MATH6380B - Advanced Topics in Derivative Pricing Models

Homework Three

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1. Apply the exchange option price formula to price the floating strike Asian call option based on the knowledge of the price formula of the fixed strike Asian call option.

   Hint: The covariance between S_T and G_T is equal to $\dfrac{\sigma^2(T-t)^2}{2}$.

2. We define the geometric average of the price path of asset price S_i, $i = 1, 2$, during the time interval $[t, t+T]$ by

$$G_i(t+T) = \exp\left(\frac{1}{T}\int_0^T \ln S_i(t+u)\,du\right).$$

   Consider an Asian option involving two assets whose terminal payoff is given by $\max(G_1(t+T) - G_2(t+T), 0)$. Show that the price formula of this European Asian option at time t is given by

$$V(S_1, S_2, t; T) = \widetilde{S}_1 N(d_1) - \widetilde{S}_2 N(d_2)$$

   where

$$\widetilde{S}_i = S_i \exp\left(-\left(\frac{r^2}{2} + \frac{\sigma^2}{12}\right)T\right),\ i = 1, 2, \quad \sigma^2 = \frac{\sigma_1^2}{3} + \frac{\sigma_2^2}{3} - \frac{2}{3}\rho\sigma_1\sigma_2,$$

$$d_1 = \frac{\ln\frac{\widetilde{S}_1}{\widetilde{S}_2} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

3. Deduce the following put-call parity relation between the prices of European fixed strike Asian call and put options under continuously monitored geometric averaging

$$\begin{aligned} & c(S,G,t) - p(S,G,t) \\ &= e^{-r(T-t)}\left\{G^{t/T}S^{(T-t)/T}\exp\left((T-t)\left[\frac{\sigma^2}{6}\left(\frac{T-t}{T}\right)^2 \right.\right.\right. \\ &\qquad \left.\left.\left. + \frac{r-q-\frac{\sigma^2}{2}}{2}\frac{T-t}{T}\right] - X\right)\right\}. \end{aligned}$$

4. Suppose continuous arithmetic averaging of the asset price is taken from $t = 0$ to T, where T is the expiration time. The terminal payoff function of the floating strike Asian call and put options are, respectively,

$$\max\left(S_T - \frac{1}{T}\int_0^T S_u\,du, 0\right) \text{ and } \max\left(\frac{1}{T}\int_0^T S_u\,du - S_T, 0\right).$$

   Show that the put-call parity relation for the above pair of European floating strike Asian options is given by

$$c - p = Se^{-q(T-t)} + \frac{S}{(r-q)T}[e^{-r(T-t)} - e^{-q(T-t)}] - e^{-r(T-t)}A_t,$$

where

$$A_t = \frac{1}{T}\int_0^t S_u\,du.$$

Suppose continuous geometric averaging of the asset price is taken, show that the corresponding put-call parity relation is given by

$$\begin{aligned} c - p = &\, Se^{-q(T-t)} - G^{t/T} S^{(T-t)/T} \\ &\exp\left(\frac{\sigma^2(T-t)^3}{6T^2} + \frac{\left(r-q-\frac{\sigma^2}{2}\right)(T-t)^2}{2T} - r(T-t)\right), \end{aligned}$$

where

$$G_t = \exp\left(\frac{1}{t}\int_0^t \ln S_u\,du\right).$$

5. Show that the put-call parity relations between the prices of floating strike and fixed strike Asian options at the start of the averaging period are given by

$$\begin{aligned} p_{f\ell}(S_0,\lambda,r,q,T) - c_{f\ell}(S_0,\lambda,r,q,T) &= \frac{S(e^{-qT}-e^{-rT})}{(r-q)T} - \lambda S_0 \\ c_{fix}(X,S_0,r,q,T) - p_{fix}(X,S_0,r,q,T) &= \frac{S(e^{-qT}-e^{-rT})}{(r-q)T} - e^{-rT}X. \end{aligned}$$

