MATH4999 —– Capstone Projects in Mathematics and Economics

Topic Four – Traffic flow models

4.1 Characteristics of traffic flows

- Traffic problems
- Velocity and traffic flows
- Flux and conservation law

4.2 Method of characteristic line

- Density wave velocity

4.3 Impact of a traffic light turning green

- Linear velocity-density relationship

4.4 Car-following models

4.1 Characteristics of traffic flows

Traffic problems

- Where to install traffic lights and stop signs?
- How long the cycle of traffic lights should be?
- How to develop a progressive traffic light system?
- Whether to change a two-way street to a one-way street?
- Where to construct entrances, exits and overpasses?
- How many lanes to build for a new highway?

Problems to be solved include

- alleviate congestion
- maximize flow of traffic

Various levels of model formulations

1. Traffic flows along a unidirectional road versus a network
2. Individual cars to be considered as particles versus observations made at fixed locations
3. Deterministic versus stochastic approach

Two methods to measure car's velocity

1. *Measure the velocity u_i of each individual car*

$$u_i = \frac{dx_i(t)}{dt}$$

Example

$$\frac{dx_1}{dt} = 45, \ t > 0 \qquad x_1(0) = L;$$
$$\frac{dx_2}{dt} = 30, \ t > 0 \qquad x_2(0) = 0.$$

The positions of the two cars are found to be

$$x_1 = 45t + L$$
$$x_2 = 30t.$$

2. *Velocity field*

We associate to each point in space and at each time a unique velocity $u(x,t)$. This is the velocity measured at time t by an observer fixed at position x.

Analogy with the temperature field

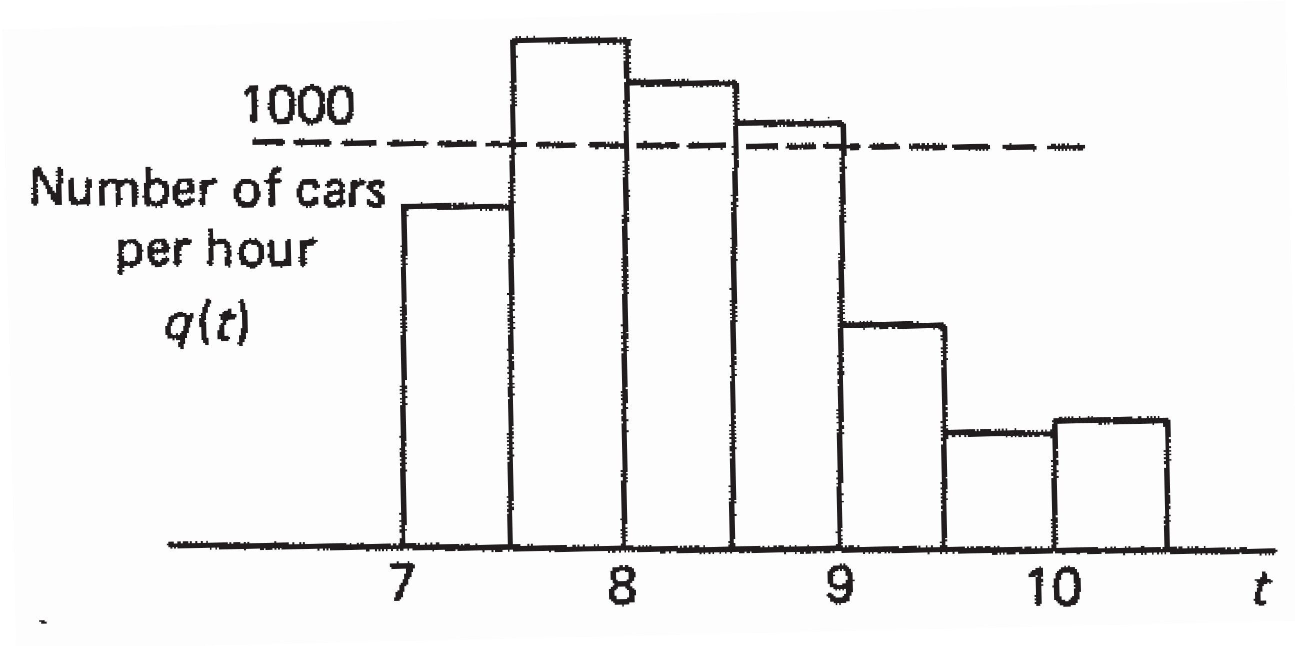
Specify the temperature at fixed position and time, rather than to associate a temperature with a moving air particle.

Traffic flow q

Average number of cars passing per hour

Time A.M.	*Number of cars passing*	*Number of cars passing per hour*
7:00– 7:30	433	866
7:30– 8:00	652	1304
8:00– 8:30	594	1188
8:30– 9:00	551	1102
9:00– 9:30	280	560
9:30–10:00	141	282
10:00–10:30	167	334

The traffic flow q is a function of time and position



Traffic flow, $q(t)$, measured every half an hour

What happen if we take measurements over each 10-second interval?

Time $\left(\begin{array}{c}\textit{in seconds}\\ \textit{after } 7:00\end{array}\right)$	*Number of cars*	*Number of cars per hour*
0-9	0	0
10-19	2	720
20-29	1	360
30-39	4	1440
40-49	1	360
50-59	4	1440

The number of cars passing fluctuates wildly over successive 10-second intervals.

Determine the measuring time intervals according to the following compromising criteria:

1. It is long enough so that many cars pass the observer (eliminating the wild fluctuations).

2. It is short enough so that variations in the traffic flows are not smoothed over by averaging over a long period.

Traffic density

Number of cars per mile

Distance along road (in miles)	*Number of cars*	*Traffic density, number of cars per mile*
$1 - 1\frac{1}{4}$	23	92
$1\frac{1}{4} - 1\frac{1}{2}$	16	64
$1\frac{1}{2} - 1\frac{3}{4}$	22	88
$1\frac{3}{4} - 2$	8	32

Example

Cars are equally spaced, all vehicles have the same length L and the distance between cars is d. The traffic density (number of cars per mile) is found to be

$$\rho = \frac{1}{L + d}.$$

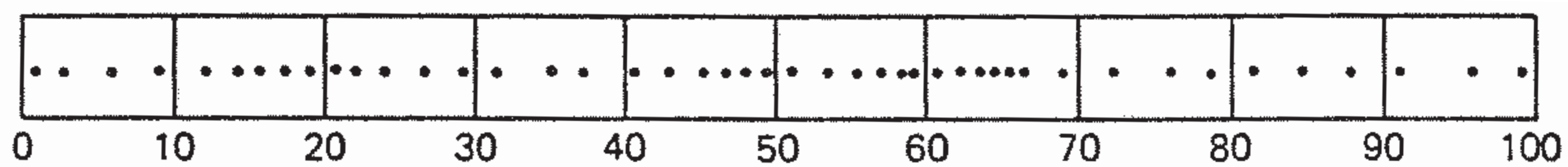
Remark

If we would like to approximate the traffic density as a continuous function of x, then densities must be measured over intervals of distance that are not too small nor too large.

Let us illustrate by an example the significance of the measuring interval. Consider a $\frac{2}{10}$ of a mile segment of a (one-lane) highway, further subdivided into one hundred smaller intervals of equal length, with boundaries labeled from 0 to 100. Suppose a photograph was taken and from it we determined that cars were located at the following positions:

$$1.0, 3.1, 6.1, 9.4, 12.7, 14.1, 15.2, 16.9, 18.9, 20.1, 21.5, 23.5,$$
$$25.8, 28.9, 31.3, 34.8, 37.0, 40.1, 43.4, 44.9, 46.4, 47.9, 49.6,$$
$$51.6, 53.3, 54.8, 56.6, 58.3, 59.6, 60.6, 61.9, 62.9, 63.7, 65.0,$$
$$66.6, 69.5, 72.1, 76.3, 78.8, 81.6, 84.2, 87.7, 90.8, 95.1, 99.3,$$

each car illustrated in the figure below as a '•':



The traffic density "appears" as the density of ink dots.

0.2 mile

divided into 100 intervals of equal length

Density at 50 = number of cars between $50 - \frac{m}{2}$ and $50 + \frac{m}{2}$ divided by the length m (converted into numbers of cars per mile)

Length of interval, m	.5	1.0	1.5	2	3	4	5	6	7	8	10	12	14	16	18	20	30	40	50
Interval of interest	$49\frac{3}{4}$ $50\frac{1}{4}$	$49\frac{1}{2}$ $50\frac{1}{2}$	$49\frac{1}{4}$ $50\frac{3}{4}$	49 51	$48\frac{1}{2}$ $51\frac{1}{2}$	48 52	$47\frac{1}{2}$ $52\frac{1}{2}$	47 53	$46\frac{1}{2}$ $53\frac{1}{2}$	46 54	45 55	44 56	43 57	42 58	41 59	40 60	35 65	30 70	25 75
Number of cars	0	1	1	1	1	2	3	3	4	5	6	7	9	9	10	12	18	22	25
Number of cars per interval	0	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{4}{7}$	$\frac{5}{8}$	$\frac{6}{10}$	$\frac{7}{12}$	$\frac{9}{14}$	$\frac{9}{16}$	$\frac{10}{18}$	$\frac{12}{20}$	$\frac{18}{30}$	$\frac{22}{40}$	$\frac{25}{50}$
Traffic density: number of cars per mile (number of cars per interval × 500)	0	500	333	250	167	250	300	250	280	312	300	292	321	281	278	300	300	275	250

Flux

Number of cars passing a fixed station per unit time:

$$q(x,t) = \rho(x,t)u(x,t),$$

or in words,

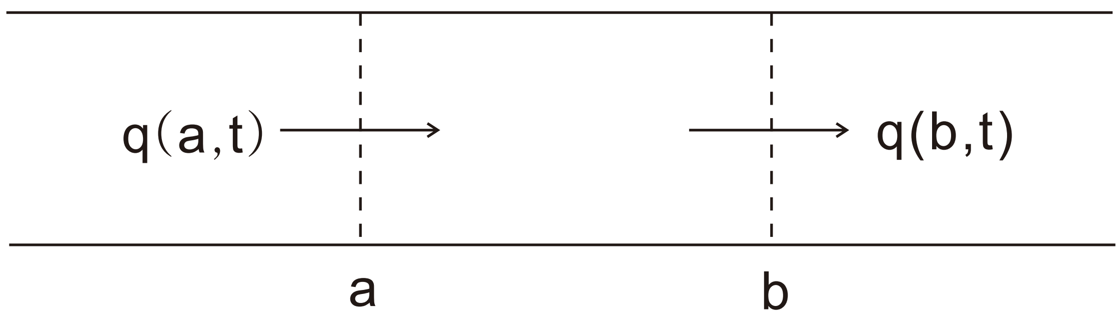
$$\text{traffic flow} = \text{traffic density} \times \text{velocity}.$$

This is like the concept of flux in physics.

Number of cars between $x = a$ and $x = b$ at time t is

$$N(t) = \int_a^b \rho(x,t)\ dx.$$

Cars entering and leaving a segment $[a, b]$ of a roadway



Useful formulas of traffic flows

1. $\dfrac{dN}{dt} = q(a,t) - q(b,t)$

   Consider the difference between the number of cars $N(t)$ within $[a,b]$ at $t = t_0$ and $t = t_1$, so

$$N(t_1) - N(t_0) = \int_{t_0}^{t_1} q(a,t)\ dt - \int_{t_0}^{t_1} q(b,t)\ dt.$$

   Since t_0 is independent of t_1, we have

$$\begin{aligned} \frac{dN(t_1)}{dt_1} &= \frac{d}{dt_1}\int_{t_0}^{t_1} [q(a,t) - q(b,t)]\ dt \\ &= q(a,t_1) - q(b,t_1). \end{aligned}$$

2. In general, since $N(t) = \int_a^b \rho(x,t)\ dx$, then

$$\frac{\partial}{\partial t}\int_a^b \rho(x,t)\ dx = q(a,t) - q(b,t).$$

   It is more appropriate to use $\dfrac{\partial}{\partial t}$ instead of $\dfrac{d}{dt}$ since a and b can be considered as additional independent variables.

Conservation equation

Assuming $\rho = \rho(x,t)$ and $q = q(x,t)$, we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0.$$

If we write $q = \rho u$, then

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0.$$

Proof

$$q(a,t) - q(b,t) = -\int_a^b \frac{\partial}{\partial x} q(x,t)\ dx$$

so that

$$\int_a^b \left[\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x}\right] dx = 0.$$

Since the above statement is valid for all values of the independently varying limits of the integral, the only function whose integral is zero for all intervals is the zero function. Therefore, we obtain

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0.$$

Consider the conservation equation

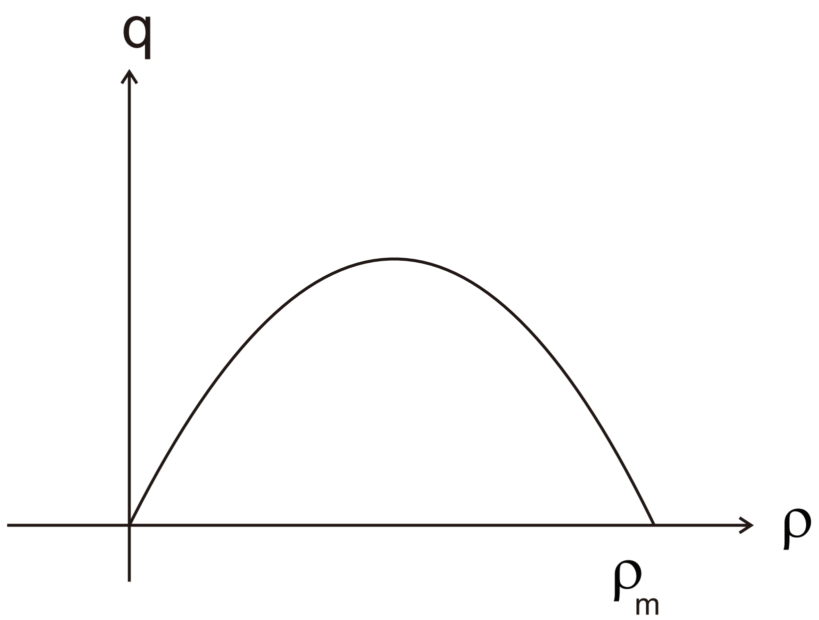
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0.$$

There are two dependent variables $\rho(x,t)$ and $u(x,t)$ in one equation. If $u(x,t)$ is know, then we have one equation for $\rho(x,t)$. Unfortunately, we do not know the velocity field without solving the equation.

