votes. For House size > 655, Gore is sure to win. Unfortunately, the House size has been fixed in 1941, at that time there was approximately one representative for every $301,000$ citizens. Based on the same ratio of representatives to people today as existed in 1941 then the House based on the 1990 census should have about 830 members.

- A direct election of the president does offer the advantage that it is independent of the House size. One drawback is that a third party candidate that draws votes disproportionately away from one candidate over the other thereby influencing the election.

Electoral College representation is sensitive to the apportionment method

	Hamilton	Jefferson	*Adams*	*Webster*	*Dean*	*Hill**
2000 E.C.	tie	Gore	Bush	Bush	Bush	Bush
Winner	269 – 269	271 – 267	274 – 264	270 – 268	272 – 266	271 – 267

- Since the Electoral College has built-in biases favoring small states, an apportionment method that partially offsets this bias might be justifiable.

- The infrequency of apportionment (once every 10 years)

  States that grow most quickly in population end up under-represented later in the life of a given apportionment.

4.4 Analysis of bias

An apportionment that gives a_1 and a_2 seats to states having populations $p_1 > p_2 > 0$ favors the larger state over the smaller state if $a_1/p_1 > a_2/p_2$ and favors the smaller state over the larger state if $a_1/p_1 < a_2/p_2$.

Over many pairs (p_1, p_2), $p_1 > p_2$, we ask whether a method tends more often to favor the larger state over the smaller or vice versa.

There are many ways to measure "bias" and there are different probabilistic models by which a tendency toward bias can be revealed theoretically.

A casual inspection shows the order: Adams, Dean, Hill, Webster, Jefferson that the apportionment methods tend increasingly to favor the larger states.

Apportionment of 6 states and 36 seats

Votes	*Adams*	*Dean/Hamilton*	*Hill*	*Webster*	*Jefferson*
27,744	10	10	10	10	11
25,178	9	9	9	9	9
19,951	7	7	7	8	7
14,610	5	5	6	5	5
9,225	3	4	3	3	3
3,292	2	1	1	1	1
100,000	36	36	36	36	36

The apportionment in any column leads to the apportionment in the next column by the transfer of one seat from a smaller state to a larger state.

Probabilistic approach

Consider a pair of integer apportionments $a_1 > a_2 > 0$ and ask

> "If the populations (p_1, p_2) has the M-apportionment (a_1, a_2), how likely is it that the small state (State 2) is favored?"

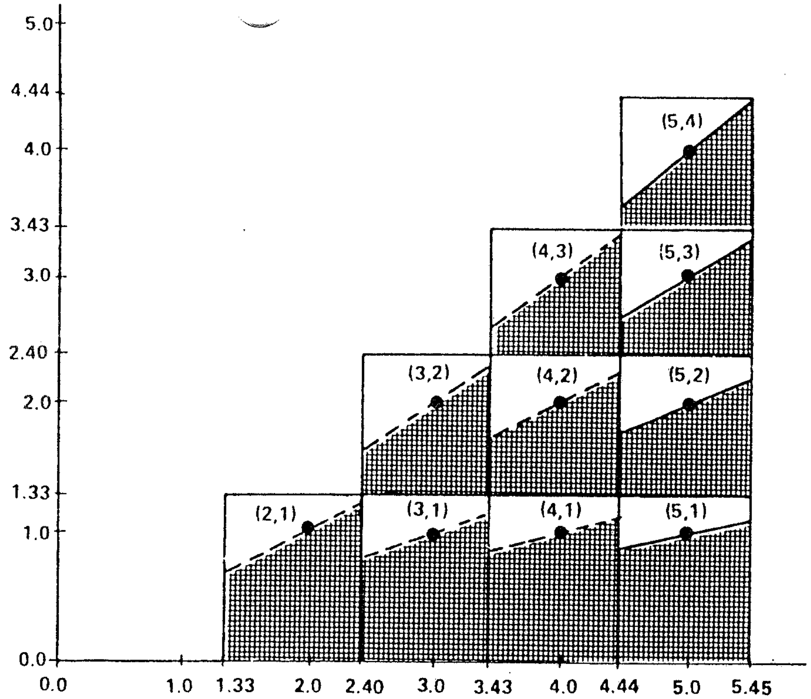
By population monotonicity, implicitly $p_1 \geq p_2$ since $a_1 > a_2$.

