Advanced Topics in Derivative Pricing Models

Topic 2 - Lookback style derivatives

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2.1 Product nature of lookback options

The payoff of a *lookback option* depends on the minimum or maximum price of the underlying asset attained during certain period of the life of the option.

Let T denote the time of expiration of the option and $[T_0, T]$ be the lookback period. We denote the minimum value and maximum value of the asset price realized from T_0 to the current time t ($T_0 \leq t \leq T$) by

$$m_{T_0}^t = \min_{T_0 \leq \xi \leq t} S_\xi$$

and

$$M_{T_0}^t = \max_{T_0 \leq \xi \leq t} S_\xi$$

- A floating strike lookback call gives the holder the right to buy at the lowest realized price while a floating strike lookback put allows the holder to sell at the highest realized price over the lookback period.

- Since $S_T \geq m_{T_0}^T$ and $M_{T_0}^T \geq S_T$ so that the holder of a floating strike lookback option always exercise the option.

- Hence, the respective terminal payoff of the lookback call and put are given by $S_T - m_{T_0}^T$ and $M_{T_0}^T - S_T$.

- A fixed strike lookback call (put) is a call (put) option on the maximum (minimum) realized price. The respective terminal payoff of the fixed strike lookback call and put are $\max(M_{T_0}^T - X, 0)$ and $\max(X - m_{T_0}^T, 0)$, where X is the strike price.

- An interesting example is the *Russian option*, which is in fact a perpetual American lookback option. The owner of a Russian option on a stock receives the historical maximum value of the asset price when the option is exercised and the option has no pre-set expiration date.

Under the risk neutral measure, the stochastic price process of the underlying asset is governed by

$$\frac{dS_t}{S_t} = r\ dt + \sigma\ dZ_t \quad \text{or} \quad d\left(\ln \frac{S_t}{S_0}\right) = dU_t = \left(r - \frac{\sigma^2}{2}\right)\ dt + \sigma\ dZ_t,$$

where $U_t = \ln \frac{S_t}{S_0}$ and $\mu = r - \frac{\sigma^2}{2}$.

2.2 Pricing formulas of European lookback options

We define the following stochastic variables

$$
\begin{aligned}
y_T &= \ln \frac{m_t^T}{S} = \min\{U_\xi, \xi \in [t, T]\} \\
Y_T &= \ln \frac{M_t^T}{S} = \max\{U_\xi, \xi \in [t, T]\},
\end{aligned}
$$

and write $\tau = T - t$. Here, S is the asset price at the current time t (dropping the subscript t for brevity).

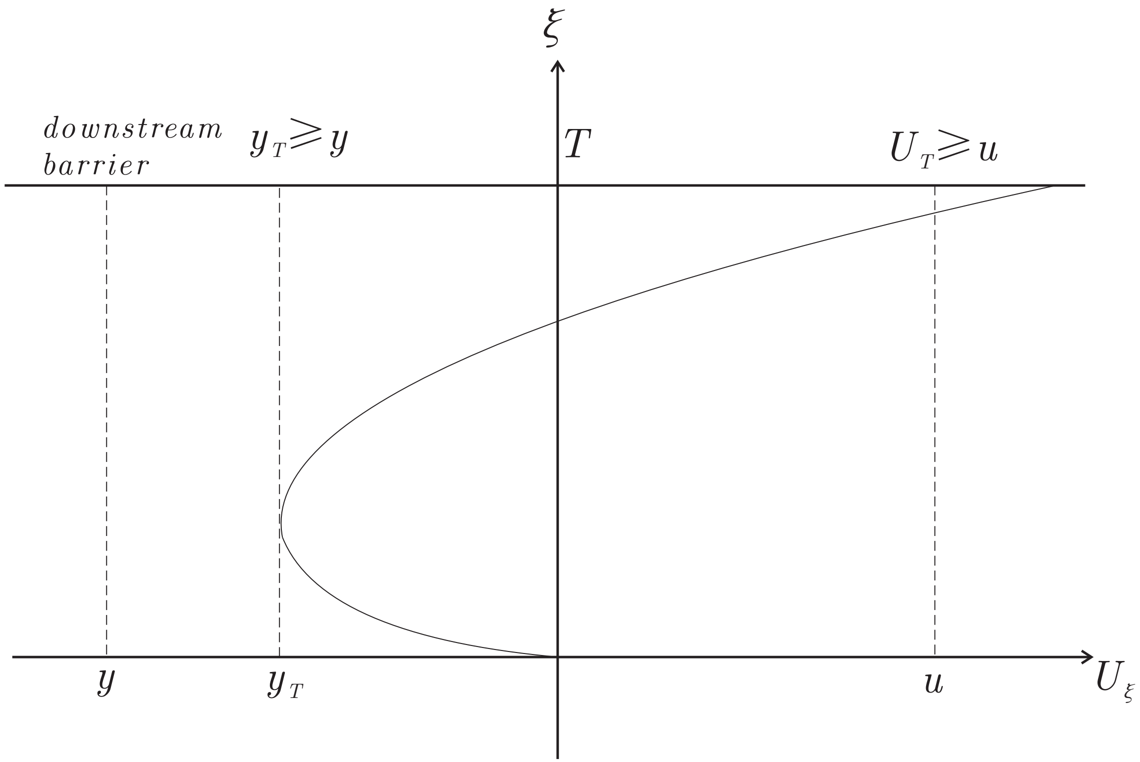
Downstream barrier

For $y \leq 0$ and $y \leq u$, we can deduce the following joint distribution function of U_T and y_T from the transition density function of the Brownian motion with the presence of a downstream barrier

$$
P[U_T \geq u, y_T \geq y] = N\left(\frac{-u + \mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-u + 2y + \mu\tau}{\sigma\sqrt{\tau}}\right).
$$

Illustration of $[U_T \geq u, Y_T \geq y]$

$U_\xi = \ln \dfrac{S_\xi}{S_0}$ is visualized as the restricted Brownian motion with constant drift rate μ and downstream absorbing barrier y.



Upstream barrier

For $y \geq 0$ and $y \geq u$, the corresponding joint distribution function of U_T and Y_T is given by

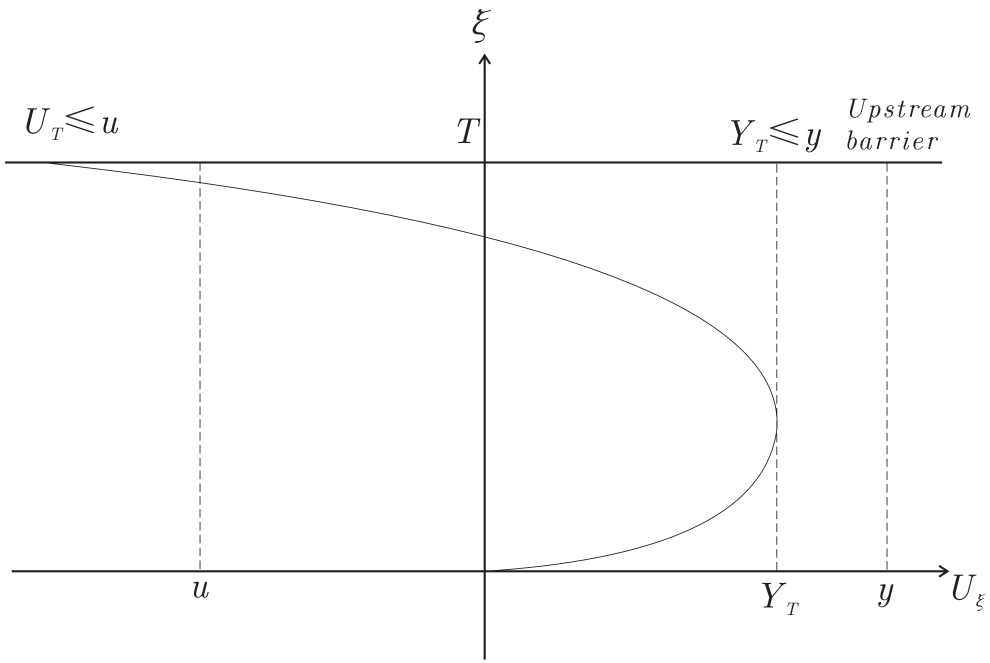
$$P[U_T \leq u, Y_T \leq y] = N\left(\frac{u-\mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{u-2y-\mu\tau}{\sigma\sqrt{\tau}}\right).$$

By taking $y = u$ in the above two joint distribution functions, we obtain the respective distribution function for y_T and Y_T

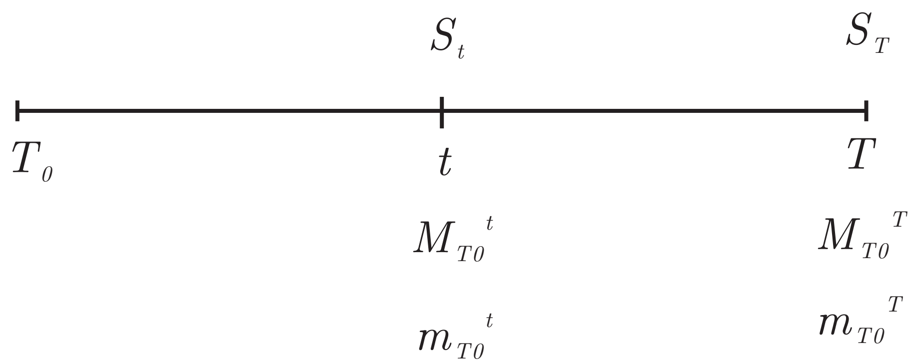
$$\begin{aligned} P(y_T \geq y) &= N\left(\frac{-y+\mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{y+\mu\tau}{\sigma\sqrt{\tau}}\right), \quad y \leq 0, \\ P(Y_T \leq y) &= N\left(\frac{y-\mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y-\mu\tau}{\sigma\sqrt{\tau}}\right), \quad y \geq 0. \end{aligned}$$

The density function of y_T and Y_T can be obtained by differentiating the above distribution functions.

Illustration of $[U_T \leq u, Y_T \leq y]$



Time frame



For convenience, we write $S = S_t$, $M = M_{T_0}^t$ and $m = m_{T_0}^t$.

European fixed strike lookback options

Consider a European fixed strike lookback call option whose terminal payoff is $\max(M_{T_0}^T - X, 0)$. The value of this lookback call option at the current time t is given by

$$c_{fix}(S, M, t) = e^{-r\tau} E\left[\max(\max(M, M_t^T) - X, 0)\right],$$

where $S_t = S, M_{T_0}^t = M$ and $\tau = T - t$, and the expectation is taken under the risk neutral measure. The payoff function can be simplified into the following forms, depending on $M \leq X$ or $M > X$:

(i) $M \leq X$

$$\max(\max(M, M_t^T) - X, 0) = \max(M_t^T - X, 0)$$

(ii) $M > X$

$$\max(\max(M, M_t^T) - X, 0) = (M - X) + \max(M_t^T - M, 0).$$

Define the function H by

$$H(S, \tau; K) = e^{-r\tau} E[\max(M_t^T - K, 0)],$$

where K is a positive constant. Once $H(S, \tau; K)$ is determined, then

$$\begin{aligned} c_{fix}(S, M, \tau) &= \begin{cases} H(S, \tau; X) & \text{if } M \leq X \\ e^{-r\tau}(M - X) + H(S, \tau; M) & \text{if } M > X \end{cases} \\ &= e^{-r\tau} \max(M - X, 0) + H(S, \tau; \max(M, X)). \end{aligned}$$

- $c_{fix}(S, M, \tau)$ is independent of M when $M \leq X$ because the terminal payoff is independent of M when $M \leq X$.

- When $M > X$, the terminal payoff is guaranteed to have the floor value $M - X$. If we subtract the present value of this guaranteed floor value, then the remaining value of the fixed strike call option is equal to a new fixed strike call but with the strike being increased from X to M.

Recall that when X is a non-negative random variable, we have

$$E[X] = \int_0^\infty [1 - F_X(t)] \; dt, \text{ if } X \text{ is continuous.}$$

Since $\max(M_t^T - K, 0)$ is a non-negative random variable, its expected value is given by the integral of the tail probabilities where

$$\begin{aligned}
& H(S,\tau;K) \\
= \; & e^{-r\tau} E[\max(M_t^T - K, 0)] \\
= \; & e^{-r\tau} \int_0^\infty P[M_t^T - K \ge x] \, dx \\
= \; & e^{-r\tau} \int_K^\infty P\left[\ln \frac{M_t^T}{S} \ge \ln \frac{z}{S}\right] dz, \quad z = x + K \\
= \; & e^{-r\tau} \int_{\ln \frac{K}{S}}^\infty S e^y P[Y_T \ge y] \, dy, \qquad y = \ln \frac{z}{S} \;\; \left[dy = \frac{1}{z} \; dz, z = Se^y\right] \\
= \; & e^{-r\tau} \int_{\ln \frac{K}{S}}^\infty S e^y \left[N\left(\frac{-y + \mu\tau}{\sigma\sqrt{\tau}}\right) + e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu\tau}{\sigma\sqrt{\tau}}\right)\right] dy
\end{aligned}$$

$$\begin{aligned} &= SN(d) - e^{-r\tau}KN(d-\sigma\sqrt{\tau}) \\ &+ e^{-r\tau}\frac{\sigma^2}{2r}S\left[e^{r\tau}N(d) - \left(\frac{S}{K}\right)^{-\frac{2r}{\sigma^2}}N\left(d - \frac{2r}{\sigma}\sqrt{\tau}\right)\right], \end{aligned}$$

where

$$d = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

The European fixed strike lookback put option with terminal payoff $\max(X - m_{T_0}^T, 0)$ can be priced in a similar manner. Write $m = m_{T_0}^t$ and define the function

$$h(S, \tau; K) = e^{-r\tau}E[\max(K - m_t^T, 0)].$$

The value of this lookback put can be expressed as

$$p_{fix}(S, m, \tau) = e^{-r\tau} \max(X - m, 0) + h(S, \tau; \min(m, X)),$$

where

$$\begin{aligned}
h(S,\tau;K) &= e^{-r\tau}\int_0^\infty P[\max(K - m_t^T, 0) \ge x]\,dx \\
&= e^{-r\tau}\int_0^K P[K - m_t^T \ge x]\,dx \qquad 0 \le \max(K - m_t^T, 0) \le K \\
&= e^{-r\tau}\int_0^K P[m_t^T \le z]\,dz, \qquad z = K - x \\
&= e^{-r\tau}\int_0^{\ln\frac{K}{S}} Se^y P[y_T \le y]\,dy, \qquad y = \ln\frac{z}{S} \\
&= e^{-r\tau}\int_0^{\ln\frac{K}{S}} Se^y \left[N\left(\frac{y - \mu\tau}{\sigma\sqrt{\tau}}\right) + e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{y + \mu\tau}{\sigma\sqrt{\tau}}\right)\right] dy \\
&= e^{-r\tau} K N(-d + \sigma\sqrt{\tau}) - SN(-d) + e^{-r\tau}\frac{\sigma^2}{2r} S \\
&\quad \left[\left(\frac{S}{K}\right)^{-2r/\sigma^2} N\left(-d + \frac{2r}{\sigma}\sqrt{\tau}\right) - e^{r\tau} N(-d)\right].
\end{aligned}$$

European floating strike lookback options

By exploring the pricing relations between the fixed and floating lookback options, we can deduce the price functions of floating strike lookback options from those of fixed strike options. Consider a European floating strike lookback call option whose terminal payoff is $S_T - m_{T_0}^T$, the present value of this call option is given by

$$\begin{aligned}
c_{f\ell}(S,m,\tau) &= e^{-r\tau}E[S_T - \min(m, m_t^T)] \\
&= e^{-r\tau}E[(S_T - m) + \max(m - m_t^T, 0)] \\
&= S - me^{-r\tau} + h(S,\tau;m) \\
&= SN(d_m) - e^{-r\tau}mN(d_m - \sigma\sqrt{\tau}) + e^{-r\tau}\frac{\sigma^2}{2r}S \\
&\qquad \left[\left(\frac{S}{m}\right)^{-\frac{2r}{\sigma^2}} N\left(-d_m + \frac{2r}{\sigma}\sqrt{\tau}\right) - e^{r\tau}N(-d_m)\right],
\end{aligned}$$

where

$$d_m = \frac{\ln\frac{S}{m} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

Consider a European floating strike lookback put option whose terminal payoff is $M_{T_0}^T - S_T$, the present value of this put option is given by

$$\begin{aligned}
p_{f\ell}(S, M, \tau) &= e^{-r\tau} E[\max(M, M_t^T) - S_T] \\
&= e^{-r\tau} E[\max(M_t^T - M, 0) - (S_T - M)] \\
&= H(S, \tau; M) - (S - Me^{-r\tau}) \\
&= e^{-r\tau} MN(-d_M + \sigma\sqrt{\tau}) - SN(-d_M) + \; e^{-r\tau} \frac{\sigma^2}{2r} S \\
&\qquad \left[e^{r\tau} N(d_M) - \left(\frac{S}{M}\right)^{-\frac{2r}{\sigma^2}} N\left(d_M - \frac{2r}{\sigma}\sqrt{\tau}\right)\right],
\end{aligned}$$

where

$$d_M = \frac{\ln \frac{S}{M} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

Remark

Through $H(S, \tau; M)$ and $H(S, \tau; X)$, we can deduce the fixed-floating relation between lookback call and put; the form of which is dependent on either $M \leq X$ or $M > X$.