We need an additional constitutive equation: $q = q(\rho)$, which reflects the peculiarities of vehicular traffic.

The flow may be zero when

1. there is no traffic, $\rho = 0$.
2. the traffic is almost not moving: $u \to 0$ at $\rho = \rho_m$.



We expect that there exists unique ρ at which q achieves maximum, so the constitutive relation to observe $\dfrac{d^2 q}{d\rho^2} < 0$.

One simple choice is the quadratic equation:

$$q = u_m \rho \left(1 - \frac{\rho}{\rho_m} \right).$$

Now the conservation equation can be rewritten as

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho} \frac{\partial \rho}{\partial x} = 0,$$

where $\frac{dq}{d\rho}$ is a known function.

Three basic quantities: $q = \rho u$;

$$\rho = \text{traffic density},\ q = \text{flow rate}, u = \text{field velocity}$$

1. Conservation law: $\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho} \frac{\partial \rho}{\partial x} = 0$

2. Constitutive relation: q is given as a function of ρ with $q = 0$ at $\rho = 0$ and $\rho = \rho_{\text{max}}$.

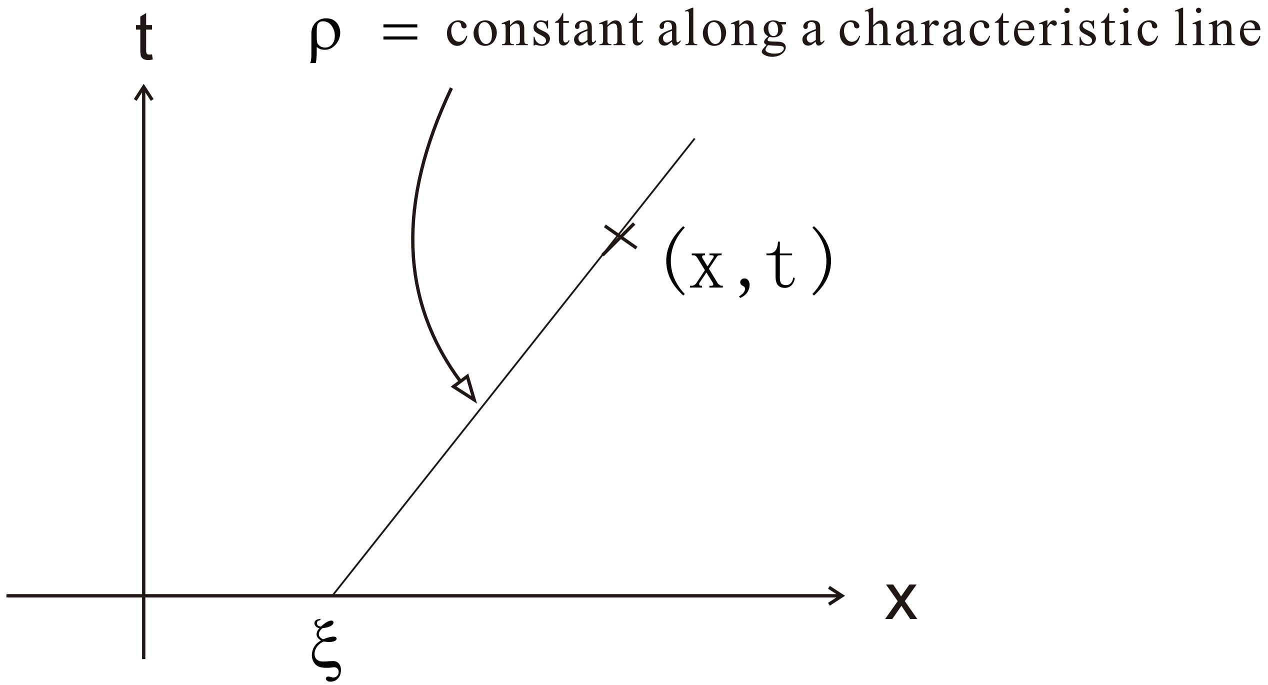
4.2 Method of characteristic line

A characteristic line connects points in the (x, t)-plane with constant traffic density ρ.

Equation of the characteristic line: $x = \xi + w(\rho)t$,

where ξ is the point of intersection with the x-axis at which $t = 0$.

How to solve for ρ in terms of x and t? It is necessary to prescribe the initial condition: $\rho(x, 0) = f(x)$. We write $\rho = f(\xi)$ at $t = 0$.



Method I: Solve for $\xi = f^{-1}(\rho)$, then substitute into the characteristic equation:

$$x = f^{-1}(\rho) + w(\rho)t.$$

Then solve implicitly (if possible) for $\rho(x, t)$.

Method II: Solve implicitly for ξ from $x = \xi + w(f(\xi))t$; then $\rho(x, t) = f(\xi(x, t))$.

Recall the two governing equations:

$$\text{conservation law: } \frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0, \; q = \rho u;$$
$$\text{constitutive relation: } q = q(\rho).$$

The conservation law and fundamental diagram of road traffic are combined to give

$$\frac{\partial \rho}{\partial t} + \frac{dq(\rho)}{d\rho}\frac{\partial \rho}{\partial x} = 0.$$

Consider an observer moving in some prescribed path $x(t)$. The density of traffic at the observer's position changes in time as governed by

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x}.$$

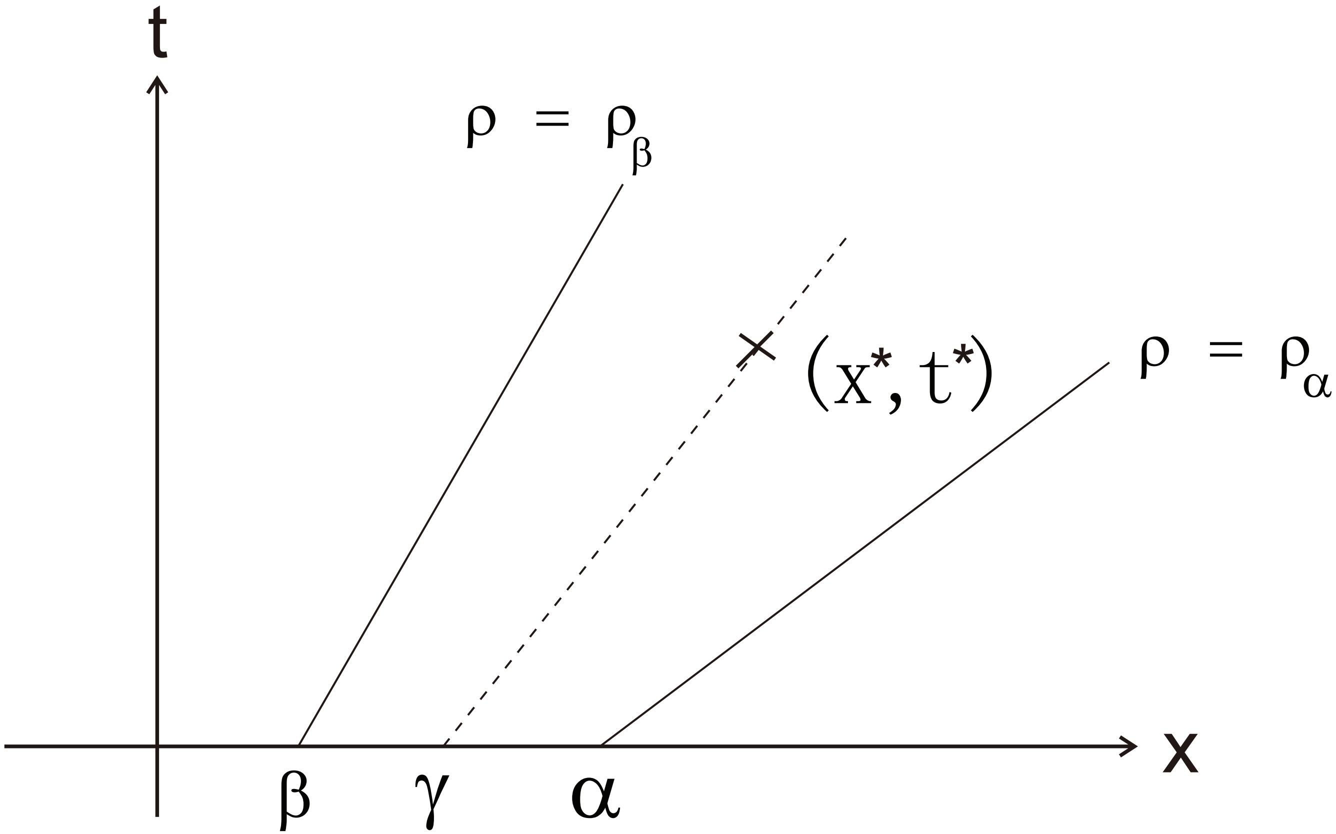
Note that the change in ρ is due to the change in observer's position.

The density will remain constant from the observer's viewpoint, as characterized by $\frac{d\rho}{dt} = 0$ or $\rho =$ constant provided that

$$\frac{dx}{dt} = q'(\rho).$$

This velocity $q'(\rho)$ depends on the density.

- If the observer moves at $q'(\rho)$, then the traffic density will appear constant to that observer.
- Each observer moves at a constant velocity on each individual characteristic line, but different observers may move at different constant velocities since they start with different initial traffic densities.



Along the curve $\dfrac{dx}{dt} = q'(\rho)$, $\dfrac{d\rho}{dt} = 0$ or $\rho =$ constant. Here, ρ equals the value $\rho(\alpha, 0) = \rho_\alpha$, a known constant. The characteristic line is given by

$$x = q'(\rho_\alpha)t + \alpha$$

and $\rho = \rho_\alpha$ on this straight line.

Given the point (x^*, t^*), we find the traffic density by determining the characteristic line that goes through the point. We have

$$\rho(x^*, t^*) = \rho(\gamma, 0)$$

where γ is the intercept on the x-axis.

Properties of the density wave velocity $dq/d\rho$

1. Since it is assumed that $\dfrac{dq}{d\rho}$ decreases as ρ increases, the wave velocity decreases as the traffic becomes denser.

2. $\dfrac{dq}{d\rho} = \rho\dfrac{du}{d\rho} + u$ as deduced from $q = \rho u(\rho)$.

   Since cars slow down as the traffic density increases, we have $\dfrac{du}{d\rho} \leq 0$, and consequently $\dfrac{dq}{d\rho} \leq u$.

Consider

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0,$$

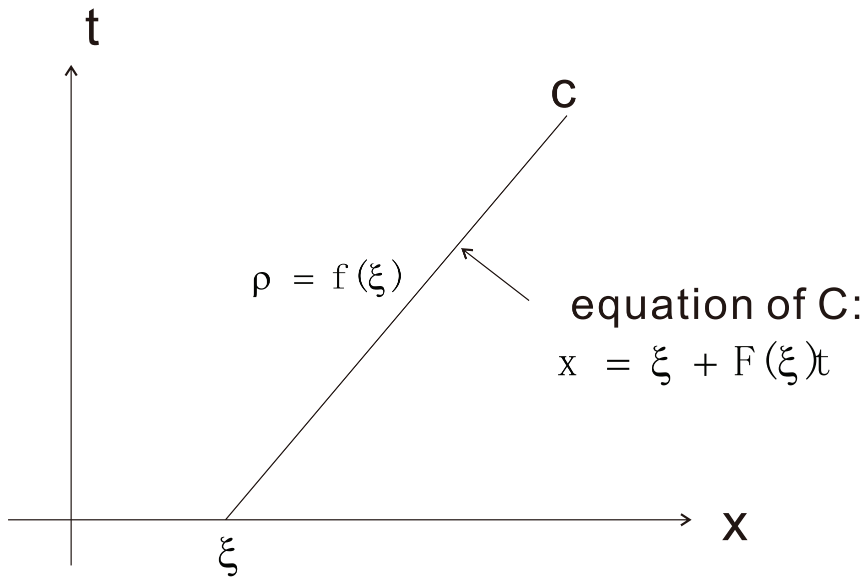
since $\dfrac{d\rho}{dt} = \dfrac{\partial \rho}{\partial t} + \dfrac{dx}{dt}\dfrac{\partial \rho}{\partial x}$ so that ρ remains constant along the line $\dfrac{dx}{dt} = q'(\rho)$.