Take as a probabilistic model that the populations $(p_1, p_2) = \boldsymbol{p} > 0$ are uniformly distributed in the positive quadrant.

$$R_X(\boldsymbol{a}) = \left\{\boldsymbol{p} > 0 : d(a_i) \geq \frac{p_i}{\lambda} \geq d(a_i - 1)\right\}, \text{ with } d(-1) = 0.$$

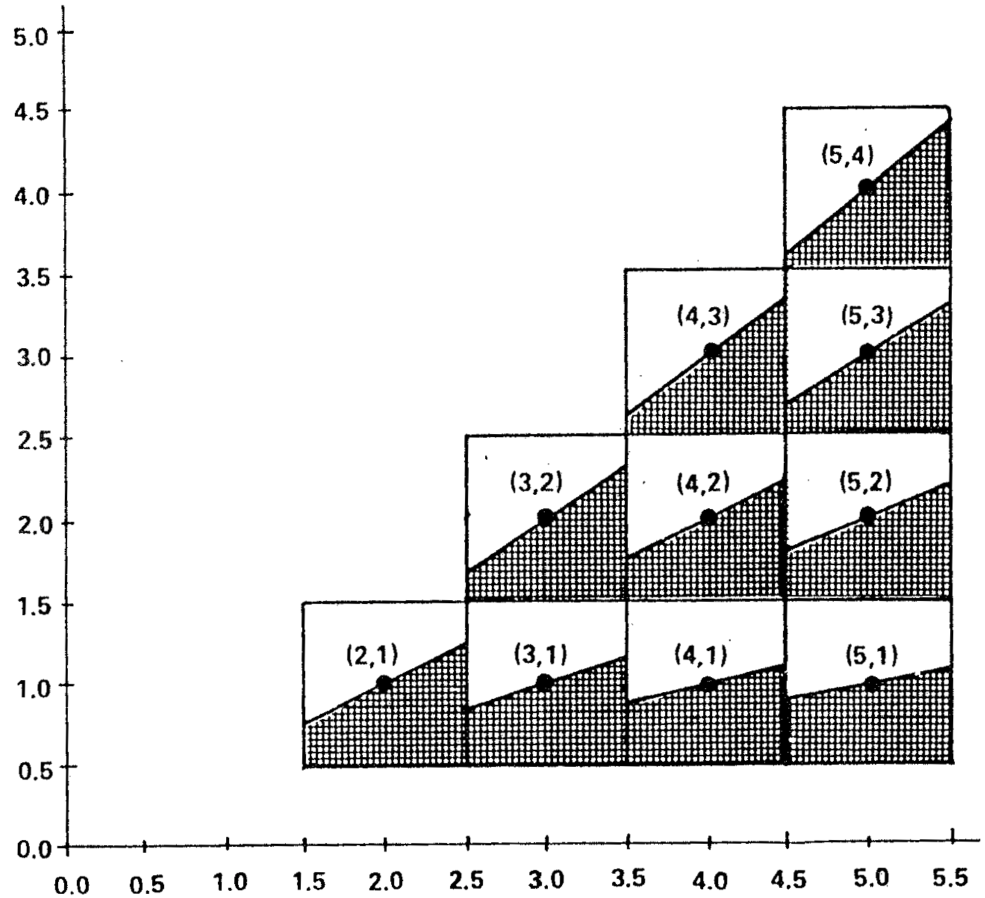
Each region $R_X(\boldsymbol{a})$ is a rectangle containing the point $\boldsymbol{a} = (a_1, a_2)$ and having sides of length $d(a_1) - d(a_1 - 1)$ and $d(a_2) - d(a_2 - 1)$.

Dean's method



The dotted line through $(a_1, a_2) = (2, 1)$ is originated from the origin $(0, 0)$. Points that are inside the shaded area satisfies $p_1/a_1 > p_2/a_2 \Leftrightarrow \dfrac{p_2/\lambda}{p_1/\lambda} < \dfrac{a_2}{a_1}$, that is, the smaller state (state 2) has smaller value in district size. The shaded area shows those populations that favor the smaller state since $\dfrac{p_2}{p_1} < \dfrac{a_2}{a_1} \Leftrightarrow \dfrac{p_2}{a_2} < \dfrac{p_1}{a_1}$.