Boundary condition at $S = m$

Consider the scenario when $S = m$, that is, the current asset price happens to be at the minimum value realized so far. The probability that the current minimum value remains to be the realized minimum value at expiration is seen to be zero. In other words, the probability that S_t touches the minimum value m once (one-touch) and remains above m at all subsequent times is zero.

Recall the distribution formula for m_t^T:

$$P[m_t^T \geq m] = N\left(\frac{-\ln\frac{m}{S} + \mu\tau}{\sigma\sqrt{t}}\right) - \left(\frac{S}{m}\right)^{1-\frac{2r}{\sigma^2}} N\left(\frac{\ln\frac{m}{S} + \mu\tau}{\sigma\sqrt{t}}\right)$$

so that $P[m_t^T \geq m] = 0$ when $S = m$.

Insensitivity of lookback option price to m when $S = m$

We can argue that the value of the floating strike lookback call should be insensitive to infinitesimal changes in m since the change in option value with respect to marginal changes in m is proportional to the probability that m will be the realized minimum at expiry

$$\left.\frac{\partial c_{f\ell}}{\partial m}(S, m, \tau)\right|_{S=m} = 0.$$

Alternatively, we may argue that the future updating of the realized minimum value does not require the current realized minimum value m. Hence, the floating strike lookback call is insensitive to m when $S = m$.

Lookback options for market entry

- Suppose an investor has a view that the asset price will rise substantially in the next 12 months and he buys a call option on the asset with the strike price set equal to the current asset price.

- Suppose the asset price drops a few percent within a few weeks after the purchase, though it does rise up strongly to a high level at expiration, the investor should have a better return if he had bought the option a few weeks later.

- Timing for market entry is always difficult to be decided. The investor could have avoided the above difficulty if he has purchased a "limited period" floating strike lookback call option whose lookback period only covers the early part of the option's life.

- It would cause the investor too much if a full period floating strike lookback call were purchased instead.

Let $[T_0, T_1]$ denote the lookback period where $T_1 < T$, T is the expiration time, and let the current time $t \in [T_0, T_1]$. The terminal payoff function of the "limited period" lookback call is $\max(S_T - m_{T_0}^{T_1}, 0)$.
We write $S_t = S, m_{T_0}^t = m$ and $\tau = T - t$. The value of this lookback call is given by

$$\begin{aligned}
c(S, m, \tau) &= e^{-r\tau} E_Q[\max(S_T - m_{T_0}^{T_1}, 0)] \\
&= e^{-r\tau} E_Q[\max(S_T - m, 0)\mathbf{1}_{\{m \le m_t^{T_1}\}}] \\
&\quad + e^{-r\tau} E_Q[\max(S_T - m_t^{T_1}, 0)\mathbf{1}_{\{m > m_t^{T_1}\}}] \\
&= e^{-r\tau} E_Q[S_T \mathbf{1}_{\{S_T > m, m \le m_t^{T_1}\}}] \\
&\quad - e^{-r\tau} m E_Q\left[\mathbf{1}_{\{S_T > m, m \le m_t^{T_1}\}}\right] \\
&\quad + e^{-r\tau} E_Q[S_T \mathbf{1}_{\{S_T > m_t^{T_1}, m > m_t^{T_1}\}}] \\
&\quad - e^{-r\tau} E_Q[m_t^{T_1} \mathbf{1}_{\{S_T > m_t^{T_1}, m > m_t^{T_1}\}}], \quad t < T_1,
\end{aligned}$$

where the expectation is taken under the risk neutral measure Q.

For the first term, the expectation can be expressed as

$$E_Q[S_T \mathbf{1}_{\{S_T > m, m \le m_t^{T_1}\}}] = \int_{\ln \frac{m}{S}}^{\infty} \int_{y}^{\infty} \int_{\ln \frac{m}{S} - x}^{\infty} S e^{xz} k(z) h(x, y)\, dz\, dx\, dy,$$

where $k(z)$ is the density function for $z = \ln \frac{S_T}{S_{T_1}}$ and $h(x, y)$ is the bivariate density function for $x = \ln \frac{S_{T_1}}{S}$ and $y = \ln \frac{m_t^{T_1}}{S}$.

The third and fourth terms can be expressed as

$$E_Q[S_T \mathbf{1}_{\{S_T > m_t^{T_1}, m > m_t^{T_1}\}}] = \int_{-\infty}^{\ln \frac{m}{S}} \int_{y}^{\infty} \int_{y-x}^{\infty} S e^{xz} k(z) h(x, y)\, dz\, dx\, dy$$

and

$$E_Q[m_t^{T_1} \mathbf{1}_{\{S_T > m_t^{T_1}, m > m_t^{T_1}\}}] = \int_{-\infty}^{\ln \frac{m}{S}} \int_{y}^{\infty} \int_{y-x}^{\infty} S e^{y} k(z) h(x, y)\, dz\, dx\, dy.$$

The price formula of the "limited-period" lookback call is found to be

$$
\begin{aligned}
& c(S,m,\tau) \\
= {} & SN(d_1) - me^{-r\tau}N(d_2) + SN_2\left(-d_1, e_1; -\sqrt{\frac{T-T_1}{T-t}}\right) \\
& + e^{-r\tau} m N_2\left(-f_2, d_2; -\sqrt{\frac{T_1-t}{T-t}}\right) \\
& + e^{-r\tau}\frac{\sigma^2}{2r} S\left[\left(\frac{S}{m}\right)^{-\frac{2r}{\sigma^2}} N_2\left(-f_1 + \frac{2r}{\sigma}\sqrt{T_1-t}, -d_1 + \frac{2r}{\sigma}\sqrt{\tau}; \sqrt{\frac{T_1-t}{T-t}}\right)\right. \\
& \qquad \left. - \; e^{r\tau} N_2\left(-d_1, e_1; -\sqrt{\frac{T-T_1}{T-t}}\right)\right] \\
& + e^{-r(T-T_1)}\left(1 + \frac{\sigma^2}{2r}\right) SN(e_2)N(-f_1), \quad t < T_1,
\end{aligned}
$$

where

$$d_1 = \frac{\ln\frac{S}{m}+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau},$$

$$e_1 = \frac{\left(r+\frac{\sigma^2}{2}\right)(T-T_1)}{\sigma\sqrt{T-T_1}}, \qquad e_2 = e_1 - \sigma\sqrt{T-T_1},$$

$$f_1 = \frac{\ln\frac{S}{m}+\left(r+\frac{\sigma^2}{2}\right)(T_1-t)}{\sigma\sqrt{T_1-t}}, \quad f_2 = f_1 - \sigma\sqrt{T_1-t}.$$

One can check easily that when $T_1 = T$ (full lookback period), the above price formula reduces to standard floating strike lookback call price formula.

Suppose the current time passes beyond the lookback period, $t > T_1$, the realized minimum value $m_{T_0}^{T_1}$ is now a known quantity. This "limited period" lookback call option then becomes a European vanilla call option with the known strike price $m_{T_0}^{T_1}$.

2.3 Rollover strategy and strike bonus premium

- The sum of the first two terms in $c_{f\ell}$ can be seen as the price function of a European vanilla call with strike price m, while the third term can be interpreted as the strike bonus premium.

Rollover strategy

At any time, we hold a European vanilla call with the strike price set at the current realized minimum asset value. In order to replicate the payoff of the floating strike lookback call at expiry, whenever a new realized minimum value of the asset price is established at a later time, one should sell the original call option and buy a new call with the same expiration date but with the strike price set equal to the newly established minimum value.

Strike bonus premium

Since the call with a lower strike is always more expensive, an extra premium is required to adopt the rollover strategy. The sum of these expected costs of rollover is termed the strike bonus premium.

The strike bonus premium can be shown to be obtained by integrating a joint probability distribution function involving m_t^T and S_T. Firstly, we observe

$$\begin{aligned}\text{strike bonus premium} &= h(S,\tau;m) + S - me^{-r\tau} - c_E(S,\tau;m) \\ &= h(S,\tau;m) - p_E(S,\tau;m),\end{aligned}$$

where $c_E(S,\tau,m)$ and $p_E(S,\tau;m)$ are the price functions of European vanilla call and put, respectively. The last result is due to put-call parity relation.

Recall

$$h(S,\tau;m) = e^{-r\tau}\int_0^m P[m_t^T \le \xi]\,d\xi$$

and

$$\begin{aligned} p_E(S,\tau;m) &= e^{-r\tau}\int_0^\infty P[\max(m - S_T, 0) \ge x]\,dx \\ &= e^{-r\tau}\int_0^m P[S_T \le \xi]\,d\xi. \end{aligned}$$

Since the two stochastic state variables satisfies $0 \le m_t^T \le S_T$, we have

$$P[m_t^T \le \xi] - P[S_T \le \xi] = P[m_t^T \le \xi < S_T]$$

so that

$$\text{strike bonus premium } = e^{-r\tau}\int_0^m P[m_t^T \le \xi < S_T]\,d\xi.$$

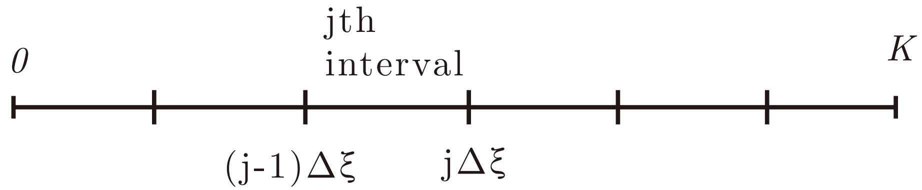
Sub-replication and replenishing premium

Replenishing premium of a European put option

$$\text{put value } = e^{-r\tau} \int_0^K P[S_T \leq \xi] \; d\xi$$

We divide the interval $[0, K]$ into n subintervals, each of equal width $\Delta\xi$ so that $n\Delta\xi = K$. The put can be decomposed into the sum of n portfolios, the j^{th} portfolio consists of long holding a put with strike $j\Delta\xi$ and short selling a put with strike $(j-1)\Delta\xi, j = 1, 2, \cdots, n$, where all puts have the same maturity date T.

Potential liabilities occur when S_T falls within $[0, K]$ (the put expires in-the-money). Hedge the exposure over successive n intervals:



$$n\Delta\xi = K \text{ and } \xi_j = j\Delta\xi$$

For the j^{th} portfolio:

$$\begin{array}{ll} \text{hold one put with strike } j\Delta\xi & \text{(more expensive)} \\ \text{short one put with strike } (j-1)\Delta\xi & \text{(less expensive)} \end{array}$$

Present value of this j^{th} portfolio

$$\begin{aligned} &= e^{-r\tau}\{E[(j\Delta\xi - S_T)\mathbf{1}_{\{S_T \le \xi_j\}}] - E[((j-1)\Delta\xi - S_T)\mathbf{1}_{\{S_T \le \xi_{j-1}\}}]\} \\ &\approx e^{-r\tau}\Delta\xi P[S_T \le \xi_j] \quad \text{(to leading order in } \Delta\xi\text{).} \end{aligned}$$

In the limit $n \to \infty$, we obtain

$$\begin{aligned} \text{put value} &= e^{-r\tau} \lim_{n\to\infty} \sum_{j=1}^{n} P[S_T \le \xi_j]\Delta\xi \\ &= e^{-r\tau} \int_0^K P[S_T \le \xi]\, d\xi. \end{aligned}$$

These n portfolios can be visualized as the appropriate replenishment to the sub-replicating portfolio in order that the writer of the put option is immunized from possible loss at the maturity of the option. The sub-replicating portfolio is taken to be the null portfolio in the current example.

- With the addition of the n^{th} portfolio [long a put with strike K and short a put with strike $(K - \Delta\xi)$] into the sub-replicating portfolio, the writer faces a loss only when S_T falls below $K - \Delta\xi$.

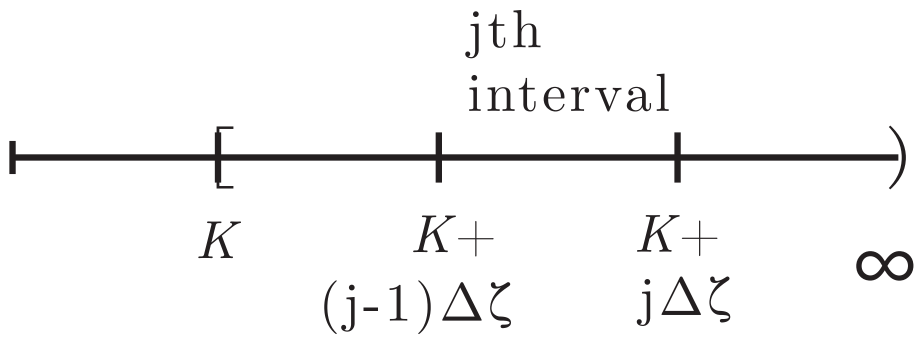
- Deductively, the protection over the interval $[(j-1)\Delta\xi, j\Delta\xi]$ in the out-of-the-money region of the put is secured with the addition of the j^{th} portfolio.

- One then proceeds one by one from the n^{th} portfolio down to the 1^{st} portfolio so that the protection over the whole interval $[0, K]$ is achieved.

- With the acquisition of all these replenishing portfolios, the writer of the put option is immunized from any possible loss at option's maturity even the put expires out-of-the-money. The cost of acquiring all these n portfolios is the *replenishing premium*.