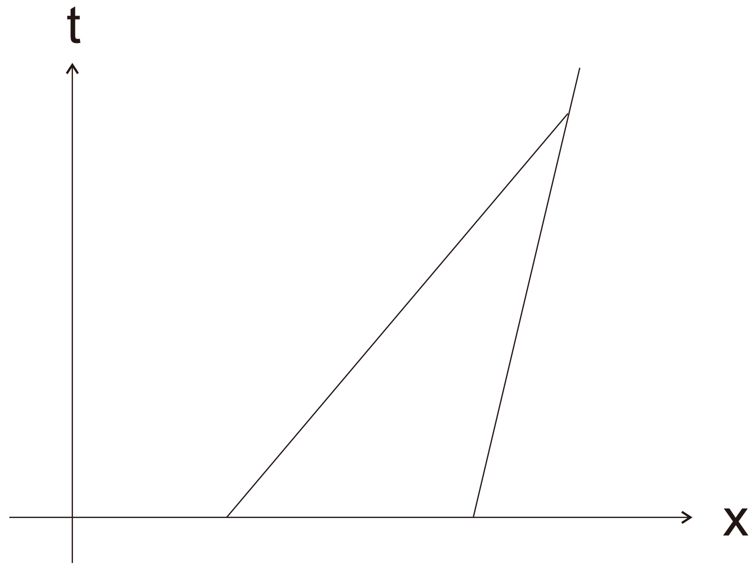
Suppose the initial condition:

$$\rho(x, 0) = f(x), \quad -\infty < x < \infty$$

is given. If one of the characteristics, say C, interests $t = 0$ at $x = \xi$, then $\rho = f(\xi)$ on the whole of the characteristic C. The slope of C is $q'(f(\xi)) = F(\xi)$. Equation of C is found to be

$$x = \xi + F(\xi)t.$$





Given any point (x_0, t_0) in the x-t plane, we solve for ξ_0 such that $x_0 = \xi_0 + F(\xi_0)t_0$, then the value of ρ at (x_0, t_0) is $\rho_0 = f(\xi_0)$.

Potential difficulties

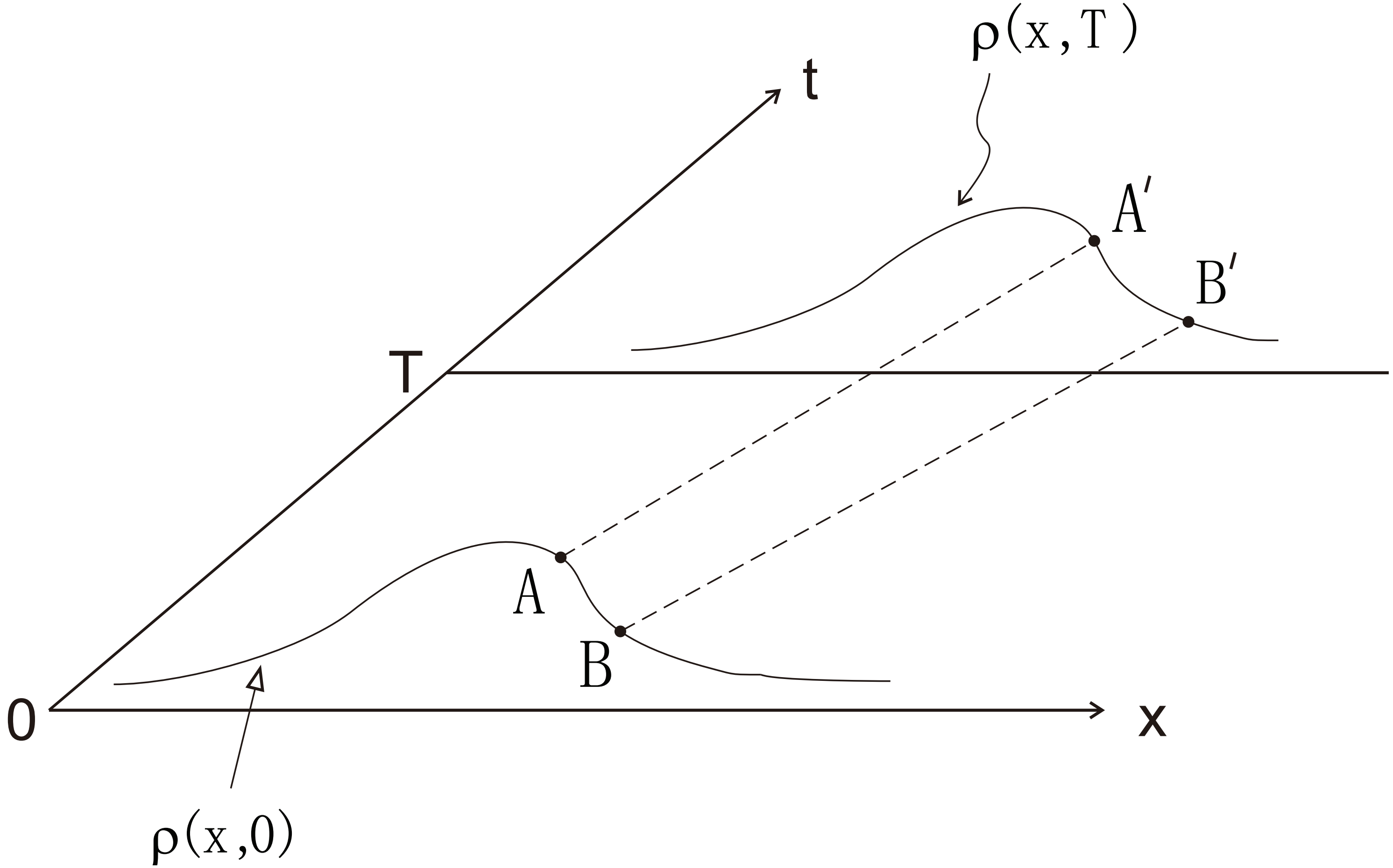
Two characteristics may intersect, then this leads to violation of single-valuedness of the traffic density function.

Wave phenomena in traffic flows

A wave is any recognizable signal that is transferred from one part of the medium to another with a recognizable velocity of propagation.

We model the propagation of traffic density as wave.

Wave velocity: $w(q) = \frac{dq}{d\rho}$; if an observer moves at the wave velocity, then he observes constant velocity of traffic flow.

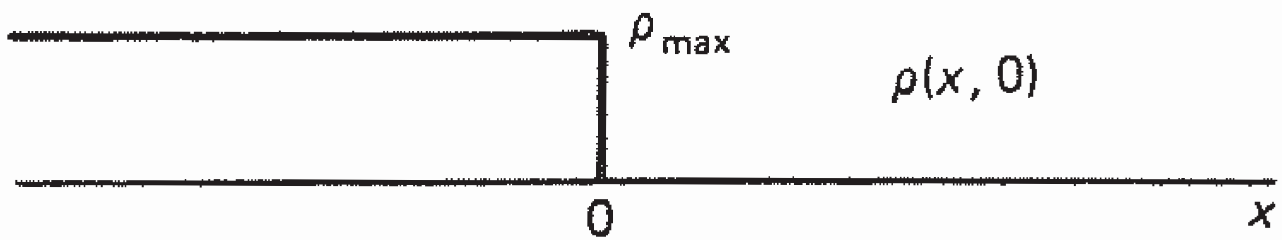


The signal of constant traffic density a is propagated from point A at time 0 to point A' at a later time T with wave velocity $q'(a)$, where $\dfrac{x_{A'} - x_A}{T - 0} = q'(a)$.

4.3 Impact of a traffic light turning green

- The position of traffic light is taken to be $x = 0$.
- Assume the cars to be bumper to bumper behind the traffic light, $\rho = \rho_{\max}$ for $x < 0$.
- Assume that there is no traffic ahead of the light (the light has stopped traffic long enough) so that

$$\rho = 0 \quad \text{for} \quad x > 0.$$



Traffic density due to an extremely long red light.

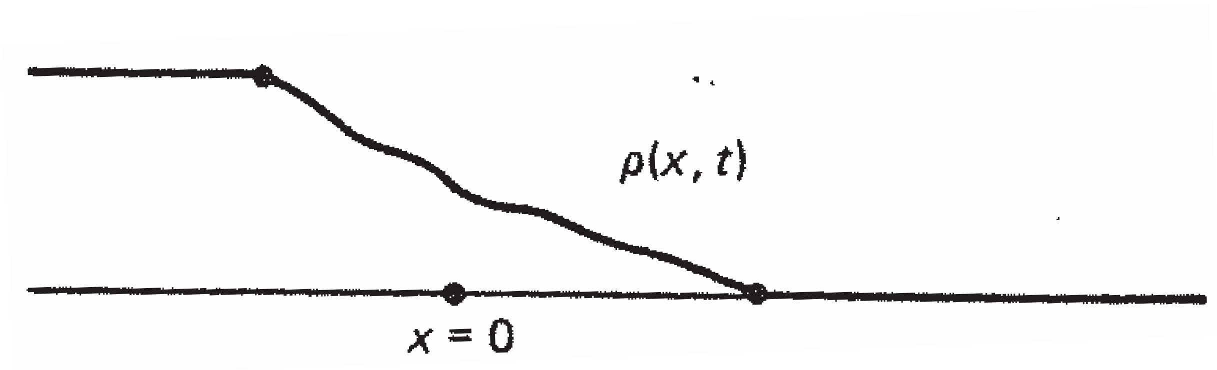
Recall that the traffic density $\rho(x,t)$ is constant along the characteristics. The governing equation is given by

$$\frac{dx}{dt} = \frac{dq(\rho)}{d\rho} = \rho\frac{du}{d\rho} + u.$$

The traffic density propagates at the velocity $\frac{dq}{d\rho}$. Since ρ remains constant, the density moves at a constant velocity. The characteristics are straight lines, namely,

$$x = t\frac{dq(\rho)}{d\rho} + k.$$

Each characteristic corresponds to a different integration constant k.



Traffic density: expected qualitative behavior after red light turns green.

Imagine you are in the first car. As soon as the light changes you observe zero density ahead of you, and therefore in this model you accelerate instantaneously to the speed u_{max}.

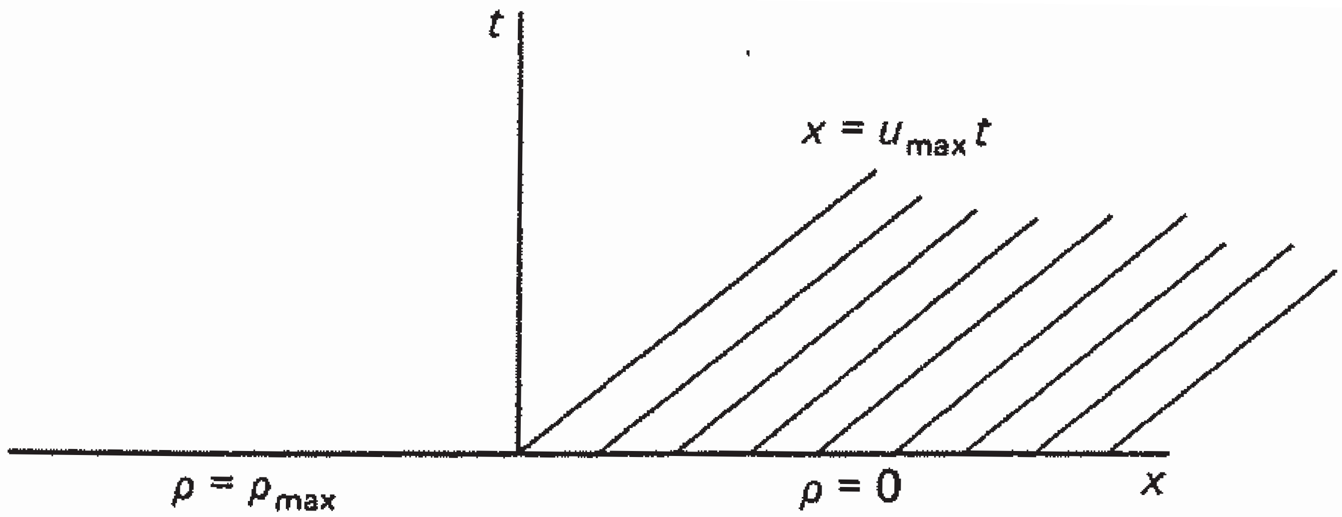
1. Consider all characteristics that intersect the initial data at $x > 0$. Note that $\rho(x, 0) = 0$ for $x > 0$.

   Hence, $\rho = 0$ along all the characteristics lines where

   $$\frac{dx}{dt} = \left.\frac{dq}{d\rho}\right|_{\rho=0} = u(0) = u_{\text{max}}.$$

   The characteristic velocity at zero density is always u_{max}, the car velocity for zero density.

You would not reach the point x until $t = x/u_{\text{max}}$ and thus there would be no cars at x for $t < x/u_{\text{max}}$.



Characteristics corresponding to no traffic.