By combining the above put-call parity relations with the fixed-floating symmetry relation between $c_{f\ell}$ and p_{fix}, deduce the following symmetry relation between c_{fix} and $p_{f\ell}$:

$$c_{fix}(X,S_0,r,q,T) = p_{f\ell}\left(S_0,\frac{X}{S_0},q,r,T\right).$$

6. Consider a self financing trading strategy of a portfolio with a dividend paying asset and money market account over the time horizon $[0,T]$. Under the risk neutral measure Q, let the dynamics of the asset price S_t be governed by

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma\,dZ_t,$$

where q is the dividend yield, $q \neq r$. We adopt the trading strategy of holding n_t units of the asset at time t, where

$$n_t = \frac{1}{(r-q)T}\left[e^{-q(T-t)} - e^{-r(T-t)}\right].$$

Let X_t denote the portfolio value at time t, whose dynamics is then given by

$$dX_t = n_t\,dS_t + r(X_t - n_tS_t)\,dt + qn_tS_t\,dt.$$

The initial portfolio value X_0 is chosen to be

$$X_0 = n_0S_0 - e^{-rT}X.$$

Show that

$$X_T = \frac{1}{T}\int_0^T S_t\,dt - X.$$

Defining $Y_t = \dfrac{X_t}{e^{qt}S_t}$, show that

$$dY_t = -(Y_t - e^{-qt}n_t)\sigma\,dZ_t^*,$$

where $Z_t^* = Z_t - \sigma t$ is a Brownian process under Q^*-measure with $e^{qt}S_t$ as the numeraire. Note that the price function of the fixed strike Asian call option with strike X is given by

$$c_{fix}(S_0, 0; X) = e^{-rT} E_Q[\max(X_T, 0)] = S_0 e^{-rT} E_{Q^*}[\max(Y_T, 0)],$$

with

$$Y_0 = \frac{X_0}{S_0} = \frac{e^{-qT} - e^{-rT}}{(r-q)T} - e^{-rT}\frac{X}{S_0}.$$

Show that

$$c_{fix}(S_0, 0; X) = S_0 u(Y_0, 0),$$

where $u(y,t)$ satisfies the following one-dimensional partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}(y - e^{-qt} n_t)^2 \sigma^2 \frac{\partial^2 u}{\partial y^2} = 0$$

with $u(y,T) = \max(y, 0)$.

7. Consider the European continuously monitored arithmetic average Asian option with terminal payoff: $\max(A_T - X_1 S_T - X_2, 0)$, where

$$A_T = \frac{1}{T}\int_0^T S_u \, du.$$

At the current time $t > 0$, the average value A_t over the time period $[0,t]$ has been realized. Let $V(S, \tau; X_1, X_2)$ denote the price function of the Asian option at the start of the averaging period. Show that the value of the in-progress Asian option is given by

$$\frac{T-t}{T} V\left(S_t, T-t; \frac{X_1 T}{T-t}, \frac{X_2 T}{T-t} - \frac{A_t - t}{T-t}\right).$$

8. Let Z_t denote the standard Brownian process. Show that the covariance matrix of the bivariate Gaussian random variable $\left(Z_t, \int_0^1 Z_u \, du\right)$ is given by

$$E\left[\left(Z_t, \int_0^1 Z_u \, du\right)^T \left(Z_t, \int_0^1 Z_u \, du\right)\right] = \begin{pmatrix} t & t\left(1 - \frac{t}{2}\right) \\ t\left(1 - \frac{t}{2}\right) & \frac{1}{3} \end{pmatrix}.$$

Also, show that the conditional distribution of Z_t given $\int_0^1 Z_u \, du = z$ is normal with mean $3t\left(1 - \frac{t}{2}\right)z$ and variance $t - 3t^2\left(1 - \frac{t}{2}\right)^2$. With Y chosen to be $\int_0^1 Z_u \, du$ and $T = 1$, show that

$$c_{fix}(S, I, 0) \geq e^{-r}\int_{-\infty}^{\infty} \sqrt{3}n(\sqrt{3}z)$$
$$\int_0^1 \max\left(Se^{3\sigma t(1-t/2)z + (r-q)t + \frac{\sigma^2}{2}\left[t - 3t^2\left(1-\frac{t}{2}\right)^2\right]} - X, 0\right) dtdz,$$

where $n(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.