Replenishing premium of a European call option

Potential liabilities occur when S_T falls within $[K, \infty)$ (the call expires in-the-money). Hedge the exposure over successive n intervals:



For the j^{th} portfolio:

hold one call with strike $K + (j-1)\Delta\xi$	(more expensive)
short one call with strike $K + j\Delta\xi$	(less expensive)

$$\begin{aligned} & \text{Present value of this } j^{\text{th}} \text{ portfolio} \\ = \; & e^{-r\tau}\{E[\{S_T - [K + (j-1)\Delta\xi]\}\mathbf{1}_{\{S_T \geq K+(j-1)\Delta\xi\}}] \\ & \quad - E[[S_T - (K + j\Delta\xi)]\mathbf{1}_{\{S_T \geq K+j\Delta\xi\}}]\} \\ \approx \; & e^{-r\tau}\Delta\xi P[S_T \geq K + j\Delta\xi] \quad \text{(to leading order in } \Delta\xi\text{).} \end{aligned}$$

In the limit $n \to \infty$, we obtain

$$\begin{aligned} \text{call value} &= e^{-r\tau} \lim_{n\to\infty} \sum_{j=1}^{n} P[S_T \geq K + j\Delta\xi]\Delta\xi \\ &= e^{-r\tau} \int_K^\infty P[S_T \geq \xi] \; d\xi. \end{aligned}$$

Since the sub-replicating portfolio has been chosen to be the null portfolio, the call value is then equal to the replenishing premium.

The above result is distribution free.

Put-call parity relations of continuously monitored floating strike and fixed strike lookback options

We let $[T_0, T]$ be the continuously monitored period for the minimum value of the asset price process. The current time t is within the monitoring period so that $T_0 < t < T$, and that the period of monitoring ends with the maturity of the lookback call option.

The terminal payoff of the continuously monitored floating strike lookback call option is given by

$$c_{f\ell}(S_T, T) = S_T - m_{T_0}^T = S_T - \min(m, m_t^T).$$

Here, m_t^T is a stochastic state variable with dependence on $S_u, u \in [t, T]$.

(i) forward as the sub-replicating instrument

Suppose we choose the sub-replicating instrument to be a forward with the same maturity and delivery price m. The terminal payoff of the sub-replicating instrument is below that of the forward only when $m_t^T < m$; otherwise, their terminal payoffs are equal. Here, m_t^T is the random variable that determines the occurrence of under replication.

Potential liabilities occur when m_t^T falls within $[0, m]$ with payoff $m - m_t^T$. Similar to the put with payoff $m - S_T$ when S_T falls within $[0, m]$, the replenishing premium is (replacing S_T in put by m_t^T)

$$e^{-r\tau} \int_0^m P[m_t^T \leq \xi] \; d\xi.$$

The replenishing premium can be visualized as the value of a European fixed strike lookback put option with fixed strike m and whose terminal payoff is

$$\left(m - m_t^T\right)^+ = \left(m - \min(m, m_t^T)\right)^+ = \left(m - m_{T_0}^T\right)^+.$$

Let $p_{fix}(S,t;K)$ denote the value of a fixed strike lookback put with strike K, whose terminal payoff is $\max(K-m,0)$. This gives the following put-call parity relation for lookback options:

$$\begin{aligned} c_{f\ell}(S,t;m) &= S - e^{-r\tau}m + e^{-r\tau}\int_0^m P[m_t^T \le \xi]\ d\xi \\ &= S - e^{-r\tau}m + p_{fix}(S,t;m), \end{aligned}$$

where S is the current asset price and $S - e^{-r\tau}m$ is the present value of the forward with delivery price m and maturity date T.

The probability distribution $P[m_t^T \le \xi]$ is given by the distribution function for the restricted asset price process with the down barrier ξ over the interval $[t,T]$.

(ii) European call option as the sub-replicating instrument

Suppose we change the sub-replication portfolio to be a European call option whose terminal payoff is $(S_T - m)^+$. Comparing $S_T - m_t^T$ with $(S_T - m)^+$, there are two sources of risks. One is the realization of lower minimum value while the other is that S_T may stay below m. The terminal payoff $c_{f\ell}(S_T, T)$ is decomposed as

$$S_T - m_t^T = \underbrace{(m - m_t^T)}_{\substack{\text{fixed strike} \\ \text{lookback} \\ \text{put with} \\ \text{strike } m}} - \underbrace{(m - S_T)}_{\substack{\text{vanilla put} \\ \text{with strike } m}}$$

When $S_T \le m$ and $m_t^T \le m$, both of the above puts are in-the-money.

$$\begin{aligned} & \text{Replenishing premium} \\ = \; & e^{-r\tau} \int_0^m \left\{ P[m_t^T \le \xi] - P[S_T \le \xi] \right\} d\xi \\ = \; & e^{-r\tau} \int_0^m P[m_t^T \le \xi < S_T] \; d\xi \\ = \; & \text{strike bonus premium} = h(S, m, t) - p_E(S, t; m). \end{aligned}$$

Remarks

There are 3 other possible cases of the relative position of m_t^T and S_T with respect to m.

1. Since $m_t^T \leq S_T$, we rule out $S_T < m$ and $m_t^T > m$.
2. The full replication is achieved when $S_T > m$ and $m_t^T > m$.
3. When $S_T > m$ and $m_t^T < m$, the amount of sub-replication is $m - m_t^T$. The corresponding replenishing premium is reduced to

$$e^{-r\tau} \int_0^m P[m_t^T \leq \xi] \; d\xi$$

since $P[S_T \leq \xi] = 0$ for $\xi \in [0, m]$.

The integral formulation of the replenishing premium on the last page covers all these 4 cases.

2.4 Partial differential equation formulation

We would like to illustrate how to derive the governing partial differential equation and the associated auxiliary conditions for the European floating strike lookback put option. First, we define the quantity

$$M_n = \left[\int_{T_0}^{t} (S_\xi)^n d\xi \right]^{1/n}, \quad t > T_0,$$

the derivative of which is given by

$$dM_n = \frac{1}{n} \frac{S^n}{(M_n)^{n-1}} \; dt$$

so that dM_n is deterministic. Taking the limit $n \to \infty$, we obtain

$$M = \lim_{n\to\infty} M_n = \max_{T_0 \le \xi \le t} S_\xi,$$

giving the realized maximum value of the asset price process over the lookback period $[T_0, t]$.

- We attempt to construct a hedged portfolio which contains one unit of a put option whose payoff depends on M_n and $-\triangle$ units of the underlying asset. Again, we choose $\triangle$ so that the stochastic components associated with the option and the underlying asset cancel.

- Let $p(S, M_n, t)$ denote the value of the lookback put option and let Π denote the value of the above portfolio. We then have

$$\Pi = p(S, M_n, t) - \triangle S.$$

- The dynamics of the portfolio value is given by

$$d\Pi = \frac{\partial p}{\partial t}dt + \frac{1}{n}\frac{S^n}{(M_n)^{n-1}}\frac{\partial p}{\partial M_n}dt + \frac{\partial p}{\partial S}dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 p}{\partial S^2}dt - \triangle dS$$

by virtue of Ito's lemma. Again, we choose $\triangle = \dfrac{\partial p}{\partial S}$ so that the stochastic terms cancel.

- Using the usual no-arbitrage argument, the non-stochastic portfolio should earn the riskless interest rate so that

$$d\Pi = r\Pi \, dt,$$

where r is the riskless interest rate. Putting all equations together, we have

$$\frac{\partial p}{\partial t} + \frac{1}{n}\frac{S^n}{(M_n)^{n-1}}\frac{\partial p}{\partial M_n} + \frac{\sigma^2}{2}S^2\frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S} - rp = 0.$$

- Next, we take the limit $n \to \infty$ and note that $S \leq M$. When $S < M$, $\lim_{n\to\infty} \frac{1}{n}\frac{S^n}{(M_n)^{n-1}} = 0$; and when $S = M$, $\frac{\partial p}{\partial M} = 0$. Hence, the second term becomes zero as $n \to \infty$.

- The governing equation for the floating strike lookback put is given by

$$\frac{\partial p}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S} - rp = 0, \quad 0 < S < M, t > T_0.$$

The domain of the pricing model has an upper bound M on S. The variable M does not appear in the equation, though M appears as a parameter in the auxiliary conditions. The final condition is

$$p(S, M, T) = M - S.$$

In this European floating strike lookback put option, the boundary conditions are applied at $S = 0$ and $S = M$. Once S becomes zero, it stays at the zero value at all subsequent times and the payoff at expiry is certain to be M.

Discounting at the riskless interest rate, the lookback put value at the current time t is

$$p(0, M, t) = e^{-r(T-t)}M.$$

The boundary condition at the other end $S = M$ is given by

$$\frac{\partial p}{\partial M} = 0 \quad \text{at} \quad S = M.$$

Partial differential equation formulation of the lookback option price function

Let S denote the stock price variable and M denote the realized maximum of the stock price recorded from the initial time of the lookback period to the current time. Let t denote the calendar time variable, T be the maturity date of the lookback option and $\tau = T - t$ be the time to expiry.

The formulation for the price function $V(S, M, \tau)$ of the one-asset European lookback option model with terminal payoff $V_T(S, M)$ is given by

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV, \quad 0 < S < M, \quad \tau > 0,$$
$$\left. \frac{\partial V}{\partial M} \right|_{S=M} = 0, \quad \tau > 0,$$
$$V(S, M, 0) = V_T(S, M), \tag{1}$$

where r is the riskless interest rate and σ is the volatility of the stock price.

- The price function is essentially two-dimensional with state variables S and M. However, the differential equation exhibits the degenerate nature in the sense that it does not involve the lookback variable M.

- M only occurs in the Neumann boundary condition $\left.\frac{\partial V}{\partial M}\right|_{S=M} = 0$ and the terminal payoff function. The Neumann boundary condition signifies that if the current stock price equals the value of the current realized maximum then the option price is insensitive to M.

- We reformulate the pricing model (1) using the following new set of variables:

$$x = \ln \frac{M}{S}, \quad y = \ln M.$$

Note that $\frac{\partial V}{\partial M} = \frac{1}{M}\left(\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\right)$ since both x and y have dependence on M.

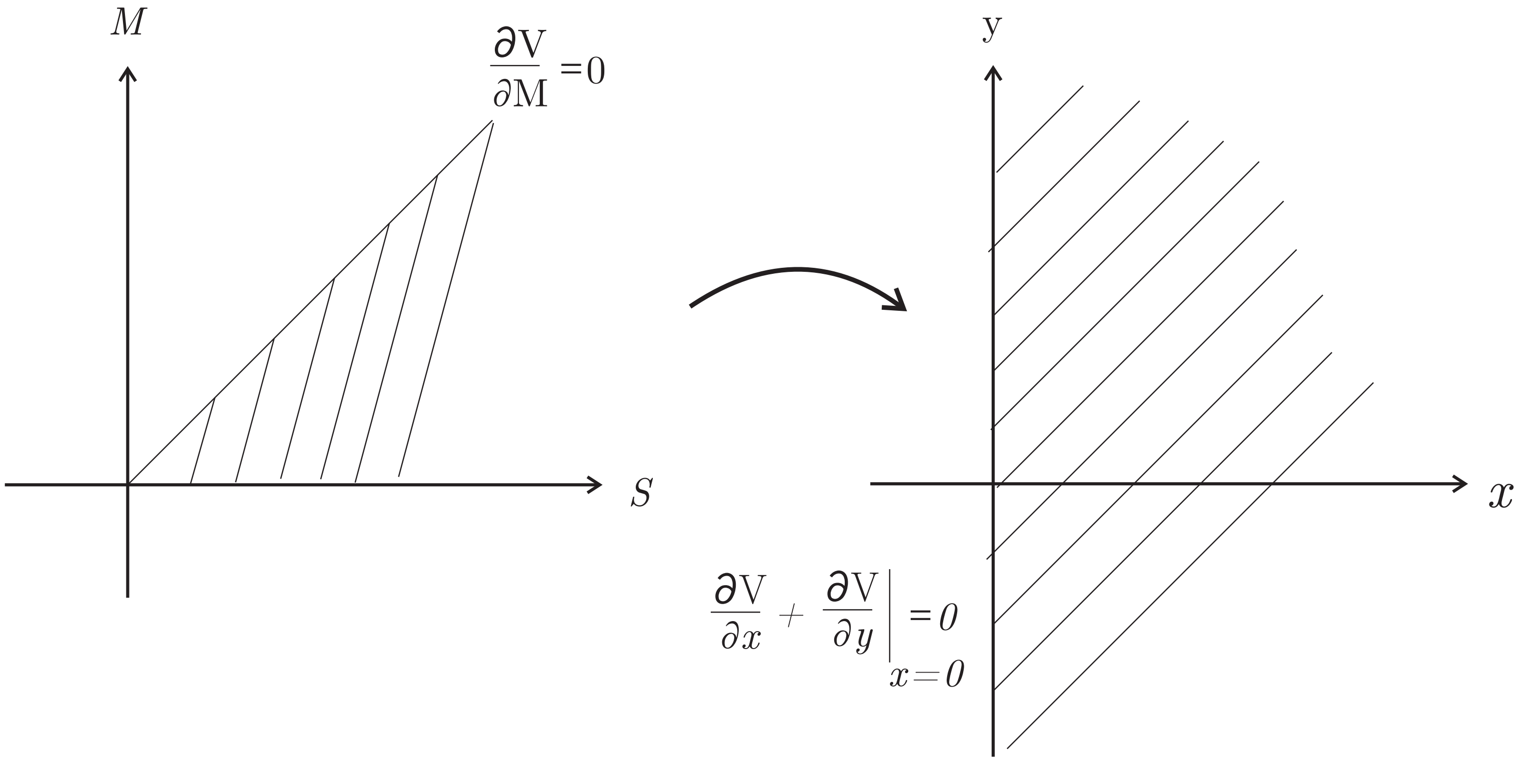
The lookback pricing model formulation can be rewritten as

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x} - rV, \quad x > 0, -\infty < y < \infty, \tau > 0,$$
$$\left.\left(\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\right)\right|_{x=0} = 0, \quad \tau > 0,$$
$$V(x, y, 0) = V_T(e^{y-x}, e^y). \qquad (2)$$

The triangular wedge shape of the original domain of definition $\mathcal{D} = \{(S, M) : 0 < S < M\}$ is now transformed into a new domain which is the semi-infinite two-dimensional plane

$$\widetilde{\mathcal{D}} = \{(x, y) : x > 0 \quad \text{and} \quad -\infty < y < \infty\}.$$

However, the boundary condition along $x = 0$ involves the function $\dfrac{\partial V}{\partial x} + \dfrac{\partial V}{\partial y}$.



Floating strike lookback options

We consider the valuation of lookback options with payoff of the form $Sf\left(\frac{M}{S}\right)$, which includes the floating strike payoff as a special example. By taking $V_T(S,M) = Sf\left(\frac{M}{S}\right)$ and applying the transformations of variables: $x = \ln\frac{M}{S}$ and $U(x,\tau) = \frac{V(S,M,\tau)}{S}$ to the pricing formulation (1), we obtain

$$\begin{aligned}
&\frac{\partial U}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} - \left(r + \frac{\sigma^2}{2}\right)\frac{\partial U}{\partial x}, \quad x > 0, \tau > 0,\\
&\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0, \quad \tau > 0\\
&U(x,0) = f(e^x).
\end{aligned}$$

Once the terminal condition is free of y (namely, $\ln M$), the dependence on y of the price function disappears.