2. Consider all characteristics that intersect the initial data at $x < 0$, where $\rho(x,0) = \rho_{\text{max}}$. Note that

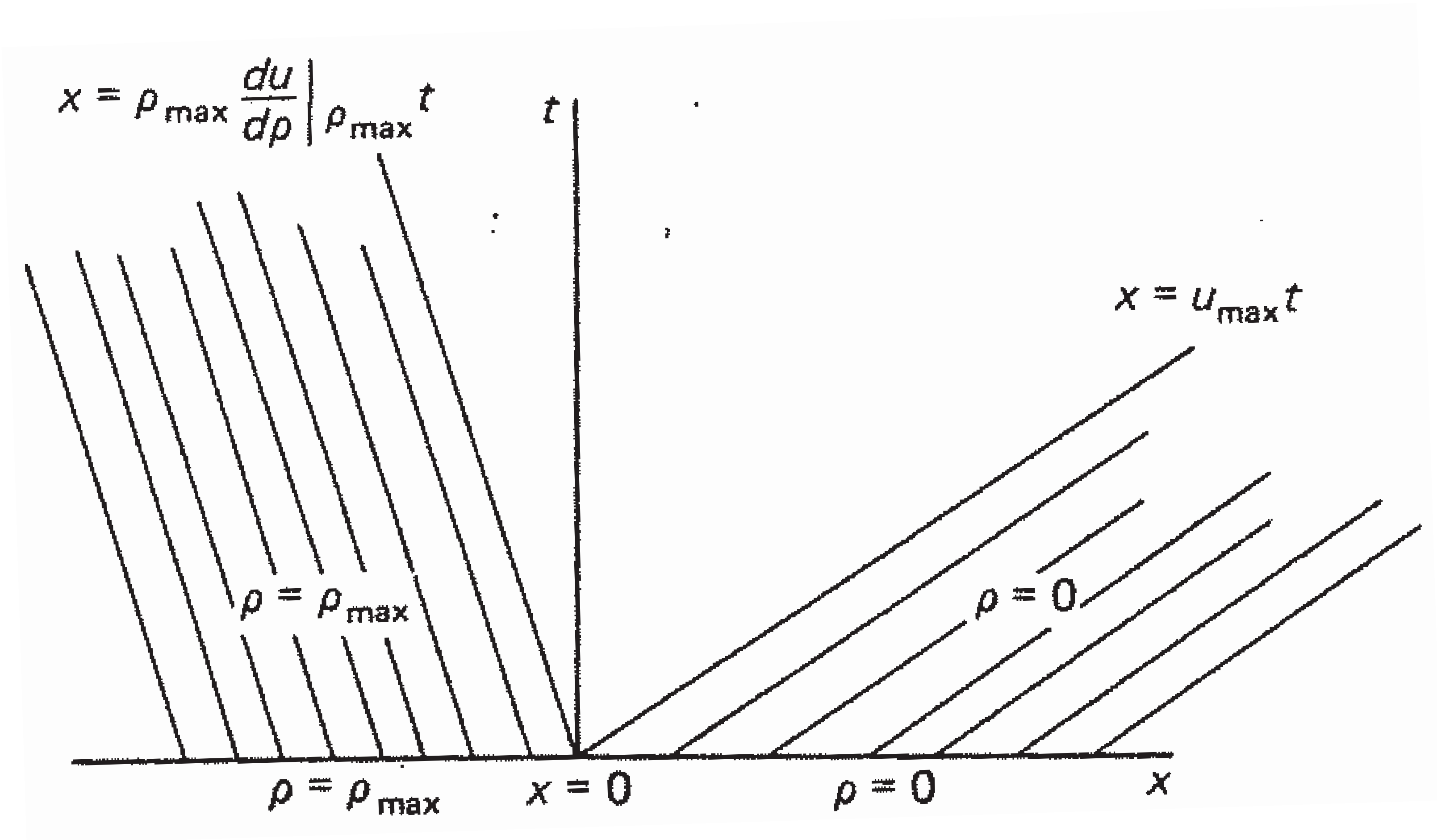
$$\frac{dx}{dt} = \left.\frac{dq}{d\rho}\right|_{\rho=\rho_{\text{max}}} = \rho_{\text{max}} \left.\frac{du}{d\rho}\right|_{\rho=\rho_{\text{max}}} = \rho_{\text{max}} u'(\rho_{\text{max}}) < 0$$

since $u(\rho_{\text{max}}) = 0$ and $u'(\rho_{\text{max}}) < 0$.

The characteristics are all parallel straight lines with negative slopes.

The cars are still facing bumper to bumper in the region: $x < \rho_{\text{max}} u'(\rho_{\text{max}}) t$.

After the light changes to green, the cars start moving and this takes a finite amount of time before each car moves.



Method of characteristics: regions of no traffic ($\rho = 0$) and bumper-to-bumper traffic ($\rho = \rho_{\text{max}}$).

Waiting times at a traffic light

Ignoring the driver's reaction and acceleration time, the n^{th} car waits an amount of time that equals

$$t = \frac{(n-1)L}{-\rho_{\text{max}} u'(\rho_{\text{max}})},$$

where L is the front-to-front distance between two consecutive cars.

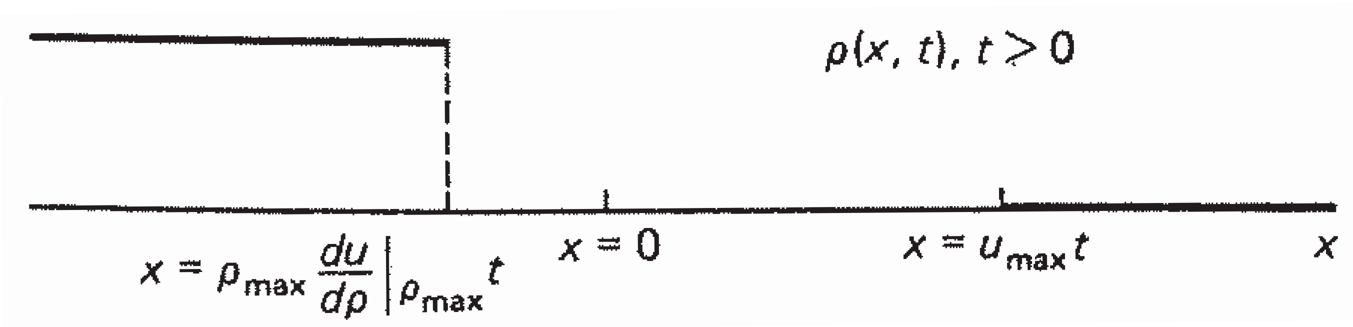
It would be interesting to measure the waiting times at a traffic light as a function of the car's position. One can perform this experiment.

- Is the waiting time roughly linearly dependent on the car's position as predicted above? Use the data to compute $u'(\rho_{\text{max}})$.
- Does $u'(\rho_{\text{max}})$ significantly vary for different road situations?

The density has not been determined in the region

$$\rho_{\text{max}} u'(\rho_{\text{max}}) t < x < u_{\text{max}} t,$$

the region in which cars actually pass through the green traffic light!



To solve the problem, we first assume that the initial traffic density is not discontinuous, but smoothly varied between $\rho = 0$ and $\rho = \rho_{\text{max}}$ in a very small distance Δx near the traffic light.

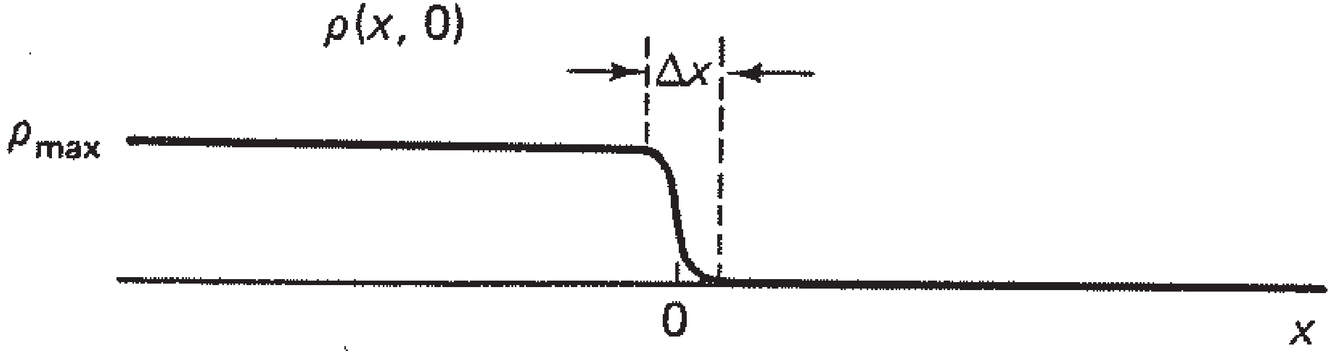
Suppose there is a very small distance Δx near the traffic light where the initial traffic density is smoothly varied between $\rho = 0$ and $\rho = \rho_{\text{max}}$. The equation of the characteristic line is

$$x = \frac{dq}{d\rho}t + x_0.$$

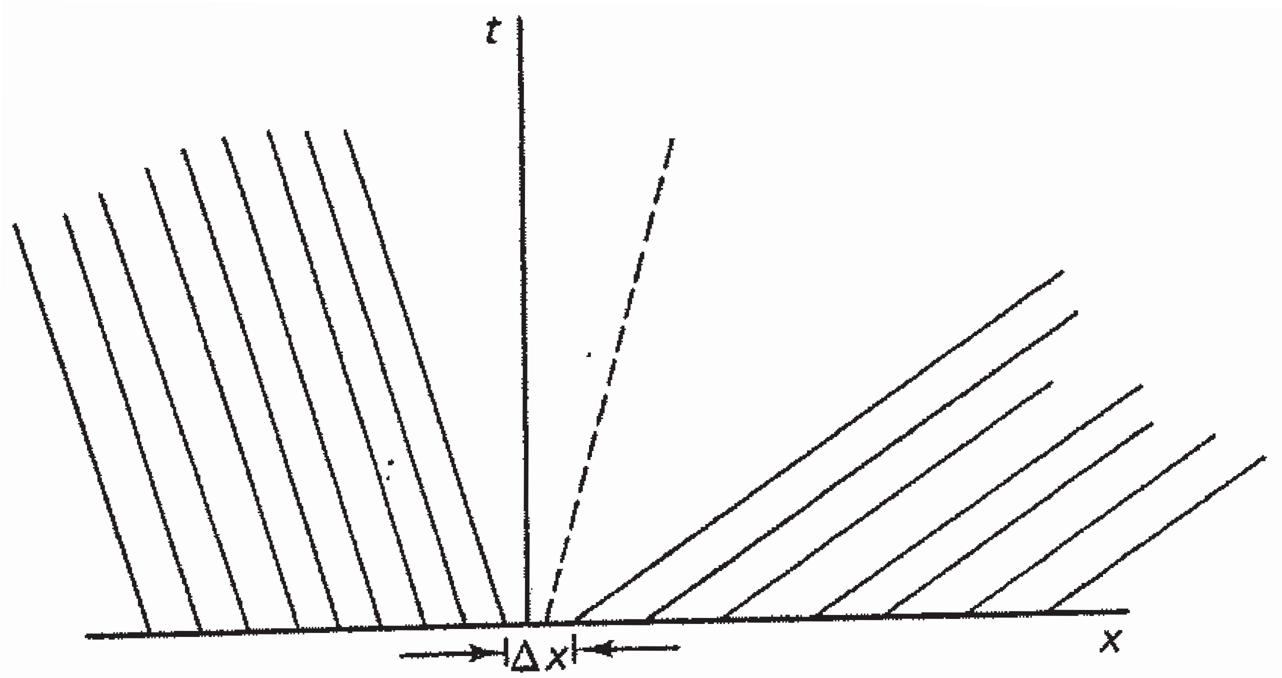
Since ρ changes continuously between $\rho = 0$ and $\rho = \rho_{\text{max}}$, the velocity $\frac{dq}{d\rho}$ is always between u_{max} and $\rho_{\text{max}} u'(\rho_{\text{max}})$.

(i) For a denser traffic, so higher ρ, then $\frac{dq}{d\rho}$ is smaller

(ii) There is a value for ρ such that $\frac{dq}{d\rho} = 0$, so q attains its maximum value.



Continuous model of the initial traffic density.



Space-time diagram for the rapid transition from no traffic to bumper-ro-bumper traffic.

As soon as the light changes from red to green, the maximum flow occurs at the light, $x = 0$, and it stays there for all future time.

If this theory is correct, that is, $u = u(\rho)$, then we position an observer at a traffic light. One then wait until the light turns red and many cars line up. When the light turns green, we simply measure the traffic flow at the light.

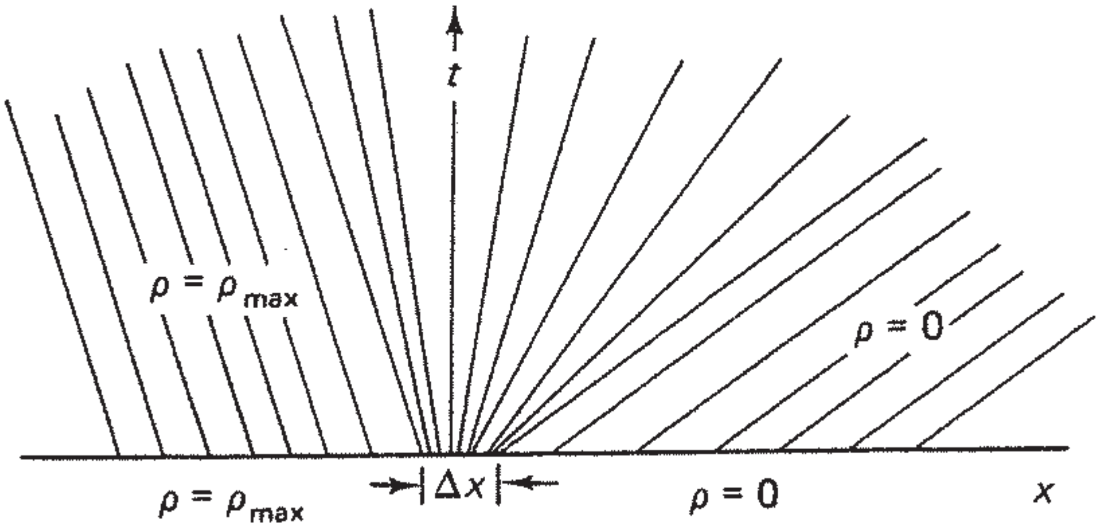
This measured traffic flow of cars will be constant and equal to the maximum possible for the road (called the capacity of the road).