Webster's method



Any quota point $(q_1, q_2) = (p_1/\lambda, p_2/\lambda)$ that falls inside the left bottom box would have the apportionment solution $(2, 1)$. The shaded area shows those populations that favor the smaller state.

Definition

A method M' favors small states relative to M if for every M-apportionment $\boldsymbol{a}$ and M'-apportionment $\boldsymbol{a}'$ for $\boldsymbol{p}$ and h,

$$p_i < p_j \Rightarrow a'_i \geq a_i \quad \text{or} \quad a'_j \leq a_j.$$

That is, it cannot happen that simultaneously a smaller state loses seats and a larger state gains seats.

Theorem

If M and M' are divisor methods with divisor criteria $d(a)$ and $d'(a)$ satisfying

$$\frac{d'(a)}{d'(b)} > \frac{d(a)}{d(b)} \text{ for all integers } a > b \geq 0,$$

then M' favors small states relative to M. For example, comparing Adams' method with $d'(a) = a$ and Jefferson's method with $d(a) = a + 1$, we observe

$$\frac{d'(a)}{d'(b)} = \frac{a}{b} > \frac{a+1}{b+1} = \frac{d(a)}{d(b)} \quad \text{for} \quad a > b.$$

Proof

By way of contradiction, for some $\boldsymbol{a} \in M(\boldsymbol{p}, h)$ and $\boldsymbol{a}' \in M'(\boldsymbol{p}, h), p_i < p_j, a'_i < a_i$ and $a'_j > a_j$. By population monotonicity of divisor methods,

$$a'_i < a_i \leq a_j < a'_j$$

so $a'_j - 1 > a'_i \geq 0$ and $d'(a'_j - 1) \geq 1$ since $a \leq d'(a) \leq a + 1$ for all a.

Using the min-max property for $\boldsymbol{a}'$, we deduce that

$$\frac{p_j}{d'(a'_j - 1)} \geq \frac{p_i}{d'(a'_i)}$$

and so $d'(a'_i) > 0$. Lastly

$$\frac{p_j}{p_i} \geq \frac{d'(a'_j - 1)}{d'(a'_i)} > \frac{d(a'_j - 1)}{d(a'_i)} \geq \frac{d(a_j)}{d(a_i - 1)}.$$

We then have $\dfrac{p_j}{d(a_j)} > \dfrac{p_i}{d(a_i - 1)}$, a contradiction to the min-max property.

Majorization ordering

Reference "A majorization comparison of apportionment methods in proportional representation," A Marshall, I. Olkin, and F. Fukelsheim, *Social Choice Welfare* (2002) vol. 19, p.885-900.

Majorization provides an ordering between two vectors

$$\boldsymbol{m} = (m_1 \cdots m_\ell) \quad \text{and} \quad \boldsymbol{m}' = (m'_1 \cdots m'_\ell)$$

with ordered elements

$$m_1 \geq \cdots \geq m_\ell \quad \text{and} \quad m'_1 \geq \cdots \geq m'_\ell,$$

and with an identical component sum

$$m_1 + m_2 + \cdots + m_\ell = m'_1 + m'_2 + \cdots + m'_\ell = M.$$

The ordering states that all partial sums of the k largest components in $\boldsymbol{m}$ are dominated by the sum of the k largest components in $\boldsymbol{m}'$.

$$
\begin{array}{rcl}
m_1 & \le & m'_1 \\
m_1 + m_2 & \le & m'_1 + m'_2 \\
 & \vdots & \\
m_1 + \cdots + m_k & \le & m'_1 + \cdots + m'_k \\
 & \vdots & \\
m_1 + \cdots + m_\ell & = & m'_1 + \cdots + m'_\ell
\end{array}
$$

$\boldsymbol{m} \prec \boldsymbol{m}', \boldsymbol{m}$ is majorized by $\boldsymbol{m}'$ or $\boldsymbol{m}'$ majorizes $\boldsymbol{m}$.

Suppose it never occurs that $m_i > m'_i$ and $m_j < m'_j$, for all $i < j$, (a larger state has more seats while a smaller state has less seats for apportionment $\boldsymbol{m}$), then apportionment $\boldsymbol{m}$ is majorized by $\boldsymbol{m}'$.

Divisor methods and signpost sequences

A divisor method of apportionment is defined through the "signpost" or "dividing point" $s(k)$ in the interval $[k, k+1]$ that splits the interval $[k, k+1]$. A number that falls within $[k, s(k)]$ is rounded down to k and it is rounded up to $k+1$ if it falls within $(s(k), k+1)$. If the number happens to hit $s(k)$, then there is an option to round down to k or to round up to $k+1$.