The *Neumann* boundary condition at $x = 0$ indicates that $x = 0$ is a *reflecting* barrier for the system particle hitting the reflecting barrier will be reflected, unlike an absorbing barrier which removes the particle from the system.

- To resolve the difficulty of dealing with the Neumann boundary condition along $x = 0$, we extend the domain of definition from the semi-infinite domain to the full infinite domain.

- This is achieved by performing continuation of the initial condition to the domain $x < 0$ such that the price function can satisfy the Neumann boundary condition.

Due to the presence of the drift term in the differential equation, the simple odd-even extension is not applicable. For the floating strike payoff $M - S$, we have $U(x, 0) = e^x - 1, x > 0$. The continuation of the initial condition to the domain $x < 0$ is found to be

$$U(x, 0) = \frac{1 - e^{(2\widetilde{\alpha}-1)x}}{2\widetilde{\alpha} - 1}, \quad x < 0, \quad \text{where} \quad \widetilde{\alpha} = \frac{r}{\sigma^2} + \frac{1}{2}.$$

To show the claim, we set

$$U(x, \tau) = \widetilde{U}(x, \tau)e^{\widetilde{\alpha}x + \widetilde{\beta}\tau},$$

where $\widetilde{\alpha} = \frac{r}{\sigma^2} + \frac{1}{2}$ and $\widetilde{\beta} = -\frac{1}{2\sigma^2}\left(r + \frac{\sigma^2}{2}\right)^2$. The transformation on U is equivalent to applying the change of measure to make the underlying price process to be drift free.

Now $\widetilde{U}(x,\tau)$ is governed by

$$\frac{\partial \widetilde{U}}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 \widetilde{U}}{\partial x^2}, \quad x > 0, \tau > 0$$

$$\left(\frac{\partial \widetilde{U}}{\partial x} + \widetilde{\alpha}\widetilde{U}\right)\Big|_{x=0} = 0, \quad \tau > 0, \text{ (Robin condition)}$$

$$\widetilde{U}(x,0) = e^{-\widetilde{\alpha}x} f(e^x) = h_+(x), \quad x > 0.$$

The fundamental solution is the free space Green function:

$$\psi(x,\tau;\xi) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{(x-\xi)^2}{2\sigma^2\tau}\right), \quad -\infty < x < \infty, \ \tau > 0.$$

Let $h_-(x)$ denote the continuation of the initial condition for $x < 0$, $\widetilde{U}(x,\tau)$ can then be formally represented by

$$\widetilde{U}(x,\tau) = \int_{-\infty}^{0} \psi(x-\xi,\tau)h_-(\xi)\ d\xi + \int_{0}^{\infty} \psi(x-\xi,\tau)h_+(\xi)\ d\xi.$$

The function $h_-(x)$ is determined by enforcing the satisfaction of the Robin boundary condition by the solution $\widetilde{U}(x,\tau)$.

We then obtain the following ordinary differential equation for $h_-(x)$:

$$h_-'(x) + \widetilde{\alpha}h_-(x) + h_+'(-x) + \widetilde{\alpha}h_+(-x) = 0,$$

with matching condition:

$$h_+(0) = h_-(0).$$

For example, suppose $f(e^x) = e^x - 1$, then $h_+(x) = e^{-\widetilde{\alpha}x}(e^x - 1)$.

By solving the above equation, we obtain

$$h_-(x) = \frac{e^{-\widetilde{\alpha}x} - e^{(\widetilde{\alpha}-1)x}}{2\widetilde{\alpha}-1}.$$

In general, the solution is found to be

$$h_-(x) = h_+(-x) + 2\widetilde{\alpha}e^{-\widetilde{\alpha}x}\int_0^x e^{\widetilde{\alpha}\xi}h(-\xi)\ d\xi.$$

We obtain the integral price formula of lookback option with payoff $Sf\left(\frac{M}{S}\right)$ as follows:

$$V(S,M,\tau) = S\left(\frac{M}{S}\right)^{\widetilde{\alpha}} e^{\widetilde{\beta}\tau}\int_1^\infty \left[\psi\left(\ln\frac{M}{S} + \ln\xi, \tau\right) + \psi\left(\ln\frac{M}{S} - \ln\xi, \tau\right)\right.$$
$$\left. + 2\alpha\int_\xi^\infty \psi\left(\ln\frac{M}{S} + \ln\eta, \tau\right)\left(\frac{\eta}{\xi}\right)^{\widetilde{\alpha}-1}\ d\eta\right]\frac{f(\xi)}{\xi^{\widetilde{\alpha}+2}}\ d\xi,$$

where $\widetilde{\beta} = -\frac{1}{2\sigma^2}\left(r + \frac{\sigma^2}{2}\right)^2.$

For the floating strike lookback option, we have $f(\xi) = \xi - 1$. The corresponding price function is found to be

$$V_{f\ell}(S, M, \tau) = Me^{-r\tau}\left[N(-d + \sqrt{\tau}) - \frac{\sigma^2}{2r}\left(\frac{M}{S}\right)^{2r/\sigma^2} N\left(d - \frac{2r}{\sigma}\sqrt{\tau}\right)\right] - S\left[N(-d) - \frac{\sigma^2}{2r}N(d)\right],$$

where

$$d = \frac{\ln\frac{M}{S} - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

2.5 Multistate lookback options

Two-asset semi-lookback option

The terminal payoff of a semi-lookback option depends on the extreme value of the price of one asset and the terminal values of the prices of other assets. Let $V^2_{semi}(S_1, S_2, t; \overline{S}_2[T_0, t])$ denote the value of the two-asset semi-lookback option whose terminal payoff is given by $(\overline{S}_2[T_0, T] - S_{1,T} - K)^+$. We write $\overline{S}_2[T_0, t] = M_2$ and $\overline{S}_2[t, T] = (M_2)_t^T$, and let the terminal payoff be expressed as

$$(\max(M_2 - K, (M_2)_t^T - K) - S_{1,T})^+.$$

We choose the sub-replicating instrument to be the put option on S_1 with strike $M_2 - K$, where the corresponding terminal payoff is

$$((M_2 - K) - S_{1,T})^+.$$

Similarly, there are two sources of risks: (i) realization of higher maximum asset value, where $(M_2)_t^T > M_2$, (ii) $S_{1,T} > M_2 - K$ and $(M_2)_t^T - K > S_{1,T} \Leftrightarrow M_2 < S_{1,T} + K < (M_2)_t^T$.

- The first scenario leads to an increase in the strike of the semi-lookback option from $M_2 - K$ to $(M_2)_t^T - K$.

- The second scenario leads to positive terminal payoff in the semi-lookback option but zero terminal payoff in the vanilla put.

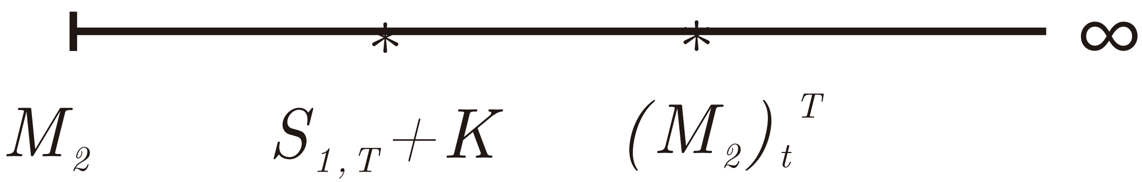
The two cases can be combined into $M_2 < S_{1,T} + K < (M_2)_t^T$. When such scenario occurs, the under replication at maturity is

$$(M_2)_t^T - K - S_{1,T} = [(M_2)_t^T - M_2] - [(S_{1,T} + K) - M_2].$$

The above terminal payoff is equivalent to the sum of those of the fixed strike lookback call on $(M_2)_t^T$ and vanilla call on $S_{1,T}+K$, both with the same strike M_2.

The required replenishing premium is given by

$$
e^{-r\tau}\left\{\int_{M_2}^{\infty} P[(M_2)_t^T > \xi]\ d\xi - \int_{M_2}^{\infty} P[S_{1,T}+K > \xi]\ d\xi\right\}
$$
$$
= e^{-r\tau}\int_{M_2}^{\infty} P[(M_2)_t^T > \xi \geq S_{1,T}+K]\ d\xi.
$$



The value of the two-asset semi-lookback option is given by

$$
V_{semi}^2(S_1, S_2, t; \overline{S}_2[T_0, t]) = p(S_1, t; M_2 - K)
$$
$$
+\ e^{-r\tau}\int_{M_2}^{\infty} P[(M_2)_t^T > \xi \geq S_{1,T}+K]\ d\xi.
$$

Discretely monitored floating strike lookback call options

Suppose the monitoring of the minimum value of the asset price takes place only at discrete time instant $t_j, j = 1, 2, \cdots, n$, where t_n is on or before the maturity date of the lookback call option. Suppose the current time is taken to be within $[t_k, t_{k+1})$. The terminal payoff of the discretely monitored floating strike lookback call option is given by

$$c_{f\ell}^{dis}(S_T, T) = S_T - \min(S_{t_1}, S_{t_2}, \cdots, S_{t_n}).$$

We use the notation $\underline{S}[i, j]$ to denote $\min(S_{t_i}, S_{t_{i+1}}, \cdots, S_{t_j}), j > i$. At the current time, $\underline{S}[1, k] = \min(S_{t_1}, S_{t_2}, \cdots, S_{t_k})$ is already known. Similar to the continuously monitored case, we choose the sub-replicating instrument to be a forward with the same maturity date T and delivery price $\underline{S}[1, k]$.

Potential liabilities occur when $\underline{S}[k+1,n]$ falls below $\underline{S}[1,k]$. This is similar to a put with the stochastic state variable $\underline{S}[k+1,n]$ and strike $\underline{S}[1,k]$.

The replenishing premium required to compensate for under replication is given by

$$\text{replenishing premium} \;=\; e^{-r\tau}\int_0^{\underline{S}[1,k]} P[\underline{S}[k+1,n] \le \xi]\ d\xi.$$

The present value of the discretely monitored European floating strike lookback call option is then given by

$$c_{f\ell}^{dis}(S,t;\underline{S}[1,k]) = S - e^{-r\tau}\underline{S}[1,k] + e^{-r\tau}\int_0^{\underline{S}[1,k]} P[\underline{S}[k+1,n] \le \xi]\ d\xi.$$

Remark Following the strike bonus approach, we can also obtain

$$c_{f\ell}^{dis}(S,t;\underline{S}[1,k]) = c_E(S,t;\underline{S}[1,k]) + e^{-r\tau}\int_0^{\underline{S}[1,k]} P[\underline{S}[k+1,n] \le \xi < S_T]\ d\xi.$$

The distribution function $P[\underline{S}[k+1,n] \leq \xi]$ can be expressed as

$$P[\underline{S}[k+1,n] \leq \xi] = \sum_{j=k+1}^{n} E\mathbf{1}_{\{S_{t_j} \leq \xi, S_{t_j}/S_{t_i} \leq 1 \text{ for all } i \neq j, k+1 \leq i \leq n\}},$$

where the indicator function in the jth term corresponds to the event that S_{t_j} is taken be the minima among $S_{t_{k+1}}, \cdots, S_{t_n}$; and j runs from $k+1$ to n.

Suppose the asset price follows the Geometric Brownian motion, then S_{t_j} and $S_{t_j}/S_{t_i}, i \neq j, k+1 \leq i \leq n$, are all lognormally distributed. The expectation values in above equation can be expressed in terms of multi-variate cumulative normal distribution functions.

One-asset lookback spread option

The terminal payoff of an one-asset lookback spread option is given by

$$c_{sp}(S_T, T; K) = \left(M_{T_0}^T - m_{T_0}^T - K\right)^+ .$$

Choice of sub-replicating portfolio:

- long holding of one unit of European lookback call and one unit of lookback put, both of floating strike;
- short holding of a riskless bond of par value K.

All these instruments have the same maturity as that of the lookback spread option.

$$\begin{aligned} & \text{Terminal payoff of this sub-replicating portfolio} \\ = \ & (M_{T_0}^T - S_T) + (S_T - m_{T_0}^T) - K = M_{T_0}^T - m_{T_0}^T - K. \end{aligned}$$

Note that

$$\begin{aligned} M_{T_0}^T - m_{T_0}^T - K &= \max(M, M_t^T) - \min(m, m_t^T) - K \\ &\geq M - m - K, \end{aligned}$$

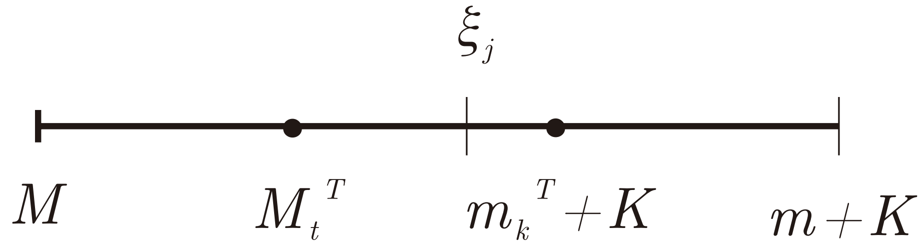
so if the lookback spread is currently in-the-money, then it will expire in-the-money. The sub-replication is a full replication when the lookback spread option is currently in-the-money.

On the other hand, if the lookback spread option is currently out-of-the-money, the terminal payoff of the sub-replicating portfolio would be less than that of the lookback spread option if the lookback spread option expires out-of-the-money. That is,

$$\max(M, M_t^T) - \min(m, m_t^T) - K < 0.$$

The sources of risks include (i) $M_t^T > M$ and $m_t^T < m$, (ii) $M_t^T - m_t^T - K < 0 \Leftrightarrow m_t^T + K > M_t^T$. These cases can be combined into

$$M < M_t^T < m_t^T + K < m + K.$$



We hedge the exposure over successive subintervals within $[M, m + K]$: $(M + (j-1)\Delta\xi, M + j\Delta\xi)$ and write $\xi_j = M + j\Delta\xi$.