Taking the limit $\Delta x \to 0$, then ρ is constant along the characteristic:

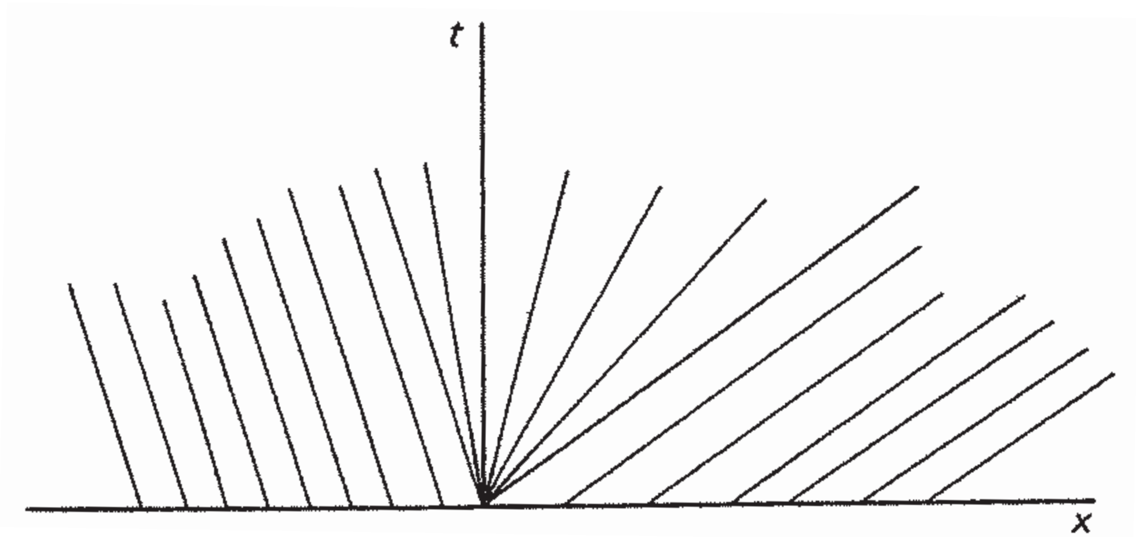
$$\frac{dx}{dt} = \frac{dq}{d\rho},$$

which are straight lines emanating from $x = 0$.

It is as though at the discontinuity ($x = 0$) that all traffic densities between $\rho = 0$ and $\rho = \rho_{\text{max}}$ are observed. The observers (following constant density) then travel at different constant velocities $\dfrac{dq}{d\rho}$ depending on which density they initially observe at $x = 0$. Along each characteristic, the density is constant.

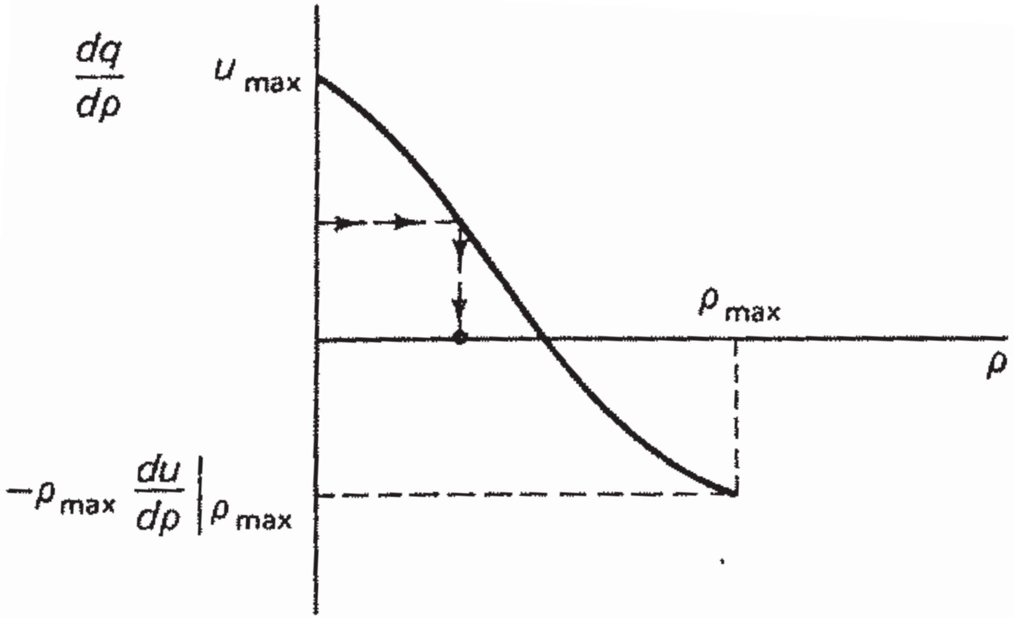


Method of characteristics: Δx is the initial distance over which density changes from 0 to $\rho_{\max}$.



Fan-shaped characteristics due to discontinuous initial data.

To obtain the density at a given x and t, we must determine which characteristic line that goes through (x, t). Assuming that $\frac{dq}{d\rho}$ depends only on ρ, we solve algebraically for ρ as a given value of $\frac{x}{t}$.



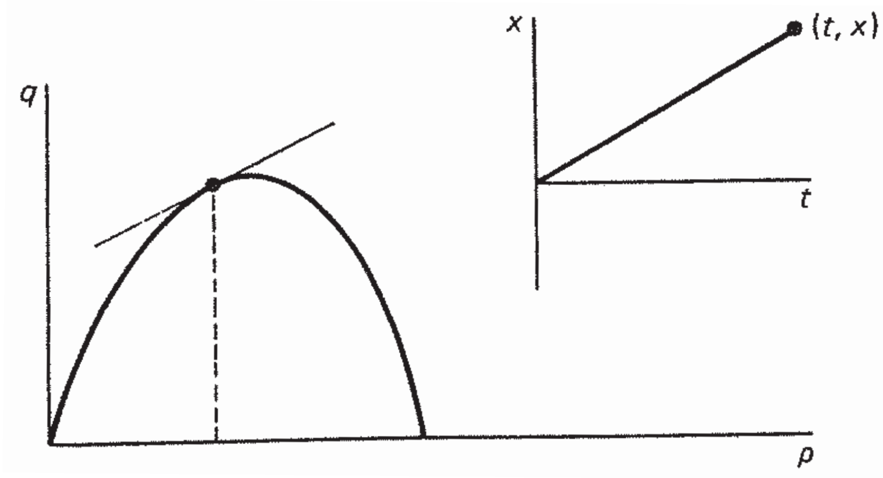
Determination of traffic density from density wave velocity.

From the graph, for a given value of $\frac{x}{t}$ for $\frac{dq}{d\rho}$, we read out the solution for ρ.

The Fundamental Diagram of Road Traffic can be used to determine graphically the density at a given position on the roadway in the region of fanlike characteristics.

Given t and x, the slope of the straight line from the origin to the point (t, x) in the figure equals $dq/d\rho$. Thus this straight line must have the same slope as the tangent to the flow-density (q-ρ) curve.

The traffic density can thus be estimated by finding the density on the q-ρ curve whose slope is the same as x/t.

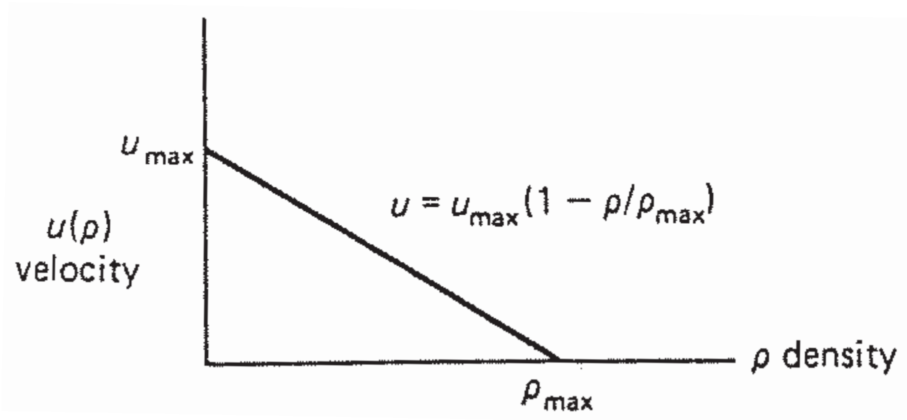


Traffic density in fan-like region of characteristics: graphical technique.

Linear velocity-density relationship

If the velocity-density relationship is assumed to be linear, then

$$u(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right).$$

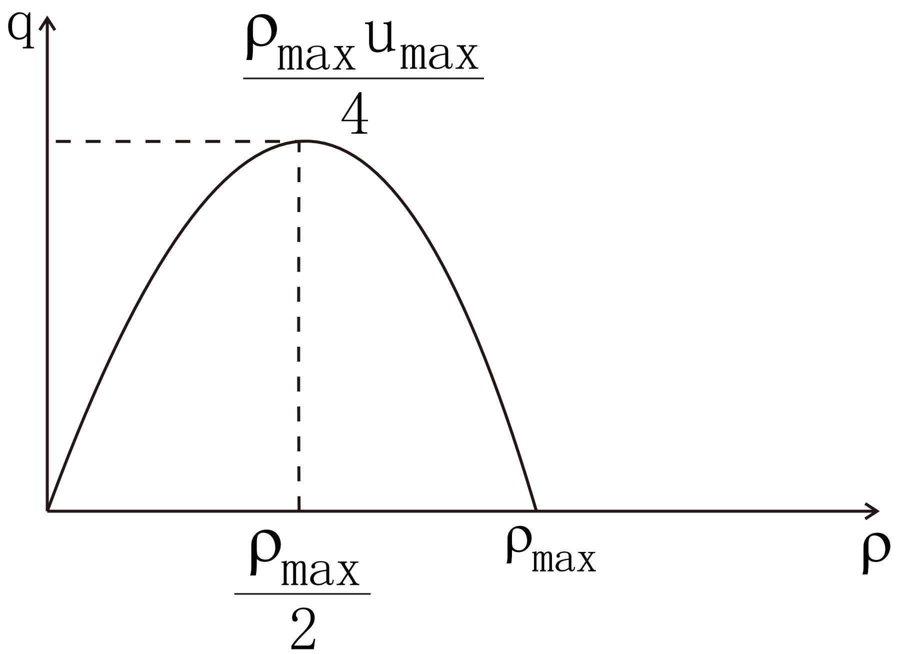


The traffic flow is given by

$$q = \rho u = u_{\max} \rho \left(1 - \frac{\rho}{\rho_{\max}} \right)$$

and

$$\frac{dq}{d\rho} = u_{\max} \left(1 - \frac{2\rho}{\rho_{\max}} \right).$$



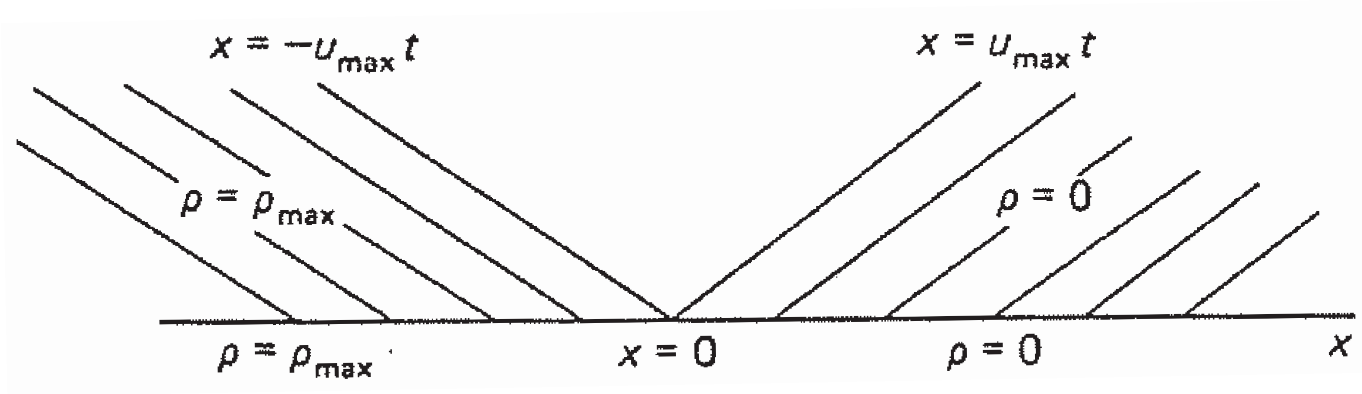
Under the linear velocity-density assumption, the density at which the traffic flow is maximized is exactly one-half the maximum density, $\rho = \frac{\rho_{\text{max}}}{2}$. The speed is one-half the maximum speed

$$u\left(\frac{\rho_{\text{max}}}{2}\right) = \frac{u_{\text{max}}}{2}.$$

The maximum traffic flow is

$$q\left(\frac{\rho_{\text{max}}}{2}\right) = \frac{\rho_{\text{max}} u_{\text{max}}}{4}.$$