Power-mean signposts

$$s(k,p) = \left[\frac{k^p}{2} + \frac{(k+1)^p}{2}\right]^{1/p}, \quad -\infty \leq p \leq \infty.$$

$p = -\infty, s(k, -\infty) = k$ (Adams); $p = \infty, s(k, \infty) = k+1$ (Jefferson); $p = 0$ (Hills); $p = -1$ (Dean); $p = 1$ (Webster).

- For Hill's method, we consider

$$
\begin{aligned}
& \ln\left(\lim_{p\to 0^+}\left[\frac{k^p}{2}+\frac{(k+1)^p}{2}\right]^{1/p}\right) \\
= & \lim_{p\to 0^+}\frac{1}{p}\ln\left(\frac{k^p}{2}+\frac{(k+1)^p}{2}\right) \\
= & \lim_{p\to 0^+}\frac{\frac{k^p}{2}\ln k+\frac{(k+1)^p}{2}\ln(k+1)}{\frac{k^p}{2}+\frac{(k+1)^p}{2}} \quad \text{(by Hospital's rule)} \\
= & \frac{1}{2}\ln k(k+1)
\end{aligned}
$$

so that

$$
\lim_{p\to 0^+}\left[\frac{k^p}{2}+\frac{(k+1)^p}{2}\right]^{1/p} = \sqrt{k(k+1)},\ k=0,1,2,\ldots.
$$

- For Jefferson's method, we consider

$$
\begin{aligned}
\lim_{p\to\infty}\left[\frac{k^p}{2}+\frac{(k+1)^p}{2}\right]^{1/p} &= \lim_{p\to\infty}[(k+1)^p]^{1/p}\lim_{p\to\infty}\left[\frac{1}{2}\left(\frac{k}{k+1}\right)^p+\frac{1}{2}\right]^{1/p} \\
&= k+1.
\end{aligned}
$$

Proposition 1

Let A be a divisor method with signpost sequence: $s(0), s(1), \cdots$, and a similar definition for another divisor method A'. Method A is majorized by Method A' if and only if the signpost ratio $s(k)/s'(k)$ is strictly increasing in k.

For example, suppose we take A to be Adams and A' to be Jefferson, then $\dfrac{s(k)}{s'(k)} = \dfrac{k}{k+1} = 1 - \dfrac{1}{k+1}$ which is strictly increasing in k.

Proposition 2

The divisor method with power-mean rounding of order p is majorized by the divisor method with power-mean rounding of order p', if and only if $p \leq p'$.

This puts the 5 traditional divisor methods into the following majorization ordering

$$\text{Adams} \prec \text{Dean} \prec \text{Hill} \prec \text{Webster} \prec \text{Jefferson}.$$

1. One can see that “is majorized by” is less demanding than “favoring small districts relative to”.

2. Since Hamilton’s apportionment is not a divisor method, how about the positioning of the Hamilton method in those ranking?

Proposition

Adams’ method favors small districts relative to Hamilton’s method while Hamilton’s method favors small districts relative to Jefferson’s method. However, Hamilton’s method is incomparable to other divisor methods such as Dean, Hill, and Webster.

Reference

“The Hamilton apportionment method is between the Adams method and the Jefferson method,” *Mathematics of Operations Research*, vol. 31(2) (2006) p.390-397.

$A >$ Hamilton $> J$, but not Hamilton $> D, H, W$.

	Population	Proportions	*A, D, H, W*	Hamilton	*J*
	603	6.70	6	7	8
	149	1.66	2	2	1
	148	1.64	2	1	1
total =	900	10.00	10	10	10

$A >$ Hamilton $> J$, but not $D, H, W >$ Hamilton.