The potential liability of $\Delta\xi$ has to be replenished when both of the following 2 events occur together:

$$M_t^T < \xi_j \quad \text{and} \quad m_t^T + K > \xi_j.$$

The expected replenishing premium over the j^{th} interval is

$$e^{-r\tau}\Delta\xi \; P[M_t^T < \xi_j < m_t^T + K].$$

Summing over all intervals, we obtain

$$e^{-r\tau}\int_M^{m+K} P[M_t^T < \xi < m_t^T + K] \; d\xi.$$

In summary

(i) $M - m - K \geq 0$ (currently in-the-money or at-the-money)

$$c_{sp}(S, t; M, m) = c_{f\ell}(S, t; m) + p_{f\ell}(S, t; M) - Ke^{-r\tau};$$

(ii) $M - m - K < 0$ (currently out-of-the-money)

$$\begin{aligned} & c_{sp}(S, t; M, m) \\ = \; & c_{f\ell}(S, t; m) + p_{f\ell}(S, t; M) - Ke^{-r\tau} \\ & + e^{-r\tau} \int_M^{m+K} P[M_t^T < \xi < m_t^T + K] \, d\xi. \end{aligned}$$

The distribution function $P[M < \xi \leq m + K]$ can be deduced from

$$
\begin{aligned}
& P[\underline{X} \geq x, \overline{X} \leq y] \\
= & \sum_{n=-\infty}^{\infty} e^{[2n\alpha(y-x)]/\sigma^2} \left\{ \left[N\left(\frac{y - \alpha t - 2n(y-x)}{\sigma\sqrt{t}} \right) \right. \right. \\
& \qquad \left. - N\left(\frac{x - \alpha t - 2n(y-x)}{\sigma_1\sqrt{t}} \right) \right] \\
& \qquad - e^{2\alpha x/\sigma^2} \left[N\left(\frac{y - \alpha t - 2n(y-x) - 2x}{\sigma\sqrt{t}} \right) \right. \\
& \qquad \left. \left. - N\left(\frac{x - \alpha t - 2n(y-x) - 2x}{\sigma\sqrt{t}} \right) \right] \right\}.
\end{aligned}
$$

Two-asset lookback spread option

Let $S_{1,u}$ and $S_{2,u}$ denote the price process of asset 1 and asset 2, respectively. Similarly, we write $\overline{S}_1[t_1, t_2]$ and $\underline{S}_2[t_1, t_2]$ as the realized maximum value of $S_{1,u}$ and realized minimum value of $S_{2,u}$ over the period $[t_1, t_2]$, respectively. The terminal payoff of a two-asset lookback spread option is given by

$$c_{sp}(S_{1,T}, S_{2,T}, T; K) = (\overline{S}_1[T_0, T] - \underline{S}_2[T_0, T] - K)^+,$$

where K is the strike price.

Replicating portfolio

Since we can express $\overline{S}_1[T_0, T] - \underline{S}_2[T_0, T] - K$ as

$$(\overline{S}_1[T_0, T] - S_{1,T}) + (S_{2,T} - \underline{S}_2[T_0, T]) + S_{1,T} - S_{2,T} - K,$$

a natural choice of the sub-replicating portfolio would consist of long holding of one European floating strike lookback put on asset 1, one European floating strike lookback call on asset 2, one unit of asset 1 and short holding of one unit of asset 2 and a riskless bond of par value K. All instruments in the portfolio have the same maturity as that of the two-asset lookback spread option.

Similar to the one-asset counterpart, the two-asset lookback spread option is guaranteed to expire in-the-money if it is currently in-the-money.

The sub-replicating portfolio will expire with a terminal payoff below that of the lookback spread option if the lookback spread option expires ont-of-the-money.

The current value of the two-asset European lookback spread option is given by

(i) $\overline{S}_1[T_0,t] - \underline{S}_2[T_0,t] - K \geq 0$

$$c_{sp}(S_1, S_2, t; \overline{S}_1[T_0,t], \underline{S}_2[T_0,t]) = p_{f\ell}(S_1, t; \overline{S}_1[T_0,t]) + c_{f\ell}(S_2, t; \underline{S}_2[T_0,t]) + S_1 - S_2 - Ke^{-r\tau};$$

(ii) $\overline{S}_1[T_0,t] - \underline{S}_2[T_0,t] - K < 0$

$$\begin{aligned} c_{sp}(S_1, S_2, t; \overline{S}_1[T_0,t], \underline{S}_2[T_0,t]) = {} & p_{f\ell}(S_1, t; \overline{S}_1[T_0,t]) + c_{f\ell}(S_2, t; \underline{S}_2[T_0,t]) \\ & + S_1 - S_2 - Ke^{-r\tau} \\ & + e^{-r\tau} \int_{\overline{S}_1[T_0,t]}^{\underline{S}_2[T_0,t]+K} P(\overline{S}_1[t,T] < \xi \leq \underline{S}_2[t,T] + K)\, d\xi. \end{aligned}$$

- The joint distribution of maximum of one asset and minimum of another asset cannot be deduced using the method of images. One has to resort to eigenfunction expansion technique, and the resulting expression involves the modified Bessel functions.

Lookbacks on two assets

- *Double Maxima*: call or put on the difference between the maximum of S_1 and the maximum of S_2:

$$\max[0, (a\overline{S}_1(T) - b\overline{S}_2(T)) - K]$$

$$\max[0, K - (a\overline{S}_1(T) - b\overline{S}_2(T))]$$

  where $a > 0$ and $b > 0$ are parameters to be chosen by investors.

  In practice, it may make sense to pick a and b such that $aS_1(0) = bS_2(0)$. For example, $a = 1/S_1(0)$ and $b = 1/S_2(0)$.

  With a and b chosen in this way, the double maxima represents an option on the difference between the maximum returns of the two stocks over a given period. When $K = 0$, the double maxima call is equivlent to an option to buy the maximum of S_1 at the maximum of S_2.

- *Double Minima:* call or put on the difference between the minimum of S_1 and the minimum of S_2:

$$\max[0, (a\underline{S}_1(T) - b\underline{S}_2(T)) - K]$$

$$\max[0, K - (a\underline{S}_1(T) - b\underline{S}_2(T))].$$

When $K = 0$, the double minima call is equivalent to an option to sell the minimum of S_1 for the minimum of S_2.

- *Double Lookback Spread:* call or put on the spread between the maximum S_1 and the minimum of S_2:

$$\max[0, (a\overline{S}_1(T) - b\underline{S}_2(T)) - K]$$

$$\max[0, K - (a\overline{S}_1(T) - b\underline{S}_2(T))].$$

Uses of these double-asset lookback options

- Double maxima/minima represent options on the difference between the maximum/ minimum returns of two stocks over a given period. They provide investors with a special vehicle to take a view on how these two stocks will perform relative to each other.

- Similarly, double lookback spreads capture the difference between the maximum upside of one stock and the maximum downside of another stock. This type of options can be an aggressive play on the volatilities of the two stocks as well as on the correlation of the two stocks.

$$V_{Dmax}(x_1, x_2) = \max\left[0, aS_1(0)e^{\max(M_1, x_1)} - bS_2(0)e^{\max(M_2, x_2)} - K\right]$$

$$V_{Dmin}(x_1, x_2) = \max\left[0, aS_1(0)e^{\min(m_1, x_1)} - bS_2(0)e^{\min(m_2, x_2)} - K\right]$$

$$V_{DLS}(x_1, x_2) = \max\left[0, aS_1(0)e^{\max(M_1, x_1)} - bS_2(0)e^{\min(m_1, x_2)} - K\right]$$

Recall
$X_i(t) = \alpha_i t + \sigma_i Z_i(t)$, $i = 1, 2$; $\text{cov}(Z_1(t), Z_2(t)) = \rho t$; ρ is a constant.

$$\underline{X}_i(t) = \min_{0 \le s \le t} X_i(s), \ \ \overline{X}_i(t) = \max_{0 \le s \le t} X_i(t).$$

The call prices C_{Dmax}, C_{Dmin}, and C_{DLS}, respectively, for double maxima, double minima, and double lookback spread options are determined as follows:

$$C_{Dmax}(x_1, x_2) = e^{-rT} \int_0^\infty dx_1 \int_0^\infty dx_2 V_{Dmax}(x_1, x_2) \frac{\partial^2 P(\overline{X}_1(t) \le x_1, \overline{X}_2(t) \le x_2)}{\partial x_1 \partial x_2}$$
$$C_{Dmin}(x_1, x_2) = e^{-rT} \int_{-\infty}^0 dx_1 \int_{-\infty}^0 dx_2 V_{Dmin}(x_1, x_2) \frac{\partial^2 P(\underline{X}_1(t) \ge x_1, \underline{X}_2(t) \ge x_2)}{\partial x_1 \partial x_2}$$
$$C_{DLS}(x_1, x_2) = e^{-rT} \int_{-\infty}^0 dx_1 \int_0^\infty dx_2 V_{DLS}(x_1, x_2) \frac{-\partial^2 P(\underline{X}_1(t) \ge x_1, \overline{X}_2(t) \le x_2)}{\partial x_1 \partial x_2}.$$

Probability density / distribution functions of the extreme values of two correlated Brownian motions.

$$
\begin{aligned}
&P(X_1(t) \in dx_1, X_2(t) \in dx_2, \underline{X}_1(t) \geq m_1, \underline{X}_2 \geq m_2) \\
&= p(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho)\, dx_1 dx_2
\end{aligned}
$$

(i) For $x_1 \geq m_1$, $x_2 \geq m_2$, where $m_1 \leq 0$, $m_2 \leq 0$,

$$
\begin{aligned}
&p(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) \\
&= \frac{e^{a_1 x_1 + a_2 x_2 + bt}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} h(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho),
\end{aligned}
$$

where

$$
\begin{aligned}
&h(x_1, x_2, t; m_1, m_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho) \\
&= \frac{2}{\beta t} \sum_{n=1}^{\infty} e^{-(r^2 + r_0^2)/2t} \sin \frac{n\pi\theta_0}{\beta} \sin \frac{n\pi\theta}{\beta} I_{(n\pi)/\beta}\left(\frac{r r_0}{t}\right).
\end{aligned}
$$

The parameter values are given by

$$a_1 = \frac{\alpha_1\sigma_2 - \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2}, \quad a_2 = \frac{\alpha_2\sigma_1 - \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2},$$

$$b = -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \rho\sigma_1\sigma_2 a_1 a_2 + \frac{1}{2}\sigma_2^2 a_2^2,$$

$$\tan\beta = -\frac{\sqrt{1-\rho^2}}{\rho}, \quad \beta \in [0,\pi],$$

$$z_1 = \frac{1}{\sqrt{1-\rho^2}}\left[\left(\frac{x_1 - m_1}{\sigma_1}\right) - \rho\left(\frac{x_2 - m_2}{\sigma_2}\right)\right], \quad z_2 = \left(\frac{x_2 - m_2}{\sigma_2}\right),$$

$$z_{10} = \frac{1}{\sqrt{1-\rho^2}}\left(-\frac{m_1}{\sigma_1} + \frac{\rho m_2}{\sigma_2}\right), \quad z_{20} = -\frac{m_2}{\sigma_2},$$

$$r = \sqrt{z_1^2 + z_2^2}, \quad \tan\theta = \frac{z_2}{z_1}, \quad \theta = [0,\beta],$$

$$r_0 = \sqrt{z_{10}^2 + z_{20}^2}, \quad \tan\theta_0 = \frac{z_{20}}{z_{10}}, \quad \theta_0 \in [0,\beta].$$

The distribution function p satisfies the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = -\alpha_1 \frac{\partial p}{\partial x_1} - \alpha_2 \frac{\partial p}{\partial x_2} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 p}{\partial x_1^2} + \rho \sigma_1 \sigma_2 \frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 p}{\partial x_2^2},$$

$$t > 0,\ m_1 < x_1 < \infty,\ m_2 < x_2 < \infty;$$

with the following initial condition:

$$p(x_1, x_2, t = 0) = \delta(x_1)\delta(x_2)$$

and absorbing boundary conditions

$$p(x_1 = m_1, x_2, t) = 0$$
$$p(x_1, x_2 = m_2, t) = 0.$$

To get rid of the drift terms, we define

$$p(x_1, x_2, t) = e^{a_1 x_1 + a_2 x_2 + bt} q(x_1, x_2, t),$$

where a_1, a_2, and b are defined as above. Then $q(x_1, x_2, t)$ satisfies

$$\frac{\partial q}{\partial t} = \frac{1}{2}\sigma_1^2 \frac{\partial^2 q}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 q}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 p}{\partial x_2^2}$$

with auxiliary conditions:

$$\begin{aligned} q(x_1, x_2, t = 0) &= \delta(x_1)\delta(x_2) \\ q(x_1 = m_1, x_2, t) &= 0 \\ q(x_1, x_2 = m_2, t) &= 0. \end{aligned}$$

This PDE can be simplified by a suitable transformation of coordinates, eliminate the cross-partial derivative and normalize the Brownian motions. Explicitly, if we define the set of new coordinates z_1 and z_2, where

$$z_1 = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_1 - m_1}{\sigma_1} - \rho\frac{x_2 - m_2}{\sigma_2}\right) \text{ and } z_2 = \frac{x_2 - m_2}{\sigma_2}, \text{ and write}$$

$$q(x_1, x_2, t) = \frac{h(z_1, z_2, t)}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}}.$$

This procedure is similar to define two uncorrelated Brownian motions from two given correlated Brownian motions. Here, $h(z_1, z_2, t)$ satisfies the following standard diffusion equation (without the cross-derivative term):

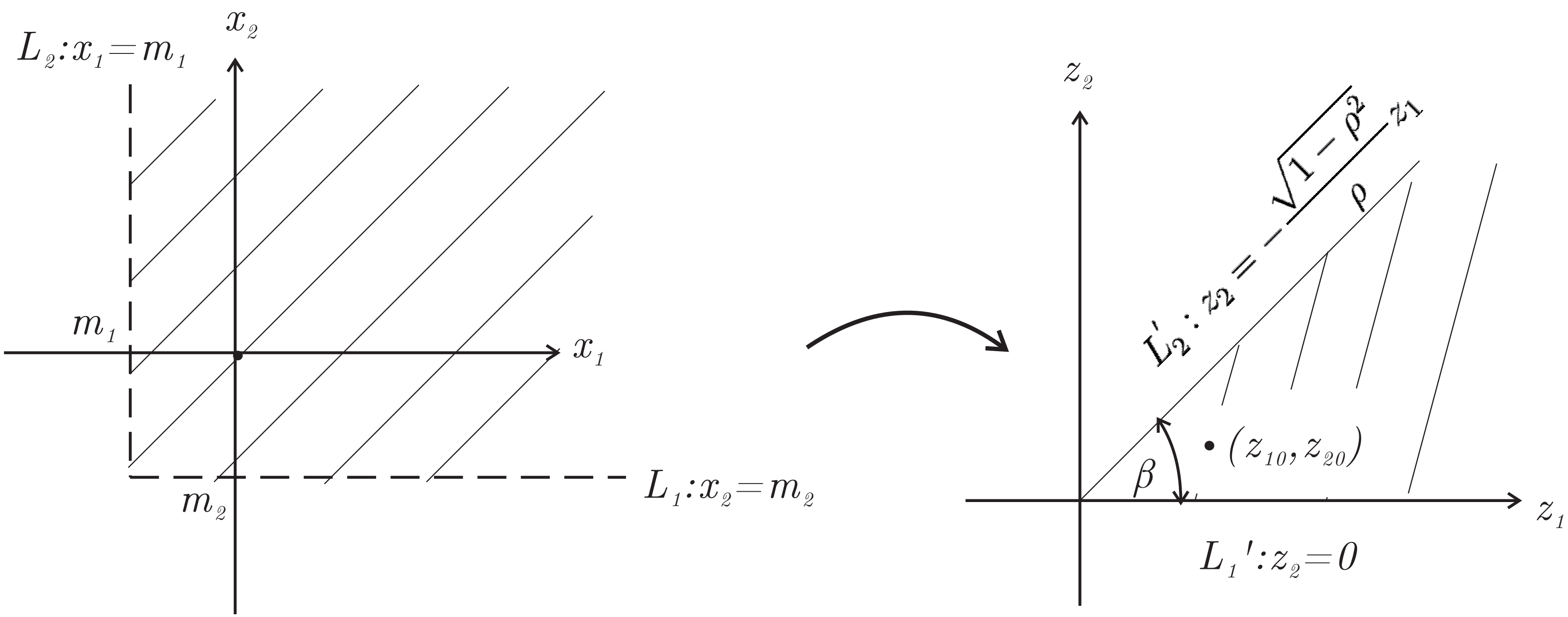
$$\frac{\partial h}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 h}{\partial z_1^2} + \frac{\partial^2 h}{\partial z_2^2}\right)$$

with boundary condition:

$$\begin{aligned} h(z_1, z_2, t) &= \delta(z_1 - z_{10})\delta(z_2 - z_{20}) \\ h(L_1, t) &= h(L_2, t) = 0, \end{aligned}$$

where $z_{10} = \frac{1}{\sqrt{1-\rho^2}}\left(-\frac{m_1}{\sigma_1} + \frac{\rho m_2}{\sigma_2}\right)$ and $z_{20} = -\frac{m_2}{\sigma_2}$; and

$$L_1 = \{(z_1, z_2) : z_2 = 0\},\ L_2 = \left\{(z_1, z_2) : z_2 = -\frac{\sqrt{1-\rho^2}}{\rho} z_1\right\}.$$



$0 \leq r < \infty, \; 0 \leq \theta \leq \beta;$

$\tan\beta = -\frac{\sqrt{1-\rho^2}}{\rho}.$

These boundary conditions along L_1 and L_2 are more conveniently expressed in polar coordinates. Introducing polar coordinates (r, θ) corresponding to (z_1, z_2) as defined above, we obtain

$$\frac{\partial h}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2}\right),\ r^2 = z_1^2 + z_2^2 \ \text{ and } \ \tan\,\theta = \frac{z_2}{z_1};$$

with boundary conditions: $h(r, \theta, t = 0) = \dfrac{1}{r_0}\delta(r - r_0)\delta(\theta - \theta_0)$,

$$h(r, \theta = 0, t) = 0,\ h(r, \theta = \beta, t) = 0.$$

Note that $\dfrac{1}{r_0}$ arises from the Jacobian of transformation from (z_1, z_2) to (r, θ).