Space-time diagram for the traffic light problem



Now, $u'(\rho_{\max})\rho_{\max} = -u_{\max}$ so that the left side characteristics are bounded by $x = -u_{\max}t$. How to calculate the density in the fanlike region:

$$-u_{\max}t < x < u_{\max}t \ ?$$

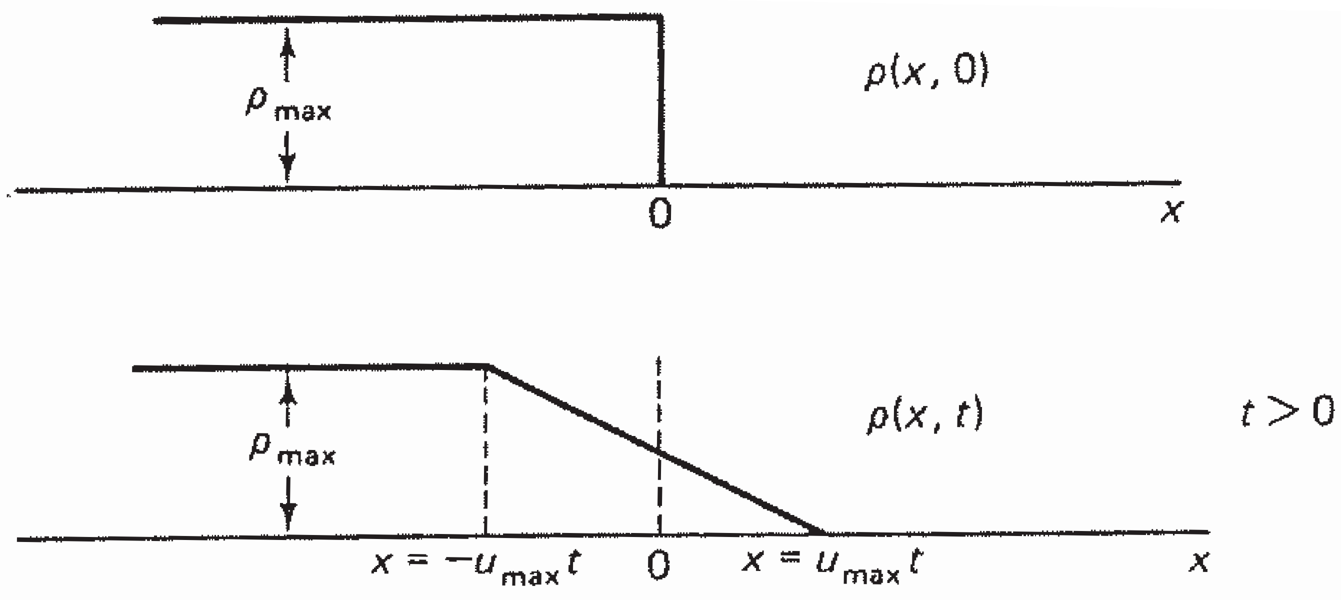
From the equation

$$\frac{x}{t} = \frac{dq}{d\rho} = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right),$$

we obtain

$$\rho = \frac{\rho_{\max}}{2}\left(1 - \frac{x}{u_{\max}t}\right).$$

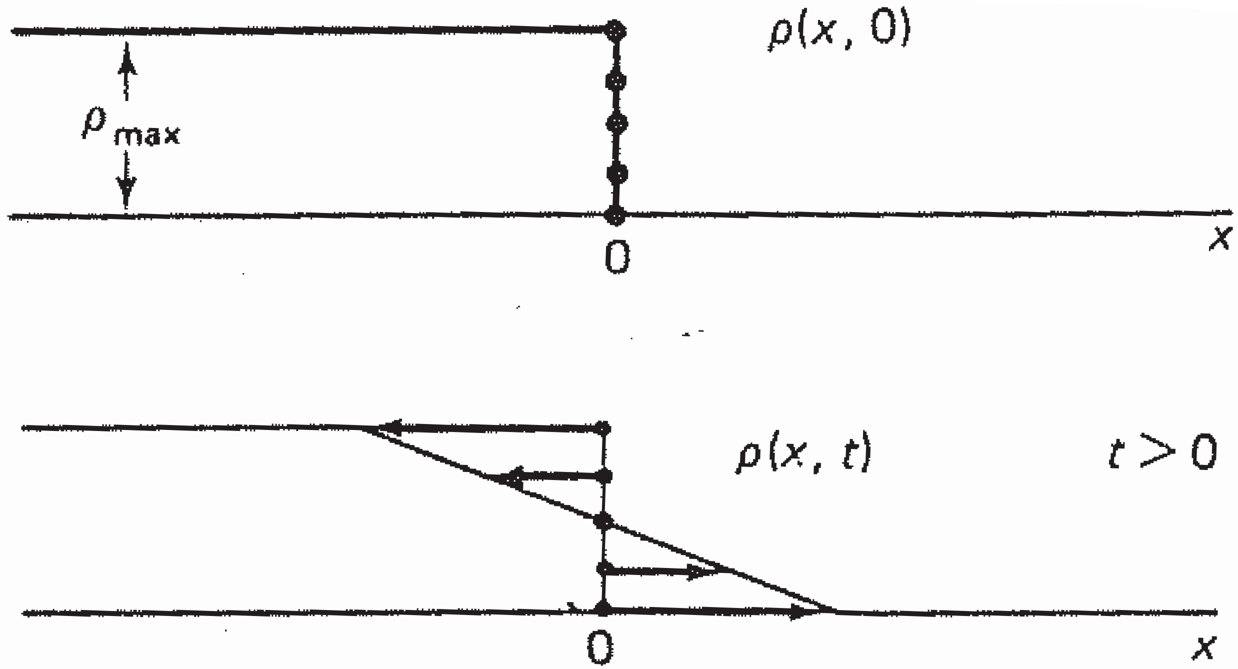
For a fixed time, the density is linearly dependent on x (in the fanlike region).



Traffic density: before and after light turns green.

For $t > 0$, and at $x = 0$, we obtain $\rho = \frac{\rho_{max}}{2}$. This is the density corresponding to the maximum flow.

Let us follow observers staying with the constant densities ρ_{max}, $3\rho_{max}/4$, $\rho_{max}/2$, $\rho_{max}/4$, and 0, marked by • on the diagram below representing the initial density. Each observer is moving at a different constant velocity. After some time (introducing an arrow showing how each observer must move), the diagram shows that the linear dependence of the wave velocity on the density yields a linear density profile.

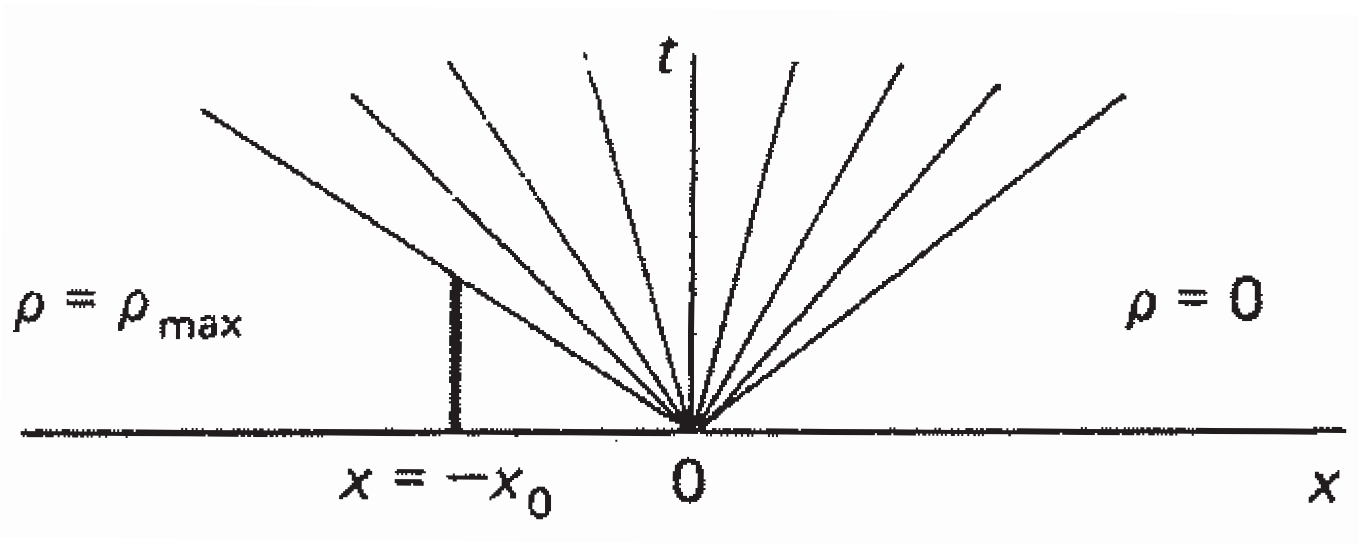


Different traffic density wave velocities.

Motion of an individual car starting at $-x_0$ ($x_0 > 0$)

The velocity of the car is given by the field velocity

$$\frac{dx}{dt} = u(x,t) = u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right).$$



Car path (while car is not moving).

The car stays still until the wave, propagating the information of the change of the light, reaches the car. After $t = x_0/u_{\text{max}}$, the car moves at the velocity given in the fanlike region.

When the car starts moving, its velocity is first zero and then slowly increases. Given $\rho = \frac{\rho_{\max}}{2}\left(1 - \frac{x}{u_{\max}t}\right)$, we have

$$\frac{dx}{dt} = \frac{u_{\max}}{2} + \frac{x}{2t}.$$

We solve for

$$t\frac{dx}{dt} - \frac{x}{2} = \frac{u_{\max}}{2}t$$

with the auxiliary condition: $t = \frac{x_0}{u_{\max}}$ and $x = -x_0$.

The solution takes the form

$$x(t) = u_{\max}t + Bt^{1/2}.$$

The arbitrary constant is determined by

$$-x_0 = x_0 + B\left(\frac{x_0}{u_{\max}}\right)^{1/2}$$

so

$$B = -2x_0\left(\frac{u_{\max}}{x_0}\right)^{1/2} = -2(x_0 u_{\max})^{1/2}.$$

Hence, the position of the car is

$$x = u_{\mathsf{max}}t - 2(x_0 u_{\mathsf{max}}t)^{1/2}$$

and the car velocity is

$$\frac{dx}{dt} = u_{\mathsf{max}} - \left(\frac{x_0 u_{\mathsf{max}}}{t}\right)^{1/2}.$$

As $t \to \infty$, the car reaches the maximum velocity u_{max}.

How long does it take the car to actually pass the light? We set $x = 0$, where

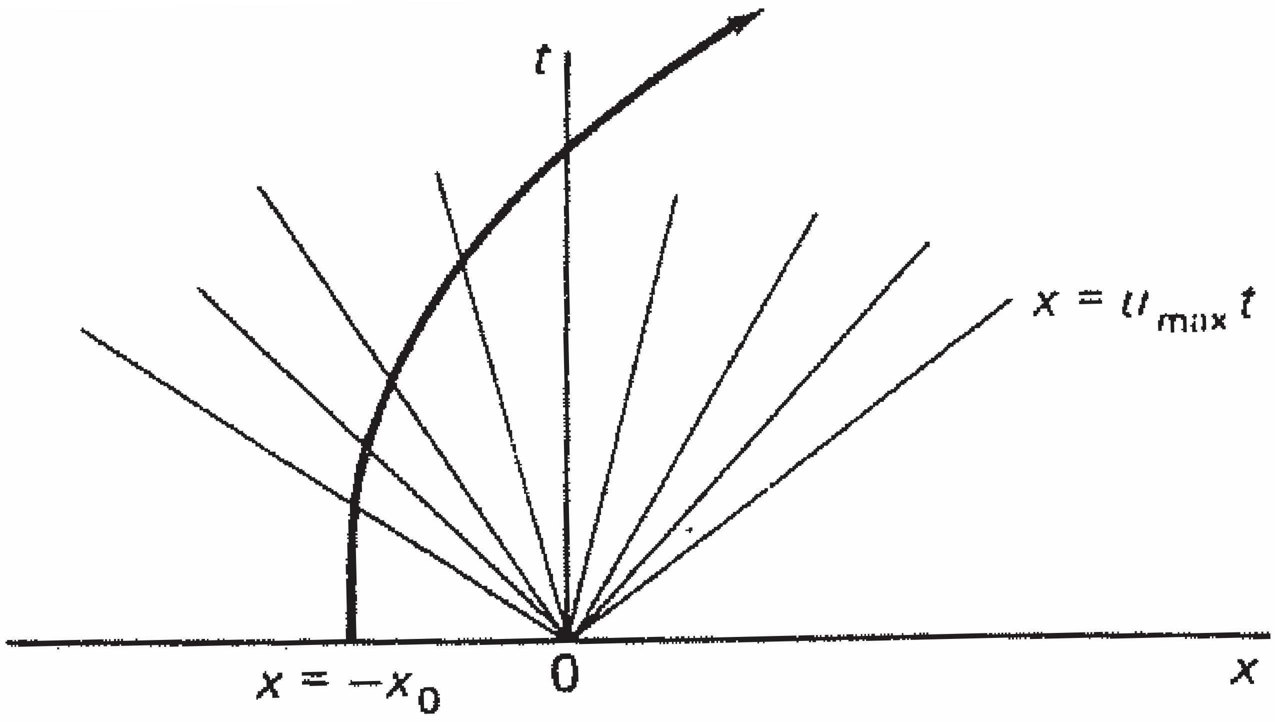
$$0 = u_{\mathsf{max}}t - 2(x_0 u_{\mathsf{max}}t)^{1/2}$$

giving

$$t = 4\frac{x_0}{u_{\mathsf{max}}},$$

At what speed is the car going when it passes the light? This is $u_{\mathsf{max}}/2$.

The path is parabolic since the relation is $x = u_{\text{max}}t - 2(x_0 u_{\text{max}} t)^{1/2}$.



Path of a car accelerating past a traffic signal.

Number of cars passing the traffic light over a given time interval

If the light stays green until time T, how many cars will pass the traffic light?

Recall that a car starting at $-x_0$ passes the traffic light at $t = 4x_0/u_{\text{max}}$. At time T, a car stating from $-u_{\text{max}}T/4$ will be at the light. The number of cars contained in that distance is $\rho_{\text{max}}\left(\frac{u_{\text{max}}T}{4}\right)$.