	Population	Proportions	*A*	Hamilton	*D, H, W, J*
	1,600	5.36	5	5	6
	1,005	3.37	3	4	3
	380	1.27	2	1	1
total =	2,985	10.00	10	10	10

Hamilton happens to be the same as Webster

	Population	Proportions	Adams	Webster	Hamilton	Jefferson
	603	6.03	5	6	6	7
	249	2.49	3	3	3	2
	148	1.48	2	1	1	1
total =	1,000	10.00	10	10	10	10

"Near the quota" property

Instead of requiring "stay within the quota", a weaker version can be stated as: It should not be possible to take a seat from one state and give it to another and simultaneously bring both of them nearer to their quotas. That is, there should be no states i and j such that

$$q_i - (a_i - 1) < a_i - q_i \quad \text{and} \quad a_j + 1 - q_j < q_j - a_j. \tag{1}$$

Alternatively, no state can be brought closer to its quota without moving another state further from its quota. The above definition is in absolute terms.

Theorem Webster's method is the unique population monotone method that is near quota.

(i) Violation of "near quota" property $\Rightarrow$ non-Webster method

If $\boldsymbol{a}$ is not near quota, that is if Eq. (1) holds for some i and j then rearranging, we have

$$1 < 2(a_i - q_i) \quad \text{and} \quad 1 < 2(q_j - a_j) \tag{2}$$

or

$$a_j + \frac{1}{2} < q_j \quad \text{and} \quad a_i - \frac{1}{2} > q_i$$

which gives

$$q_j/(a_j + \frac{1}{2}) > 1 > q_i/(a_i - \frac{1}{2}).$$

Hence the min-max inequality for Webster's method is violated, so $\boldsymbol{a}$ could not be a Webster apportionment. Therefore, Webster's method is near quota.

(ii) non-Webster method $\Rightarrow$ "non-near quota" property

Conversely, let M be a population monotone method (property satisfied by a divisor method) different from Webster's. Then *there exists* a 2-state problem (p_1, p_2) in which the M-apportionment is uniquely (a_1+1, a_2), whereas the W-apportionment is uniquely (a_1, a_2+1). By the latter, we deduce the property:

$$p_2/(a_2+1/2) > p_1/(a_1+1/2).$$

At $h = a_1 + a_2 + 1$, the quota of state 1 is

$$\begin{aligned} q_1 &= \frac{p_1 h}{p_1+p_2} \\ &= \frac{p_1(a_1+1/2+a_2+1/2)}{p_1+p_2} < \frac{p_1(a_1+1/2)+p_2(a_1+1/2)}{p_1+p_2} \\ &= a_1+1/2. \end{aligned}$$

State 2's quota is $q_2 = (a_1+a_2+1) - q_1 > a_2+1/2$. These pair of inequalities are equivalent to (2) with $a_i = a_1+1$ and $a_j = a_2$. Therefore the M-apportionment (a_1+1, a_2) is not near quota.

4.5 Matrix apportionment: proportionality in both districts and parties

The Zurich Canton Parliament is composed of seats that represent the electoral districts as well as political parties.

- Each district, $j = 1, 2, \ldots, n$, is represented by a number of seats r_j that is proportional to its population (preset before the election).
- Each political party, $i = 1, 2, \ldots, m$, gets c_i seats proportional to its total number of votes (constitutional requirement).
- The vote count in district j of party i is denoted by v_{ij}. The vote counts are assembled into a vote matrix $V \in \mathbb{N}^{m \times n}$.

Vote Numbers for the Zurich City Council Election on February 12, 2006

		District									
		1 + 2	3	4 + 5	6	7 + 8	9	10	11	12	
Party	125	12	16	13	10	17	16	12	19	10	Total
SP	44	28,518	45,541	26,673	24,092	61,738	42,044	35,259	56,547	13,215	333,627
SVP	24	15,305	22,060	8,174	9,676	27,906	31,559	19,557	40,144	10,248	184,629
FDP	19	21,833	10,450	4,536	10,919	51,252	12,060	15,267	19,744	3,066	149,127
Greens	14	12,401	17,319	10,221	8,420	25,486	9,154	9,689	12,559	2,187	107,436
CVP	10	7,318	8,661	4,099	4,399	14,223	11,333	8,347	14,762	4,941	78,083
EVP	6	2,829	2,816	1,029	3,422	10,508	9,841	4,690	11,998	0	47,133
AL	5	2,413	7,418	9,086	2,304	5,483	2,465	2,539	3,623	429	35,760
SD	3	1,651	3,173	1,406	1,106	2,454	5,333	1,490	6,226	2,078	24,917
Total		92,268	117,438	65,224	64,338	199,050	123,789	96,838	165,603	36,164	960,712
Total no. of voters		7,891	7,587	5,269	6,706	12,180	7,962	8,344	9,106	3,793	68,838

- The number of seats allocated to each district is based on population figures of the districts.
- The number of seats allocated to each party is based on total votes casted on the parties for the whole Zurich.