To solve this PDE for $h(r,\theta,t)$, we look for separable solutions of the form $R(r)\Theta(\theta)T(t)$. Plugging this into the PDE, we find

$$\frac{T'}{T} = \frac{1}{2}\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right) = -\lambda^2/2,$$

where the separation constant is negative because the solutions must decay as $t \to \infty$.

Hence, we have the eigenfunction for $T(t)$ to be an exponential function:

$$T(t) \sim e^{-\lambda^2 t/2}$$

and

$$\overbrace{\left(r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda^2 r^2\right)}^{k^2} + \overbrace{\left(\frac{\Theta''}{\Theta}\right)}^{-\,k^2} = 0.$$

Defining $\Theta''/\Theta = -k^2$, we obtain

$$\Theta(\theta) \sim A\sin k\theta + B\cos k\theta.$$

The boundary conditions require that $\Theta(0) = \Theta(\beta) = 0$, and hence k must be real, $B = 0$, and $\sin k\beta = 0$.

This last requirement restricts k to discrete values of the form: $k_n = \dfrac{n\pi}{\beta}$, $n = 1, 2, \ldots$.

Thus the most general angular solution consistent with the boundary conditions is

$$\Theta(\theta) \sim \sin \frac{n\pi\theta}{\beta}, \; n = 1, 2, \ldots .$$

Finally, the radial part of the solution is

$$r^2 R'' + rR' + (\lambda^2 r^2 - k_n^2)R = 0.$$

Defining $y = \lambda r$, we can rewrite this in the standard form

$$y^2 \frac{d^2R}{dy^2} + y\frac{dR}{dy} + (y^2 - k_n^2)R = 0.$$

This is the Bessel equation, with the well known fundamental solutions $J_{k_n}(y)$ and $I_{k_n}(y)$. Since $I_{k_n}(0)$ diverges and we require $R(0)$ to be well-behaved, the $I_{k_n}(x)$ solution is not permitted. Hence the general radial solution is $R(r) \sim J_{k_n}(\lambda r)$.

In summary, the most general solution to the PDE for $h(r, \theta, t)$ consistent with the absorbing boundary conditions:

$$h(r, 0, t) = h(r, \beta, t) = 0,$$

is given by

$$h(r, \theta, t) = \int_0^\infty \sum_{n=1}^{\infty} c_n(\lambda) e^{-\lambda^2 t/2} \sin\left(\frac{n\pi\theta}{\beta}\right) J_{n\pi/\beta}(\ \lambda r)\ d\lambda.$$

Note that $T_n(t)$ has a continuum of eigenvalues λ.

The next step is to find the coefficients $c_n(\lambda)$ which fit the initial condition:

$$h(r, \theta, 0) = r_0^{-1} \delta(r - r_0) \delta(\theta - \theta_0).$$

Setting $t = 0$, we have

$$\frac{1}{r_0}\delta(r - r_0)\delta(\theta - \theta_0) = h(r, \theta, 0) = \int_0^\infty \sum_{n=1}^{\infty} c_n(\lambda) \sin \frac{n\pi\theta}{\beta} J_{\frac{n\pi}{\beta}}(\lambda r)\ d\lambda.$$

Next, we multiply the previous equation at $t = 0$ by $\sin(m\pi\theta/\beta)$ and integrate over the interval $[0, \beta]$ in θ. From the orthogonality relation, we obtain

$$r_0^{-1}\delta(r - r_0) \sin\left(\frac{m\pi\theta_0}{\beta}\right) = \frac{\beta}{2}\int_0^\infty c_m(\lambda) J_{m\pi/\beta}(\lambda r)\ d\lambda.$$

Using the following relation:

$$\int_0^\infty x J_v(ax) J_v(bx) dx = a^{-1}\delta(a - b),$$

and comparing the equations, we ontain

$$c_m(\lambda') = \frac{2\lambda'}{\beta} \sin\left(\frac{m\pi\theta_0}{\beta}\right) J_{m\pi/\beta}(\ \lambda' r_0).$$

Putting all the results together, we finally obtain

$$h(r,\theta,t) = \int_0^\infty \left(\frac{2\lambda}{\beta}\right) \sum_{n=1}^{\infty} e^{-\lambda^2 t/2} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin\left(\frac{n\pi\theta}{\beta}\right) J_{n\pi/\beta}(\lambda r_0) J_{n\pi/\beta}(\lambda r)\ d\lambda.$$

The integration with respect to λ can be performed explicitly using the relation:

$$\int_0^\infty x e^{-c^2x^2} J_v(ax) J_v(bx)\ dx = \frac{1}{2c^2} e^{-(a^2+b^2)/4c^2} I_v\left(\frac{ab}{2c^2}\right).$$

We then obtain the final expression, where

$$h(r,\theta,t) = \frac{2}{\beta t} \sum_{n=1}^{\infty} e^{-(r^2+r_0^2)/2t} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin\left(\frac{n\pi\theta}{\beta}\right) I_{n\pi/\beta}\left(\frac{rr_0}{t}\right).$$

Based on the symmetry relations, we obtain

(ii) For $x_1 \geq m_1$, $x_2 \leq M_2$, where $m_1 \leq 0$, $M_2 \geq 0$, we have

$$
\begin{aligned}
& P(X_1(t) \in dx_1, X_2(t) \in dx_2, \underline{X}_1(t) \geq m_1, \overline{X}_2 \leq m_2) \\
= & \, p(x_1, -x_2, t; m_1, -M_2, \alpha_1, -\alpha_2, \sigma_1, \sigma_2, -\rho) \, dx_1 dx_2.
\end{aligned}
$$

(iii) For $x_1 \leq M_1$, $x_2 \leq M_2$, where $M_1 \geq 0$, $M_2 \geq 0$, we have

$$
\begin{aligned}
& P(X_1(t) \in dx_1, X_2(t) \in dx_2, \overline{X}_1(t) \leq m_1, \overline{X}_2 \leq m_2) \\
= & \, p(-x_1, -x_2, t; -M_1, -M_2, -\alpha_1, -\alpha_2, \sigma_1, \sigma_2, \rho) \, dx_1 dx_2.
\end{aligned}
$$

From the joint density function of correlated restricted Brownian motions, we derive the distribution function of joint maximum of the two-dimensional correlated Brownian motions as follows:

Integrating over the density functions and applying a change of polar-coordinates, we obtain the distribution function:

$$P(\overline{X}_1(t) \leq x_1, \overline{X}_2(t) \leq x_2) = e^{a_1 x_1 + a_2 x_2 + bt} f(r', \theta', t),$$

where

$$f(r', \theta', t) = \frac{2}{\alpha' t} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta'}{\alpha'}\right) e^{-r'^2/2t} \int_0^{\alpha'} \sin\left(\frac{n\pi\theta}{\alpha'}\right) g_n(\theta)\, d\theta,$$

with

$$g_n(\theta) = \int_0^\infty r e^{-r^2/2t} e^{-b_1 r \cos(\theta-\alpha) - b_2 r \sin(\theta-\alpha)} I_{n\pi/\alpha}\left(\frac{rr'}{t}\right) dr,$$

$$r' = \frac{1}{\sqrt{1-\rho^2}} \left(\frac{x_1^2}{\sigma_1^2} - \frac{2\rho x_1 x_2}{\sigma_1 \sigma_2} + \frac{x_2^2}{\sigma_2^2} \right)^{1/2}$$

$$\theta' = \theta + \alpha, \quad \text{with } \cos\theta = \frac{x_1}{\sigma_1 r'}$$

$$\tan\alpha = \frac{\rho}{\sqrt{1-\rho^2}}, \quad \alpha' = \alpha + \frac{\pi}{2},$$

$$b_1 = a_1\sigma_1 + a_2\sigma_2\rho \quad \text{and} \quad b_2 = a_2\sigma_2\sqrt{1-\rho^2}.$$

Similar expressions can be derived for

$$P(\underline{X}_1(T) \geq x_1, \overline{X}_2(t) \leq x_2) \quad \text{and} \quad P(\underline{X}_1(t) \geq x_1, \underline{X}_2(t) \geq x_2).$$

Corollary

When the correlation ρ can take on only the special values

$$\rho_\pi = -\cos\frac{\pi}{n}, \; n = 1, 2, \ldots,$$

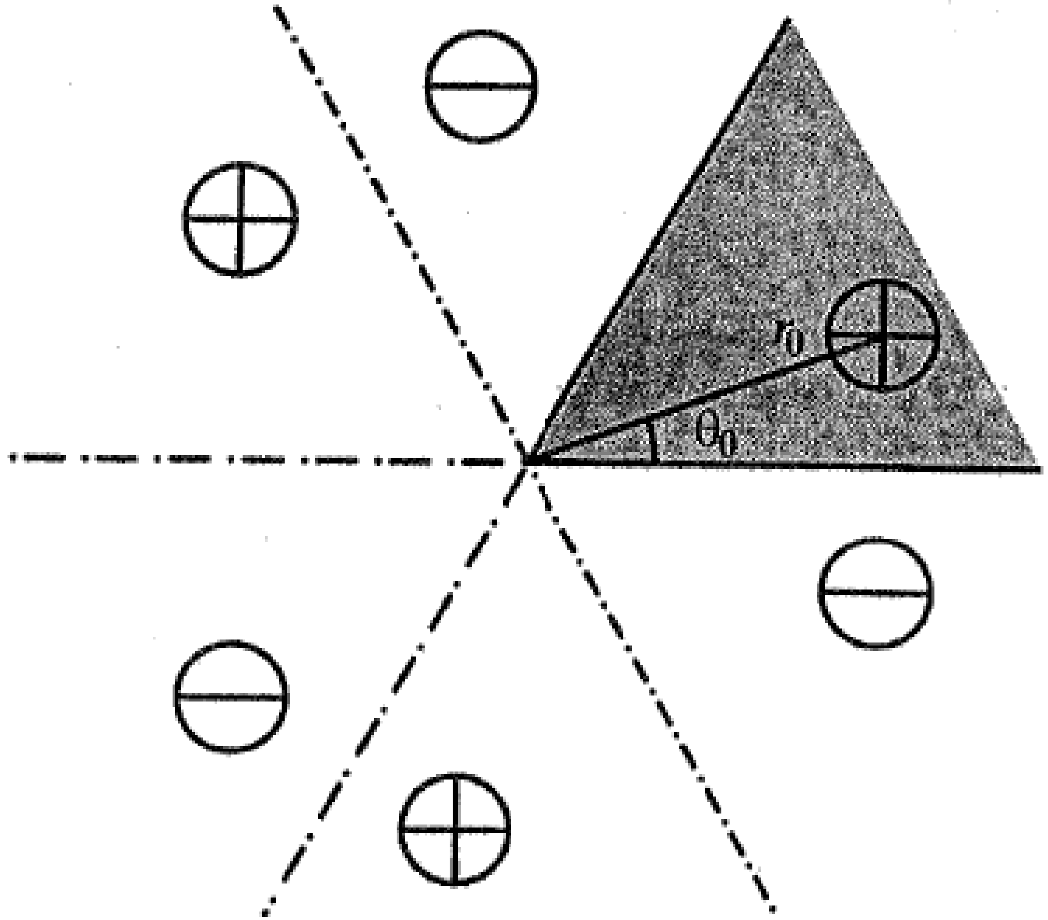
then $\tan\beta = -\frac{\sqrt{1-\cos^2\frac{\pi}{n}}}{-\cos\frac{\pi}{n}} = \tan\frac{\pi}{n}$ so that $\beta = \frac{\pi}{n}$. In this case, the density function p has the special form

$$p(x_1, x_2, t) = \frac{e^{a_1 x_1 + a_2 x_2 + bt}}{\sigma_1 \sigma_2 \sqrt{1-\rho_n^2}} h(z_1, z_2, t),$$

where h is a finite sum of bivariate normal densities

$$h(z_1, z_2, t) = \sum_{k=0}^{n-1} [g_k^+(z_1, z_2, t) + g_k^-(z_1, z_2, t)]$$

$$g_k^\pm(z_1, z_2, t) = \pm\frac{1}{2\pi}\exp\left(-\frac{1}{2}\left[\left(z_1 - r_0\cos\left(\frac{2k\pi}{n} \pm \theta_0\right)\right)^2 + \left(z_2 - r_0\sin\left(\frac{2k\pi}{n} \pm \theta_0\right)\right)^2\right]\right).$$



$\rho = -1/2$
$\beta = \pi/3$

A method of images solution for the two-dimensional case.

When $\rho_n = -\cos(\pi/n)$, the angles between the lines L_1 and L_2 take the special values

$$\beta_n = \pi/n, \; n = 1, 2, \ldots .$$

For these angles, a method of images solution to the PDE is possible. Note that

$$g^{\pm}(z_1, z_2, t; a_1, a_2) = \pm \frac{1}{2\pi t} \exp\left(-\frac{1}{2}[(z_1 - a_1)^2 + (z_2 - a_2)^2]\right)$$

satisfies the PDE with the initial condition

$$g^{\pm}(z_1, z_2, 0; a_1, a_2) = \pm\delta(z_1 - a_1)\delta(z_2 - a_2).$$

2.6 Dynamic fund protection

Provides an investor with a floor level of protection during the investment period.

- This feature generalizes the concept of a put option, which provides only a floor value at a particular time.

- The dynamic fund protection ensures that the fund value is upgraded if it ever falls below a certain threshold level.