Numerical computation

For a one-minute light, using $\rho_{\text{max}} = 225$ and $u_{\text{max}} = 40$mph, the number of cars passing$= \frac{225}{4} \cdot 40 \cdot \frac{1}{60} \approx 37.5$ cars.

Solution for $\rho(x,t)$

1. Parameterizing the initial position as a function of x and t.

   Each characteristic is labeled by its position x_0 at $t = 0$. Hence, for given values of x and t, we try to find x_0. This is done by eliminating ρ from the two equations:

$$x = u_{\text{max}}\left(1 - \frac{2\rho}{\rho_{\text{max}}}\right)t + x_0 \quad \text{and} \quad \rho = f(x_0) \Rightarrow x_0 = x_0(x,t).$$

Example

Supposing $\rho(x,0) = \dfrac{\rho_{\text{max}}}{1 + e^{x/L}}$, then

$$x = u_{\text{max}}\left(1 - \frac{2}{1 + e^{x_0/L}}\right)t + x_0.$$

Since the density at the point (x,t) depends on x_0, then

$$\rho(x,t) = \rho(x_0,0) = f(x_0) = f(x_0(x,t)).$$

2. Parameterizing the initial position as a function of the initial density

   First, we determine x_0 as a function of ρ; that is, $x_0 = x_0(\rho)$.

   We then substitute $x_0 = x_0(\rho)$ into $x = u_{\text{max}} \left(\frac{1 - 2\rho}{\rho_{\text{max}}} \right) t + x_0$, we obtain an equation involving only x, t and ρ, giving ρ's dependence on x and t.

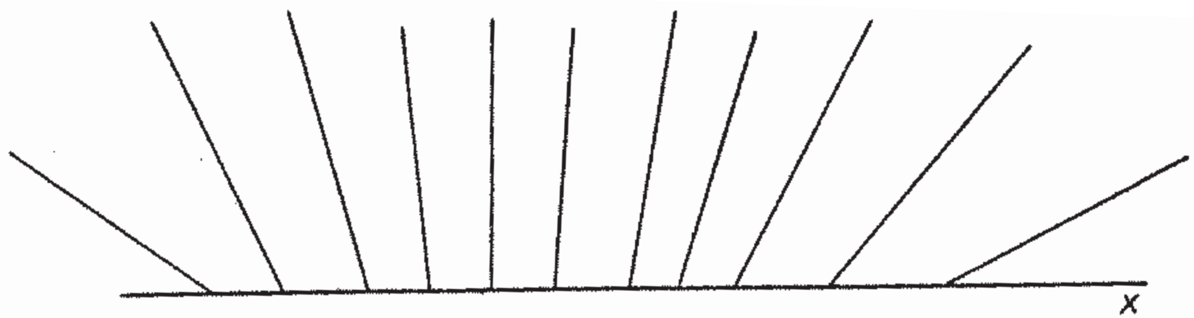
Suppose the initial density varies in a prescribed way: $\rho(x, 0) = f(x)$.
Assume $u(\rho) = u_{\text{max}}\left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$, then

$$\frac{dx}{dt} = \frac{dq}{d\rho} = u_{\text{max}}\left(1 - \frac{2\rho}{\rho_{\text{max}}}\right).$$

The characteristic starting from $x = x_0$ is

$$x = u_{\text{max}}\left(1 - \frac{2\rho}{\rho_{\text{max}}}\right)t + x_0.$$

Along the above characteristic, density is constant and its value is $f(x_0)$. Here, we assume that the characteristics do not intersect.



Nonparallel nonintersecting characteristics.

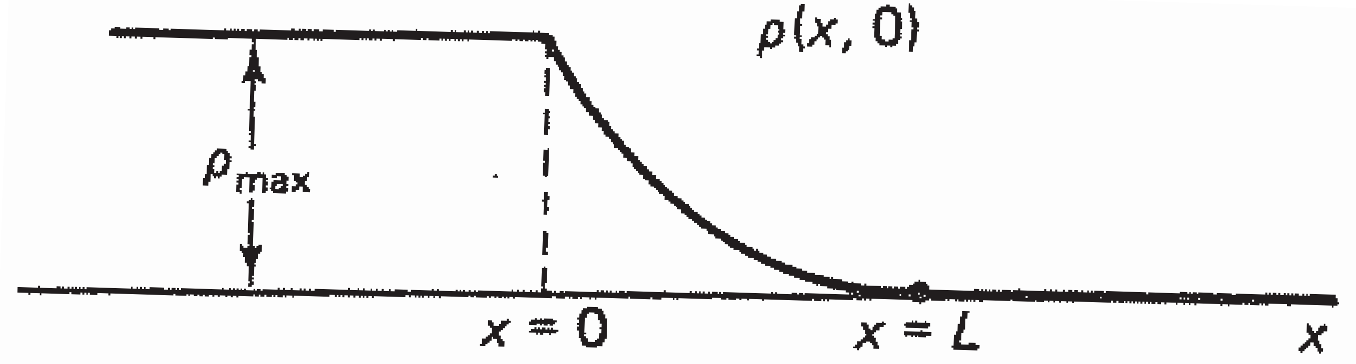
Example

Consider the velocity function

$$u(\rho) = u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$$

and the density function

$$\rho(x,0) = \begin{cases} \rho_{\text{max}} & x < 0 \\ \rho_{\text{max}} \frac{(x-L)^2}{L^2} & 0 < x < L \\ 0 & x > L \end{cases} .$$



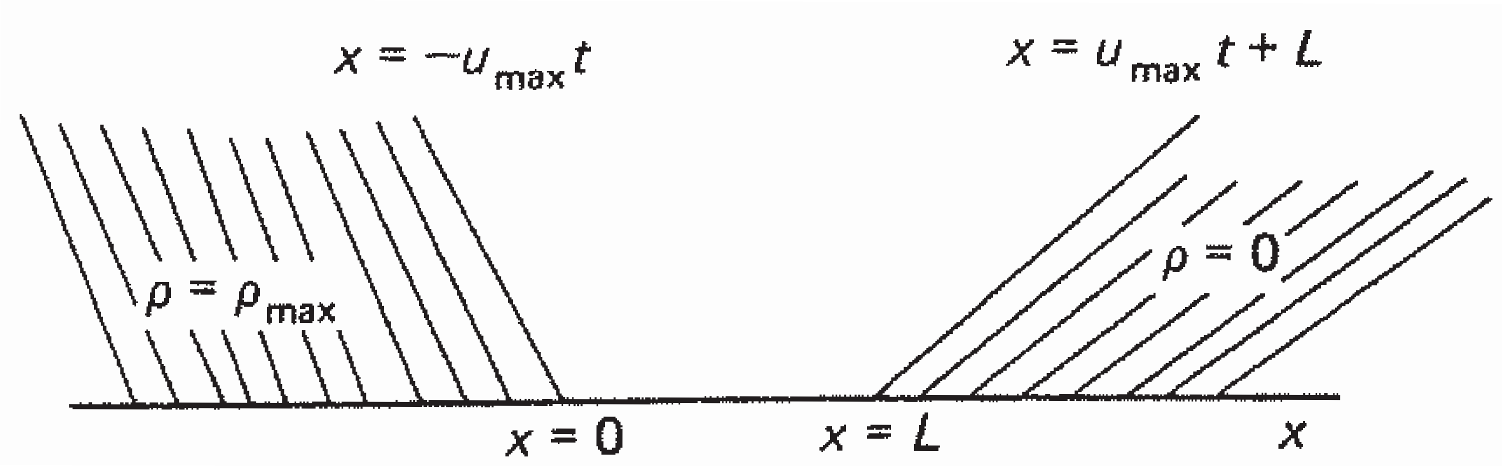
Initial traffic density.

If $x_0 > L$ or $x_0 < 0$, then the characteristics start from a region of constant density. The corresponding density wave velocities are given by

$$\left.\frac{dq}{d\rho}\right|_{\rho=0} = u_{\text{max}} \quad \text{and} \quad \left.\frac{dq}{d\rho}\right|_{\rho=\rho_{\text{max}}} = -u_{\text{max}}.$$

We obtain the density function

$$\rho = \begin{cases} 0 & \text{for } x > u_{\text{max}}t + L \\ \rho_{\text{max}} & \text{for } x < -u_{\text{max}}t \end{cases}.$$



The two set of characteristic lines at $x \leq 0$ and $x \geq L$.

Method I: $x_0 = x_0(x,t)$

For the characteristics that start from $0 < x_o < L$, we have

$$\rho = \frac{\rho_{\text{max}}(x_0 - L)^2}{L^2},$$

so the equation for this characteristic is

$$x = u_{\text{max}}\left[1 - \frac{2}{L^2}(x_0 - L_0)^2\right]t + x_0, \qquad 0 < x_0 < L.$$

Solving for x_0:

$$(x_0 - L)^2\frac{2u_{\text{max}}t}{L^2} - (x_0 - L) + x - L - u_{\text{max}}t = 0,$$

we obtain

$$x_0 - L = \frac{1 - \sqrt{1 - \frac{8u_{\text{max}}t}{L^2}(x - L - u_{\text{max}}t)}}{4u_{\text{max}}t/L^2},$$

where the negative sign is chosen.

Since $\rho = \dfrac{\rho_{\text{max}}(x_0 - L)^2}{L^2}$, we obtain

$$\rho(x,t) = \frac{\rho_{\text{max}}}{L^2} \frac{\left[1 - \sqrt{1 - \frac{8u_{\text{max}}t}{L^2}(x - L - u_{\text{max}}t)}\right]^2}{16u_{\text{max}}^2 t^2/L^4}$$

So that

$$\frac{\rho(x,t)}{\rho_{\text{max}}} = \left[\frac{-L + \sqrt{L^2 - 8u_{\text{max}}t(x - L - u_{\text{max}}t)}}{4u_{\text{max}}t}\right]^2.$$

As a check of the satisfaction of the auxiliary conditions, we observe

(i) as $x \to u_{\text{max}}t + L$, $\rho \to 0$

(ii) as $x \to -u_{\text{max}}t$, $\rho \to \dfrac{\rho_{\text{max}}}{L^2} \dfrac{\left[1 - \sqrt{\left(1 + \frac{4u_{\text{max}}t}{L}\right)^2}\right]^2}{16u_{\text{max}}^2 t^2/L^4} = \rho_{\text{max}}$.

(iii) as $t \to 0$, using $\sqrt{1-t} \approx 1 - \dfrac{t}{2}$ for small t, we have

$$\rho(x,t) = \frac{\rho_{\text{max}}}{L^2} \frac{\left[1 - \left(1 - \frac{4u_{\text{max}}t}{L}(x-L)\right)\right]^2}{16u_{\text{max}}^2 t^2/L^4} = \frac{\rho_{\text{max}}(x-L)^2}{L^2}.$$

This is precisely the initial condition: $\rho(x,0)$, $0 < x < L$.

Method II: $x_0 = x_0(\rho)$

First, we determine x_0 as a function of ρ,

$$(x_0 - L)^2 = \frac{L^2\rho}{\rho_{\max}} \quad \text{or} \quad x_0 = L \pm L\sqrt{\frac{\rho}{\rho_{\max}}}.$$

The negative sign must be chosen so that

$$x_0 = L - L\sqrt{\rho/\rho_{\max}} = L\left(1 - \sqrt{\frac{\rho}{\rho_{\max}}}\right).$$

As ρ varies between 0 and $\rho_{\max}$, x_0 varies between 0 and L.

The equation for the density becomes

$$x = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)t + L\left(1 - \sqrt{\frac{\rho}{\rho_{\max}}}\right).$$

If we treat the above equation as a quadratic equation for $\sqrt{\rho/\rho_{\text{max}}}$, then we obtain

$$\left(\sqrt{\frac{\rho}{\rho_{\text{max}}}}\right)^2 2u_{\text{max}}t + L\sqrt{\frac{\rho}{\rho_{\text{max}}}} + (x - L - u_{\text{max}}t) = 0.$$

Solving the quadratic equation, we obtain

$$\sqrt{\frac{\rho}{\rho_{\text{max}}}} = \frac{-L + \sqrt{L^2 - 8u_{\text{max}}t(x - L - u_{\text{max}}t)}}{4u_{\text{max}}t},$$

where the positive sign of the square not has been chosen since $\sqrt{\frac{\rho}{\rho_{\text{max}}}} > 0$.

Summary of the calculation procedures

Given the point (x, t), find the characteristic that originates from the point $(x_0, 0)$. On the characteristic, the density ρ is constant.