- The district magnitudes are based on the population counts, which are known prior to the election. For example, district 9 has 16 seats.

- Each voter has as many votes as there are seats in the corresponding district. Voters in district 9 can cast up to 16 votes.

- The table does not include parties that do not pass the threshold of 5% of the votes in at least one district. So, the total number of votes in Table is less than the total number of actual votes.

District marginals

District 12 has 5.5% of the voters ($3,793$ out of $68,838$), but is set to receive 8.0% of the seats (10 out of 125). This is because the *population counts* form the basis for the allocation of seats to districts.

District quota

This is the proportion of seats that a party should receive within each district. For example, the Greens received $9,154$ votes out of $123,789$ votes in district 9; so

$$\begin{aligned} & \text{district quota for the Greens in district 9} \\ = \; & 16 \times \frac{9,154}{123,789} = 1.18. \end{aligned}$$

- Summing all district quota for the Greens across all 12 districts gives the sum 13.92.

District Quotas for the Zurich City Council Election on February 12, 2006

		District									
		1 + 2	3	4 + 5	6	7 + 8	9	10	11	12	
Party	125	12	16	13	10	17	16	12	19	10	Total
SP	44	3.71	6.20	5.32	3.74	5.27	5.43	4.37	6.49	3.65	44.19
SVP	24	1.99	3.01	1.63	1.50	2.38	4.08	2.42	4.61	2.83	24.45
FDP	19	2.84	1.42	0.90	1.70	4.38	1.56	1.89	2.27	0.85	17.81
Greens	14	1.61	2.36	2.04	1.31	2.18	1.18	1.20	1.44	0.60	13.92
CVP	10	0.95	1.18	0.82	0.68	1.21	1.46	1.03	1.69	1.37	10.41
EVP	6	0.37	0.38	0.21	0.53	0.90	1.27	0.58	1.38	0.00	5.62
AL	5	0.31	1.01	1.81	0.36	0.47	0.32	0.31	0.42	0.12	5.13
SD	3	0.21	0.43	0.28	0.17	0.21	0.69	0.18	0.71	0.57	3.47
Total		12.00	16.00	13.00	10.00	17.00	16.00	12.00	19.00	10.00	125.00

At simple level, one may solve a sequence of vector apportionments in all districts based on the district quota data.

However, a Greens candidate in District 12 (less voters turnout) has comparative advantage over his partymates in other districts with more voter turnout. It may occur that a candidate in District 12 wins but receives less number of votes than a losing candidate in another District with larger voter turnout. The biapportionment scheme should try to avoid such occurrence.

- The percentage of population count of each district is *not* the same as the district's percentage of voters count, reflecting the varying levels of engagement in politics in the districts. District 12 has the least percentage of population coming to vote (politically less engaged).

- Suppose we use the total aggregate votes across all districts (whole city) as the basis for computing the quota for the Greens, we obtain

$$\begin{aligned}&\text{eligible quota for the Greens (out of 125 seats)}\\=\;&\frac{107,436}{960,712}\times 125 = 13.97 \text{ (slightly different from 13.92).}\end{aligned}$$

$$\begin{aligned}&\text{eligible quota for the Greens in district 9}\\=\;&\frac{9,154}{960,712}\times 125 = 1.19.\end{aligned}$$

First step: Determine the eligible quota for each party

- Party seats are allocated on the basis of the total party ballots in the whole electoral region.

- Respond to the constitutional demand that all voters contribute to the electoral outcome equally, no matters whether voters cast their ballots in districts that are large or small.

- For a given party, we divide the vote counts in each district by its corresponding district magnitude (rounding to the nearest integer), and sum over all districts. This gives the *support size* for each party – *number of voters supporting a party*. The data on the support sizes of the parties for the whole city are used to determine the eligible quota for each party. This would satisfy the mandate that the vote contributions to the parties are independent of the districts which the voters cast their votes.