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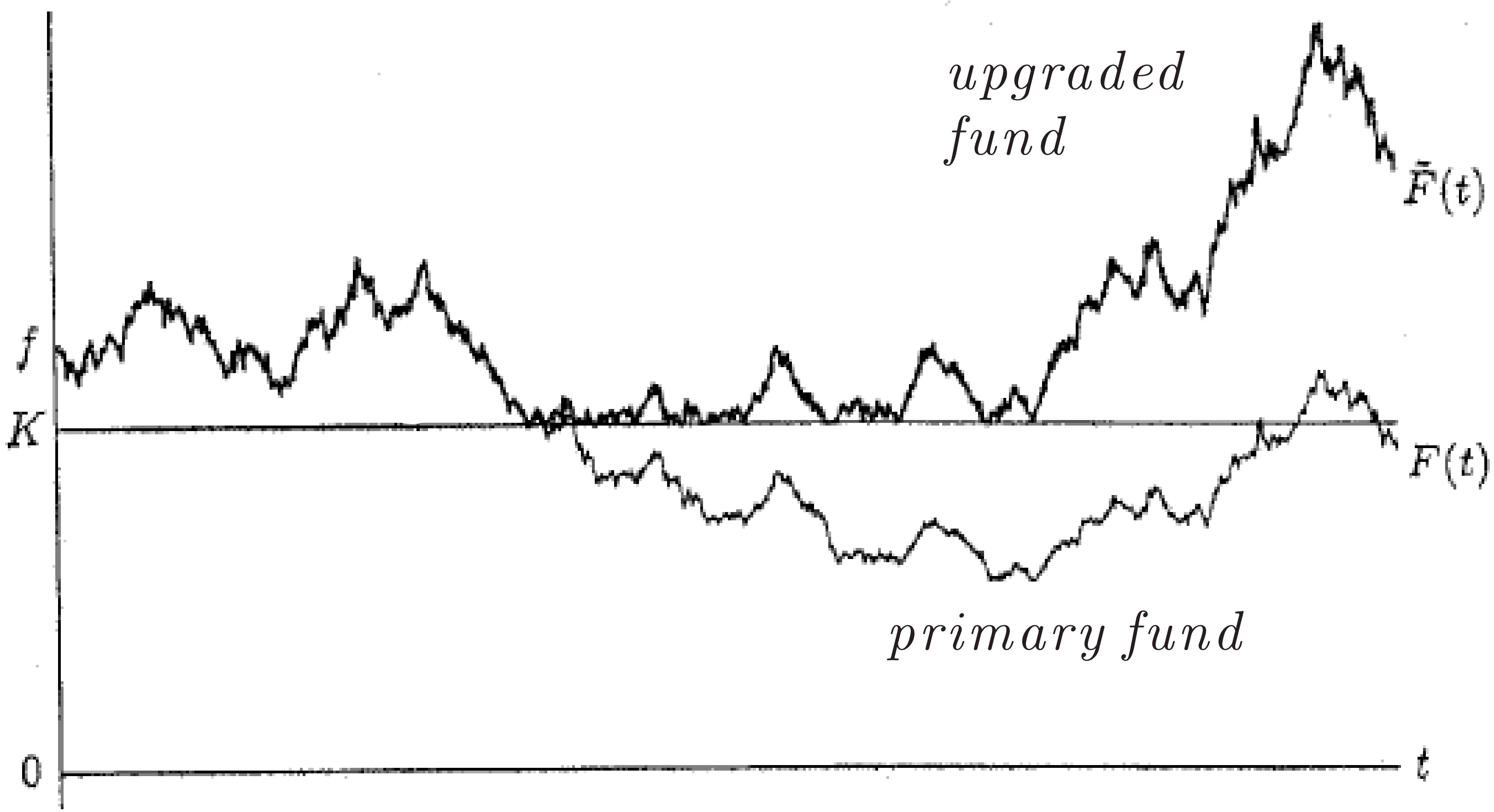
Typical sample path of the fund unit values



1. $\widetilde{F}(0) = F(0)$

2. $\widetilde{F}(t) = F(t) \max \left\{ 1, \max_{0 \le s \le t} \frac{K}{F(s)} \right\}$

- Whenever $\widetilde{F}(t)$ drops below K, just enough money will be added so that the upgraded fund unit value does not fall below K.

- Write $M(t) = \max \left\{ 1, \frac{K}{\min_{0 \le s \le t} F(s)} \right\}$, then

$$\widetilde{F}(T) = F(T) \max \left\{ M(t), \frac{K}{\min_{t \le s \le T} F(s)} \right\}.$$

Here, $M(t)$ is the number of units of primary fund acquired at time t. The lookback feature of the dynamic protection is revealed by $M(t)$.

Protection with reference to a stock index under finite number of resets

$F(t)$: value of the primary fund

$I(t)$: value of the reference stock index

$\widetilde{F}(t)$: value of the protected fund

When the investor makes its reset decision, the sponsor of the protected fund has to purchase additional units of the primary fund so that the protected fund value is upgraded to that of the reference index.

F as the numeraire

At the reset instant ξ_i, $i = 1, 2, \ldots$, we have

$$\widetilde{F}(\xi_i) = I(\xi_i) = n(\xi_i)F(\xi_i),$$

where $n(\xi_i) = I(\xi_i)/F(\xi_i) > 1$ is the new number of units of the primary fund in the investment fund.

It is obvious that $n(\xi_1) < n(\xi_2) < \cdots$. The normalized upgraded fund value $\widetilde{F}/F$ is related to I/F. This motivates us to use F as the numeraire.

Fund value dynamics

Under the risk neutral valuation framework, we assume that the primary fund value $F(t)$ and the reference index value $I(t)$ follow the Geometric Brownian motions:

$$\begin{aligned} \frac{dF}{F} &= (r - q_p)dt + \sigma_p \, dZ_p, \\ \frac{dI}{I} &= (r - q_i)dt + \sigma_i \, dZ_i, \end{aligned}$$

where r is the riskless interest rate, q_p and q_i are the dividend yield of the primary fund and stock index, respectively, σ_p and σ_i are the volatility of the primary fund value and reference index value, respectively, and $dZ_p dZ_i = \rho \, dt$. Here, ρ is the correlation coefficient between the primary fund process and reference index process.

Pricing formulation of the protected fund with n resets

Let $V_n(F, I, t)$ denote the value of the investment fund with dynamic protection with respect to a reference stock index, where the investor has n reset rights outstanding. We first consider the simpler case, where there has been no prior reset. That is, the number of units of the primary fund is equal to one at current time t.

The dimension of the pricing model can be reduced by one if F is chosen as the numeraire. We define the stochastic state variable

$$x = \frac{I}{F}$$

where F is the fund value at the grant date (same as the primary fund value).

Note that x follows the Geometric Brownian motion

$$\frac{dx}{x} = (q_p - q_i)dt + \sigma \; dZ,$$

where $\sigma^2 = \sigma_p^2 - 2\rho\sigma_p\sigma_i + \sigma_i^2$. Accordingly, we define the normalized fund value function with F as the numeraire by

$$W_n(x,t) = \frac{V_n(F,I,t)}{F}.$$

Whenever a reset has occurred, the upgraded fund value $\widetilde{F}$ will be used as the numeraire so that $x = \frac{I}{\widetilde{F}}$. Note that $F(t)$ and $\widetilde{F}(t)$ have the same dynamics equation since $\widetilde{F}(t)$ is scalar multiple of $F(t)$.

- The investor should never reset when $F(t)$ stays above $I(t)$.

- With only a finite number of reset rights, he does not reset immediately when $F(t)$ just hits the level of $I(t)$.

With reference to the variable x, the investor resets when x reaches some sufficiently high threshold value (denoted by x_n^*). The value of x_n^* is not known in advance, but has to be solved as part of the solution to the pricing model.

Upon reset at $x = x_n^*$, the sponsor has to increase the number of units of the primary fund so that the new value of the investment fund equals I.

The corresponding number of units should then be x_n^*, which is the ratio of the reference index value to the primary fund value right before the reset moment.

After the reset, we have

- the number of resets outstanding is reduced by one,
- the value of x becomes one since the ratio of the reference index value to the newly upgraded fund value is one.

We then have

$$W_n(x_n^*, t) = x_n^* W_{n-1}(1, t).$$

To show the claim, note that

$$\begin{aligned} W_{n-1}(1, t^+) &= \frac{V_{n-1}(\widetilde{F}, \widetilde{F}, t^+)}{\widetilde{F}(t^+)} = \frac{V_n(\widetilde{F}(t^-), x_n^*(t^-)\widetilde{F}(t^-), t^-)}{x_n^*(t^-)\widetilde{F}(t^-)} \\ &= \frac{1}{x_n^*(t^-)} W_n(x, t^-). \end{aligned}$$

Smooth pasting condition

We should have the smooth pasting condition at $x = x_n^*$, based on optimality consideration:

$$W_n'(x_n^*, t) = W_{n-1}(1, t).$$

This extra smooth pasting condition determines the value of x_n^* such that the investment fund value is maximized.

The terminal payoff of the investment fund is simply equal to F, if no reset has occurred throughout the life of the fund. This gives $W_n(x, T) = 1$.

Governing equation

In the continuation region, inside which the investor chooses not to exercise the reset right, the value function $V_n(F, I, t)$ satisfies the Black-Scholes equation with the two state variables, F and I.

In terms of x, the governing equation and the associated auxiliary conditions for $W_n(x, t)$ are given by

$$\frac{\partial W_n}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 W_n}{\partial x^2} + (q_p - q_i) x \frac{\partial W_n}{\partial x} - q_p W_n = 0, \; t < T, \; x < x_n^*(t),$$
$$W_n(x_n^*, t) = x_n^* W_{n-1}(1, t) \quad \text{and} \quad W_n'(x_n^*, t) = W_{n-1}(1, t), \; W_n(x, T) = 1,$$

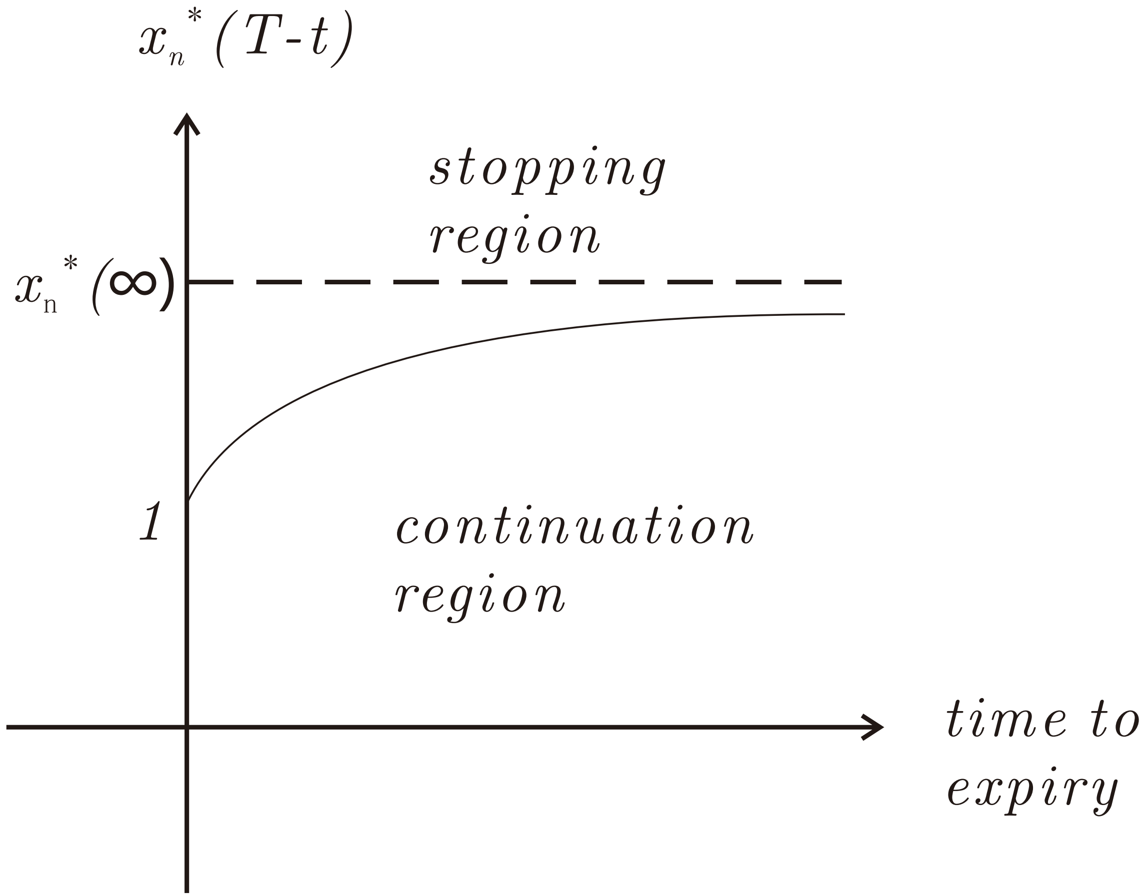
where $x_n^*(t)$ is the time-dependent threshold value at which the investor optimally exercises the reset right.

Free boundary value problem

The pricing model leads to a free boundary value problem with the free boundary $x_n^*(t)$ separating the continuation region $\{(x,t) : x < x_n^*(t), t < T\}$ and the stopping region $\{(x,t) : x \geq x_n^*(t), t < T\}$. The free boundary is not known in advance but has to be determined as part of the solution of the pricing model. This is a *multiple* optimal stopping problem.

At times close to expiry, the investor should choose to reset even when $F(t)$ is only slightly below $I(t)$, so we deduce that $x_n^*(T^-) = 1$.

- The free boundary $x_n^*(t)$ is a monotonically decreasing function of t since the holder should reset at a lower threshold fund value as time is approaching maturity. At fixed value of t, $x_n^*(t)$ is a monotonically decreasing function of n since the investor should become more conservative with less number of resets outstanding.



The exercise payoff is the value of the dynamic protected fund with outstanding reset rights reduced by one. When $n \to \infty$, $x_n^*(\infty)$ tends to 1 since the investor chooses to reset whenever $\widetilde{F}(t)$ falls to $I(t)$. As $x_n^*(\infty) > x_n^*(t) > 1$, so $x_\infty^*(t) = 1$.

Simplified pricing model under limit of infinite number of resets - automatic reset

With $x^*_\infty(t) = 1$ for all $t < T$. This gives $W'_\infty(x^*_\infty, t) = W_\infty(1, t)$. For convenience, we define

$$W_\infty(y,t) = \frac{V_\infty(\widetilde{F}, I, t)}{\widetilde{F}}, \text{ where } y = \ln x = \ln \frac{I}{\widetilde{F}},\ \tau = T - t.$$

Note that the free boundary $x^*_n(t)$ becomes the fixed boundary $y = \ln x^*_\infty(t) = \ln 1 = 0$.

The governing equation and auxiliary conditions for $W_\infty(y,\tau)$ are reduced to

$$\frac{\partial W_\infty}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 W_\infty}{\partial y^2} + \mu \frac{\partial W_\infty}{\partial y} - q_p W_\infty,\ \tau > 0,\ y < 0;$$
$$\frac{\partial W_\infty}{\partial y}(0,\tau) = W_\infty(0,\tau),\ W_\infty(y,0) = 1,$$

where $\mu = q_p - q_i - \dfrac{\sigma^2}{2}$ [Note that $y < 0 \Longleftrightarrow F(t) > I(t)$].