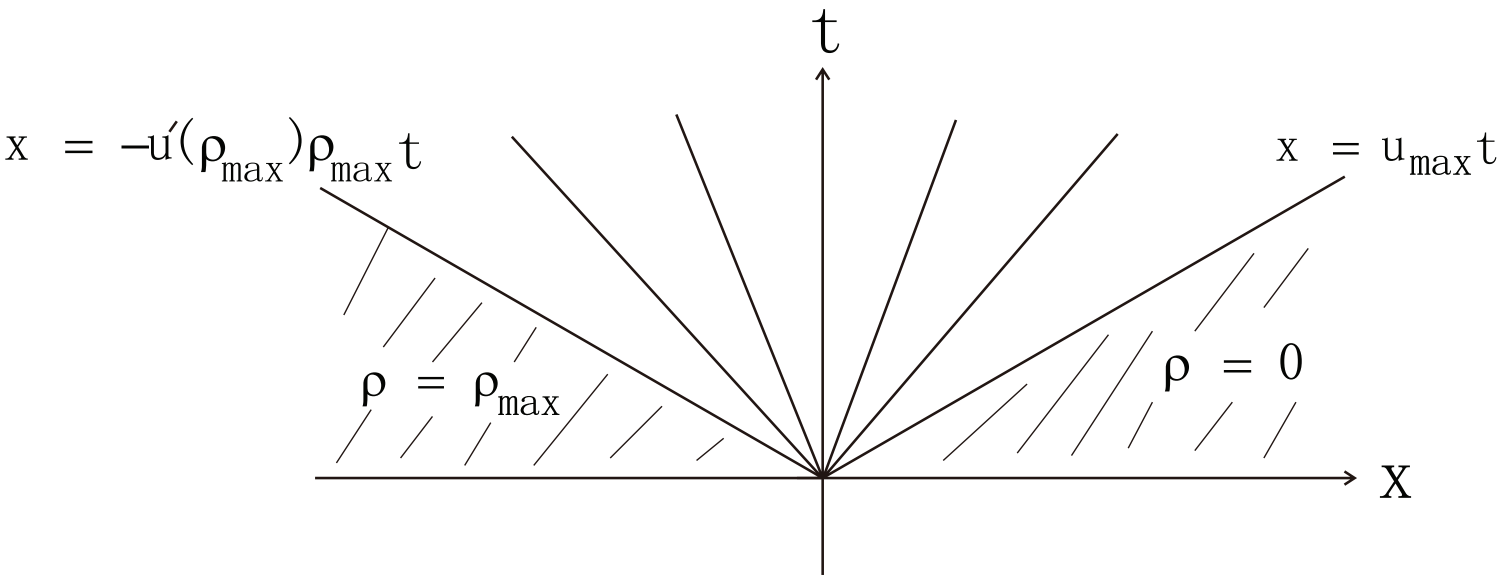
Equation of the characteristic: $\dfrac{dx}{dt} = \dfrac{dq}{d\rho}$.

Let $\rho(x, 0) = f(x)$, then the equation of the characteristic that passes through (x, t) is

$$\frac{dx}{dt} = \left.\frac{dq}{d\rho}\right|_{\rho=f(x_0)},$$

where $q = \rho u$ and $u = u(\rho)$ is given (from empirical observations). The characteristics are straight lines.

Fan-like characteristics



When the characteristics resemble the fan-like rays bounded by two boundary characteristics, we have $\frac{dx}{dt} = \frac{x}{t}$ since the characteristics are straight lines through the origin.

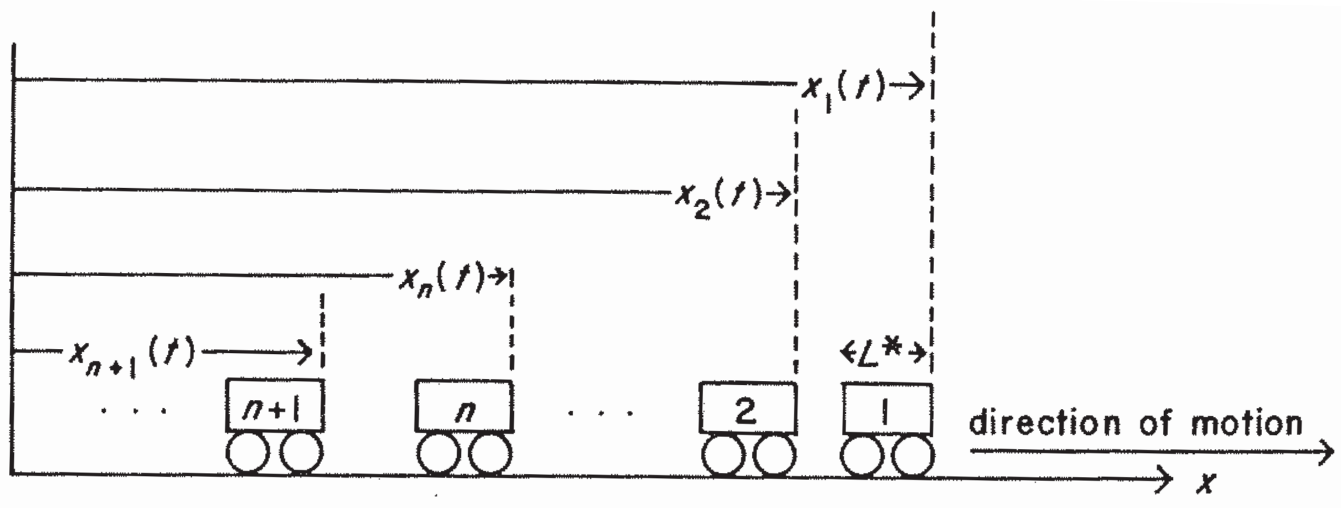
Kinematic equation: $\frac{dx}{dt} = u(\rho)$

We try to obtain ρ as a function of x and t.

4.4 Car-following models

Car-following theory

Model the single-lane (no passing) relatively dense traffic on long straight highways.



Geometry of Car-Following Models

Homogeneity assumptions

All drivers in the line drive the same type of cars which are in the same mechanical condition and all drivers react the same way to a given situation. Quantities considered: speed, acceleration and separation distances.

Pipes' Model

California Vehicle Code statement: A good rule for following another vehicle at a safe distance is to allow yourself the length of a car (about 15 feet) for every 10 miles per hour (14.67 ft/sec) you are traveling.

Define $T^* = 15\text{ft}/[14.67\text{ft/sec}] \approx 1.02s$ (take it to be 1 sec as an approximation)

Let $L^* = 15$ft be the length of the vehicles, then

$$x_n(t) = x_{n+1}(t) + \frac{L^*}{14.67} v_{n+1}(t) + L^* + b^*, \tag{1}$$

where $v_{n+1}(t)$ is the speed of the $(n+1)^{\text{st}}$ vehicle (in units of 10 miles per hour).

The quantity b^* is added, which is the legal distance at rest. If $b^* = 0$, this would mean the car behind will touch the bumper of the car ahead when the vehicles are at rest. Differentiating with respect to t in eq.(1), we obtain

$$v_n(t) = v_{n+1}(t) + T^* \frac{dv_{n+1}(t)}{dt}. \tag{2}$$

If we set $T^* = 1$, we have

$$a_{n+1}(t) = \frac{dv_{n+1}(t)}{dt} = v_n(t) - v_{n+1}(t).$$

The trailing car adjusts the acceleration $a_{n+1}(t)$ in response to the relative velocity between the vehicle and the vehicle ahead.

$$\begin{array}{ccccc} \text{response} & = & \text{sensitivity} & \times & \text{stimulus} \\ \uparrow & & \uparrow & & \uparrow \\ \text{acceleration } a_{n+1}(t) & & \frac{1}{T^*} & & \text{relative velocity } v_n - v_{n+1} \end{array}$$

Other possible modification

Recall that the California Vehicle Code dictated

$$x_n(t) - x_{n+1}(t) - b^* - L^* - T^* v_{n+1}(t) = 0. \qquad (1)$$

Suppose there is a fluctuation in the behavior of the lead car, as a result of response lags, that causes Eq(1) to be violated. Define $s_{n+1}(t) = x_n(t) - x_{n+1}(t) - b^* - L^* - T^* v_{n+1}(t)$. Say, when $s_{n+1}(t) > 0$, the $(n+1)^{\text{th}}$ driver would accelerate according to

$$a_{n+1}(t+T) = \lambda_0 [x_n(t) - x_{n+1}(t) - b^* - L^* - T^* v_{n+1}(t)].$$

Delay response model

Consider the delay response model

$$\frac{d^2 x_n(t+T)}{dt^2} = -\lambda \left[\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt} \right],$$

where T is the reaction time and λ measures the sensitivity of the two-car interaction. Integrating the equation, we obtain

$$\frac{dx_n(t+T)}{dt} = -\lambda[x_n(t) - x_{n-1}(t)] + dn.$$

Imagine a steady state situation where all cars are equidistant apart, and so they are moving at the same speed. That is,

$$\frac{dx_n(t+T)}{dt} = \frac{dx_n(t)}{dt}$$

and let $d = dn$ (independent of n) be the common distance.

We simplify the equation as follows:

$$\frac{dx_n(t)}{dt} = -\lambda[x_n(t) - x_{n-1}(t)] + d.$$

Suppose we define the traffic density ρ by the relation

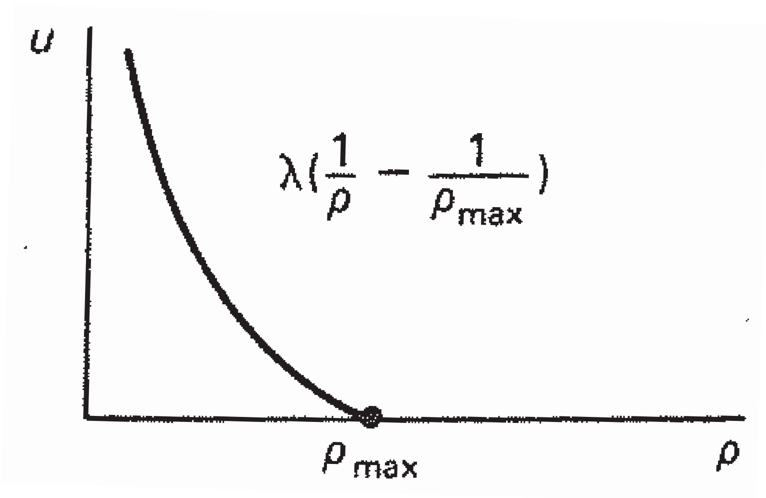
$$x_{n-1}(t) - x_n(t) = \frac{1}{\rho},$$

we obtain the velocity-density relation

$$u = \frac{\lambda}{\rho} + d.$$

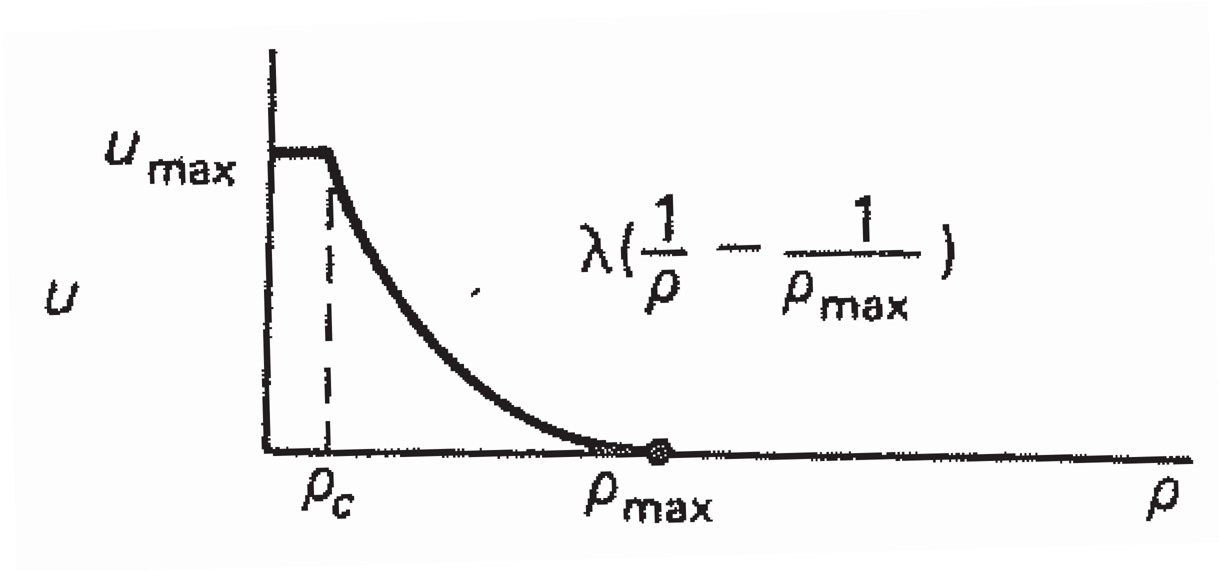
We choose d by noting that at the maximum density (bumper-to-bumper traffic) $u = 0$. Finally, we obtain

$$u = \lambda \left(\frac{1}{\rho} - \frac{1}{\rho_{\max}} \right).$$



To avoid infinite velocity at zero value of ρ, we may modify the velocity-density relation as

$$u = \begin{cases} u_{\max} & \rho < \rho_c \\ \lambda\left(\frac{1}{\rho} - \frac{1}{\rho_{\max}}\right) & \rho > \rho_c \end{cases} .$$



Example

Suppose $\lambda = \dfrac{c}{x_{n-1}(t) - x_n(t)}$, then the revised equation is

$$\frac{d^2 x_n(t+T)}{dt^2} = c\frac{\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt}}{x_n(t) - x_{n-1}(t)}.$$

Upon integrating, we obtain

$$\frac{dx_n(t+T)}{dt} = c \ln |x_n(t) - x_{n-1}(t)| + dn.$$

Assume steady state condition, this leads to

$$u = -c \ln \rho + d.$$

We choose the integration constant such that the velocity is zero at the maximum velocity. We then obtain

$$u = -c \ln \frac{\rho}{\rho_{\text{max}}}.$$

Furthermore, we consider

$$q = \rho u = -c\rho \ln \frac{\rho}{\rho_{\text{max}}},$$

so that

$$0 = \frac{dq}{d\rho} = -c\left(\ln \frac{\rho}{\rho_{\text{max}}} + 1\right).$$

One can check that maximum q occurs at

$$\rho = \frac{\rho_{\text{max}}}{e} \text{ and } u\left(\frac{\rho_{\text{max}}}{e}\right) = c.$$