Biapportionment of the Zurich City Parliament election of 12 February 2006:

	SP	SVP	FDP	Greens	CVP	EVP	AL	SD	City divisor
Support size	23180	12633	10300	7501	5418	3088	2517	1692	530
Seats 125	44	24	19	14	10	6	5	3	

For example, consider Party SP:

$$\frac{28,518}{12} + \underset{\substack{\uparrow \\ \text{each voter} \\ \text{in district 3} \\ \text{has 16 votes}}}{\frac{45,541}{16}} + \cdots + \frac{56,547}{19} + \frac{13,215}{10} \approx 23,180$$

Apply the divisor 530 so that

$$\begin{aligned} & \left[\frac{23,180}{530}\right] + \left[\frac{12,633}{530}\right] + \cdots + \left[\frac{2,517}{530}\right] + \left[\frac{1,692}{530}\right] \\ = \; & [43.7] + [23.8] + \cdots + [4.7] + [3.19] \\ = \; & 44 + 24 + \cdots + 5 + 3 = 125. \end{aligned}$$

How to deal with the allocation of the seats to the parties within the districts? Each vote count of a party in a district is divided by its corresponding district divisor and party divisor. The quotient is rounded using the standard apportionment schemes to obtain the seat number.

Mathematical formulation

$\boldsymbol{r} = (r_1 \ldots r_m) > \mathbf{0}$ and $\boldsymbol{c} = (c_1 \ldots c_n) > \mathbf{0}$ are integer-valued vectors whose sums are equal. That is,

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j = h = \text{total number of seats.}$$

We need to find the row multipliers λ_i and column multipliers μ_j such that

$$x_{ij} = [\lambda_i v_{ij} \mu_j], \text{ for all } i \text{ and } j,$$

such that the row-sum and column-sum requirements are fulfilled. Here, [ ] denotes some form of rounding.

An apportionment solution is a matrix $X = (x_{ij})$, where $x_{ij} > 0$ and integer-valued, such that

$$\sum_{j=1}^{n} x_{ij} = r_i \text{ for all } i \text{ and } \sum_{i=1}^{m} x_{ij} = c_j \text{ for all } j.$$

- Assign integer values to the elements of a matrix that are proportional to a given input matrix, such that a set of row-sum and column-sum requirements are fulfilled.

- In a divisor-based method for biproportional apportionment, the problem is solved by computing the appropriate row-divisors and column-divisors, and by rounding the quotients.

Result of Zurich City Council Election on February 12, 2006

		District									
		1 + 2	3	4 + 5	6	7 + 8	9	10	11	12	
Party	125	12	16	13	10	17	16	12	19	10	Divisor $1/\lambda_i$
SP	44	4	7	5	4	5	6	4	6	3	1.006
SVP	24	2	3	2	1	2	4	3	4	3	1.002
FDP	19	3	1	1	2	5	2	2	2	1	1.010
Greens	14	2	3	2	1	2	1	1	1	1	0.970
CVP	10	1	1	1	1	1	1	1	2	1	1.000
EVP	6	0	0	0	1	1	1	1	2	0	0.880
AL	5	0	1	2	0	1	0	0	1	0	0.800
SD	3	0	0	0	0	0	1	0	1	1	1.000
Divisor $1/\mu_j$		7,000	6,900	5,000	6,600	11,200	7,580	7,800	9,000	4,000	

The divisors are those that were published by the Zurich City administration. In district $1+2$, the Greens had $12,401$ ballots and were awarded by two seats. This is because $12,401/(7,000 \times 0.97) \approx 1.83$, which is rounded up to 2.

- For the politically less active districts, like district 12, the divisor (number of voters represented by each seat) is smaller ($1/\mu_j = 4,000$).

- The matrix apportionment problem can be formulated as an integer programming problem with constraints, which are given by the row sums and column sums. We solve for the multipliers λ_i and μ_j through an iterative algorithm.

Greatest remainder biproportional rounding method

There are $n = 56$ electoral regions in the Greek parliamentary elections in 2007 and 288 seats are allotted to m parties (these are parties that receive more than 3% of the national votes) based on the votes received in the 56 electoral regions.

List of notations

- v_{ij} number of votes received by party i in region j, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, 56$
- t_j total votes of eligible parties in region j, $j = 1, 2, \ldots, 56$; equals $\sum_{i=1}^{m} v_{ij}$
- s_j number of seats in region j, $j = 1, 2, \ldots, 56$
- μ_j number of voters represented by each seat in region j, $j = 1, 2, \ldots, 56$; equals t_j/s_j
- k_{ij} number of seats allotted to party i in region $j, i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, 56$
- e_i number of seats allotted to party i (whole nation), $i = 1, 2, \ldots, m$

- s_j (seats in region j) is determined based on population census.
- e_i (total seats received by party i in all regions) is determined based on vector apportionment of support sizes of the parties.

Each seat in region j represents μ_j votes. Note that $\mu_j = t_j/s_j$ may not be an integer. Seats are allotted in two rounds. In the first distribution for each region j, we set

$$k_{ij}^A = \left\lfloor \frac{v_{ij}}{\mu_j} \right\rfloor, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n,$$

as the lower bound for k_{ij}. This is like allocating the lower quota in the vector apportionment among all parties in region j. The seat allotment in the second distribution is defined as

$$k_{ij}^B = k_{ij} - k_{ij}^A$$

based on the $v_{ij} - k_{ij}^A \mu_j$ left-over eligible votes. Here, k_{ij}^B is either 0 or 1.

We define

$$r_{ij} = \frac{v_{ij}}{\mu_j} - \left\lfloor \frac{v_{ij}}{\mu_j} \right\rfloor$$

as a measure of inequity. The integer programming problem is formulated as the minimization of the aggregate inequity among all parties and regions:

$$\begin{aligned} & \min_{k_{ij}^B} \sum_{j=1}^{n} \sum_{i=1}^{m} (r_{ij} - k_{ij}^B)^2 \\ = \; & \sum_{j=1}^{n} \sum_{i=1}^{m} [r_{ij}^2 + (k_{ij}^B)^2] + 2 \min_{k_{ij}^B} \sum_{j=1}^{n} \sum_{i=1}^{m} (-r_{ij} k_{ij}^B). \end{aligned}$$

Note that

$$\sum_{j=1}^{n}\sum_{i=1}^{m} r_{ij}^2 = \text{constant}$$

and since $k_{ij}^B \in \{0,1\}$, we have

$$\sum_{j=1}^{n}\sum_{i=1}^{m} (k_{ij}^B)^2 = \sum_{j=1}^{n}\sum_{i=1}^{m} k_{ij}^B,$$

which is the total number of seats in the second distribution. It is the difference between the total number of seats and the seats allocated in the first distribution, a known quantity.

The problem becomes maximization of the sum of remainders, where

$$\epsilon^* = \max_{k_{ij}^B} \sum_{j=1}^{n}\sum_{i=1}^{m} r_{ij}k_{ij}^B.$$

Recall that e_i, $i = 1, 2, \ldots, m$ and s_j, $j = 1, 2, \ldots, n$, are determined using some vector apportionment method based on party votes and population census, respectively. There are $m + n$ additional constraints for the integer-valued maximization problem:

$$\sum_{i=1}^{n} k_{ij}^{B} = e_i - \sum_{j=1}^{n} k_{ij}^{A} = e_i^{B}, \quad i = 1, 2, \ldots, m;$$
$$\sum_{j=1}^{m} k_{ij}^{B} = s_j - \sum_{i=1}^{m} k_{ij}^{A} = s_j^{B}, \quad j = 1, 2, \ldots, n.$$

e_i^B gives the allocation of remaining seats available to party i across different districts and s_j^B gives the allocation of remaining seats available to district j across different parties. Based on the input data of k_{ij}^A, we determine e_i^B and s_j^B.

The solution procedure is to allocate the remaining seats to party i in region j with the largest value in r_{ij} sequentially until the respective constraints stated above are met. Eligibility of assignment of a new seat requires constraints on row sum or column sum have not been met.

Example 4-region and 3-party

	Regions				
$\underbrace{\text{Party}}$	A	B	C	D	$\underbrace{e_i^B}$
a	0.42	**0.60**	**0.71**	0.27	2
b	**0.85**	0.38	0.18	**0.62**	2
c	**0.73**	0.02	0.11	0.11	1
	2	1	1	1	$\leftarrow s_j^B$

Naturally, the seats are allotted to those party-region slots that have the large values of r_{ij}, respecting the $4 + 3$ constraints on e_i^B and s_j^B. In the second round distribution, $e_1^B = 2$ means 2 seats can be allocated to party 1 and $s_2^B = 1$ means 1 seat can be allocated in region 2.

Reference

Ch. Tsitouras, Greatest remainder bi-proportional rounding and the Greek parliamentary election of 2007, *Applied Mathematics and Computation*, vol. 217, p.9254-9260 (2011).