The Robin boundary condition: $\dfrac{\partial W_\infty}{\partial y}(0,\tau) = W_\infty(0,\tau)$ can be expressed as $\dfrac{\partial V_\infty}{\partial F}(I,\widetilde{F},t) = 0$ at $\widetilde{F} = I$. If the index value is taken to be the constant value K, then the Neumann boundary condition: $\dfrac{\partial V_\infty}{\partial F}(\widetilde{F},t)\bigg|_{\widetilde{F}=K} = 0$ is equivalent to the *reflecting* boundary condition with respect to the upgraded fund $\widetilde{F}$ placed at the guarantee level K.

The Robin boundary condition at $y = 0$ leads to a slight complication in the solution procedure. The analytic representation of the solution $W_\infty(y,\tau)$ admits different forms, depending on $q_p \neq q_i$ or $q_p = q_i$.

Let $\alpha = 2(q_i - q_p)/\sigma^2$ and $\widetilde{\mu} = \mu + \sigma^2$, the price function $V_\infty(F, I, t)$ is found to be

(i) $q_p \neq q_i$

$$\begin{aligned} & V_\infty(F, I, t) \\ = \;& I e^{-q_i \tau}\left(1 - \frac{1}{\alpha}\right) N\left(\frac{\ln(I/F) + \widetilde{\mu}\tau}{\sigma\sqrt{\tau}}\right) + \frac{I}{\alpha}\left(\frac{I}{F}\right)^{\alpha} e^{-q_p \tau} N\left(\frac{\ln(I/F) - \mu\tau}{\sigma\sqrt{\tau}}\right) \\ & + F e^{-q_p \tau} N\left(\frac{-\ln(I/F) - \mu\tau}{\sigma\sqrt{\tau}}\right), \; F > I. \end{aligned}$$

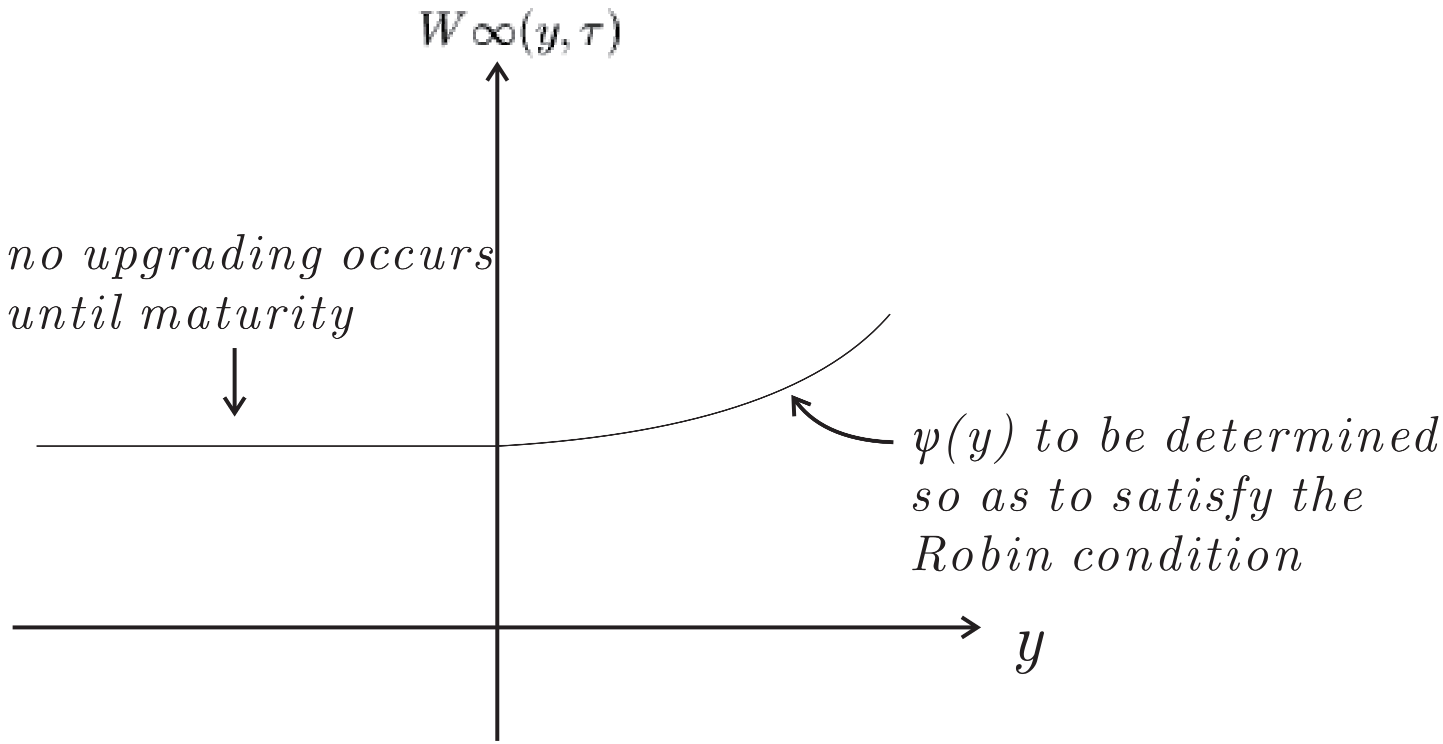
(ii) $q_p = q_i$ (write the common dividend yield as q)

$$\begin{aligned} & V_\infty(F, I, t) \\ = \;& I e^{-q\tau}\sigma\sqrt{\tau}\, n\left(\frac{\ln(I/F) + (\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right) + I e^{-q\tau}\left(\ln\frac{I}{F} + 1 + \frac{\sigma^2\tau}{2}\right) N\left(\frac{\ln(I/F) + (\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right) \\ & + F e^{-q\tau} N\left(\frac{-\ln(I/F) + (\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right), \; F > I. \end{aligned}$$

Derivation of $W_\infty(y, \tau)$

We perform the continuation of the initial condition to the whole domain $(-\infty, \infty)$, where

$$W_\infty(y, 0) = \begin{cases} 1 & \text{if } y < 0, \\ \Psi(y) & \text{if } y \geq 0. \end{cases}$$



The function $\Psi(y)$ is determined such that the Robin boundary condition is satisfied. Let $g(y, \tau; \xi)$ denote the fundamental solution to the governing differential equation, where

$$g(y, \tau; \xi) = \frac{e^{-q_p \tau}}{\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{(\xi - y - \mu\tau)^2}{2\sigma^2\tau}\right).$$

The following relations are useful in later derivation procedure.

$$\int_{-\infty}^{0} g(y, \tau; \xi)\ d\xi = e^{-q_p \tau}\left[1 - N\left(\frac{y + \mu\tau}{\sigma\sqrt{\tau}}\right)\right]$$

and

$$\frac{\partial}{\partial y}\int_{-\infty}^{0} g(y, \tau; \xi)\ d\xi = -e^{-q_p \tau} n\left(\frac{y + \mu\tau}{\sigma\sqrt{\tau}}\right) = -g(y, \tau; 0).$$

The solution to $W_\infty(y,\tau)$ can be formally expressed as

$$\begin{aligned} W_\infty(y,\tau) &= \int_{-\infty}^{\infty} W_\infty(\xi,0)g(y,\tau;\xi)\ d\xi \\ &= e^{-q_p\tau}\left[1 - N\left(\frac{y+\mu\tau}{\sigma\sqrt{\tau}}\right)\right] + \int_0^\infty \Psi(\xi)g(y,\tau;\xi)\ d\xi. \end{aligned}$$

Performing the differentiation with respect to y on both sides of the above equation and integration by parts, we have

$$\begin{aligned} \frac{\partial W_\infty}{\partial y}(y,\tau) &= -g(y,\tau;0) - \int_0^\infty \Psi(\xi)\frac{\partial g}{\partial \xi}(y,\tau;\xi)\ d\xi \\ &= \int_0^\infty \Psi'(\xi)g(y,\tau;\xi)\ d\xi. \end{aligned}$$

Note that

$$W_\infty(0,\tau) = \int_0^\infty \Psi(\xi) g(0,\tau;\xi)\ d\xi + \int_0^\infty g(0,\tau;-\xi)\ d\xi.$$

We apply the Robin boundary condition: $\dfrac{\partial W_\infty}{\partial y}(0,\tau) = W_\infty(0,\tau)$ to obtain

$$\int_0^\infty \{[\Psi'(\xi) - \Psi(\xi)] g(0,\tau;\xi) - g(0,\tau;-\xi)\}\ d\xi = 0$$

This relation works for any choice of g. In order that the Robin boundary condition is satisfied, $\Psi(\xi)$ and $g(0,\tau;\xi)$ have to observe the following relation:

$$\Psi'(\xi) - \Psi(\xi) = \frac{g(0,\tau;-\xi)}{g(0,\tau;\xi)} = e^{(\alpha+1)\xi},\ \text{where } \alpha = \frac{2(q_i - q_p)}{\sigma^2}.$$

Interestingly, $g(0,\tau;-\xi)/g(0,\tau;\xi)$ has no dependence on τ. If otherwise, this procedure does not work.

The auxiliary condition for $\Psi(\xi)$ is obtained by observing continuity of $W(y,0)$ at $y = 0$, giving $\Psi(0) = 1$. The solution of $\Psi(\xi)$ depends on $\alpha \neq 0$ or $\alpha = 0$, namely,

(i) when $\alpha \neq 0$,

$$\Psi(\xi) = e^{\xi}\left(\frac{e^{\alpha\xi}}{\alpha} + 1 - \frac{1}{\alpha}\right);$$

(ii) when $\alpha = 0$, $\Psi(\xi) = e^{\xi}\left[1 + \lim_{\alpha\to 0} \frac{e^{\alpha\xi} - 1}{\alpha}\right] = (1+\xi)e^{\xi}$.

By substituting the known solution of $\Psi(\xi)$, we obtain

(i) when $\alpha \neq 0$,

$$\begin{aligned} W_\infty(y,\tau) &= e^{y-q_i\tau}\left(1-\frac{1}{\alpha}\right)N\left(\frac{y+\widetilde{\mu}\tau}{\sigma\sqrt{\tau}}\right)+\frac{1}{\alpha}e^{(1+\alpha)y-q_p\tau}N\left(\frac{y-\mu\tau}{\sigma\sqrt{\tau}}\right) \\ &\quad + e^{-q_p\tau}N\left(\frac{-y-\mu\tau}{\sigma\sqrt{\tau}}\right),\ y<0; \end{aligned}$$

(ii) when $\alpha = 0$ (write q as the common dividend yield),

$$\begin{aligned} W_\infty(y,\tau) &= e^{-q\tau}N\left(\frac{-y+(\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right)+e^{y-q\tau}\sigma\sqrt{\tau}n\left(\frac{y+(\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right) \\ &\quad + e^{y-q\tau}\left(y+1+\frac{\sigma^2\tau}{2}\right)N\left(\frac{y+(\sigma^2\tau/2)}{\sigma\sqrt{\tau}}\right),\ y<0. \end{aligned}$$

Remark

When $q_p = 0$, the primary fund does not pay dividend. The dynamic protection has no value, so we expect

$$\lim_{F\to\infty} V_\infty(F,I,t) = F.$$

Mid-contract valuation

Let M denote the path dependent state variable which represents the realized maximum value of the state variable x from the grant-date to the mid-contract time t, that is,

$$M = \max_{0 \leq u \leq t} \frac{I(u)}{F(u)}.$$

At any mid-contract time, the number of units of primary fund held in the investment fund is given by

$$n(t) = \begin{cases} 1 & \text{if } M \leq 1, \\ M & \text{if } M > 1. \end{cases}$$

If the primary fund value has been staying above or at the reference stock index so far (corresponding to $M \leq 1$), then upgrade has never occurred, so the number of units of primary fund remains at one. Otherwise, the number of units is upgraded to M.

Governing differential equation

Let $V_{\text{mid}}(F, I, M, t)$ denote the mid-contract investment fund value at time t, with dependence on the state variable M. Like the governing equation for the lookback option price function, M only enters into the auxiliary conditions but not the governing differential equation. Both $V_{\text{mid}}(F, I, M, t)$ and $V_\infty(F, I, t)$ satisfy the same two-state Black-Scholes equation, namely,

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{F,I}\right) V_{\text{mid}}(F, I, M, t) = 0 \text{ and } \left(\frac{\partial}{\partial t} + \mathcal{L}_{F,I}\right) V_\infty(F, I, t) = 0. \tag{A.1}$$

where

$$\mathcal{L}_{F,I} = \frac{\sigma_p^2}{2} F^2 \frac{\partial^2}{\partial F^2} + \rho \sigma_p \sigma_i F I \frac{\partial^2}{\partial F \partial I} + \frac{\sigma_i^2}{2} I^2 \frac{\partial^2}{\partial I^2} + (r - q_p) F \frac{\partial}{\partial F} + (r - q_i) I \frac{\partial}{\partial I} - r.$$

Terminal payoff

The terminal payoff of the investment fund value at maturity T is given by $F \max(M, 1)$, a payoff structure that involves both F and M. The valuation of the mid-contract value may seem to be quite involved, but economic intuition gives the mid-contract value $V_{\text{mid}}(F, I, M, t)$ in terms of the grant-date value $V_\infty(F, I, t)$.

Relationship between $V_{mid}(F, I, M, t)$ and $V_\infty(F, I, t)$

When $M > 1$, the number of units of primary fund is increased to M so that the investment fund is equivalent to one unit of "new" primary fund having fund value MF. When $M \leq 1$, V_{mid} is insensitive to M since the terminal payoff value will not be dependent on the current realized maximum value M. That is, V_{mid} remains constant at different values of M, for all $M \leq 1$. By continuity of the price function with respect to the variable M, V_{mid} at $M = 1$ is equal to the limiting value of V_{mid} (corresponding to the regime: $M > 1$) as $M \to 1^+$.

Mid-contract value function

$$\begin{aligned} V_{\text{mid}}(F, I, M, t) &= V_\infty(\max(M,1)F, I, t) \\ &= \begin{cases} V_\infty(F, I, t) & M \le 1, \\ V_\infty(MF, I, t) & M > 1. \end{cases} \end{aligned} \tag{A.2}$$

- For $M \le 1$, $V_{\text{mid}}(F, I, M, t) = V_\infty(F, I, t)$ is obvious since there has been no upgrading of fund value occurs. It is only necessary to show

$$V_{\text{mid}}(F, I, M, t) = V_\infty(MF, I, t) \text{ for } M > 1.$$

Proof of the formula

When the current index value I equals MF, then V_{mid} is insensitive to M. We impose the usual auxiliary condition for a lookback option:

$$\left.\frac{\partial V_{\text{mid}}}{\partial M}\right|_{M=I/F} = 0.$$

One can check that $V_{\text{mid}}(F, I, M, t)$ given in Eq.(A.2) satisfies the governing differential equation (A.1) since the multiplier M is canceled in the differential equation when F is multiplied by M. It suffices to show that $V_{\text{mid}}(F, I, M, t)$ satisfies the terminal payoff condition and the above auxiliary condition.

Firstly, when $M > 1$, it is guaranteed that M at maturity must be greater than one so the terminal payoff becomes $F \max(1, M) = MF$. Since $V_\infty(F, I, T) = F$ so that $V_\infty(MF, I, T) = MF$, hence the terminal payoff condition is satisfied.

Recall: $W_\infty\left(\ln\frac{I}{F},\tau\right)=\frac{V_\infty(F,I,t)}{F}$ so that

$$V_\infty(MF,I,t)=MFW_\infty\left(\ln\frac{I}{MF},\tau\right).$$

We perform differentiation with respect to M to obtain

$$\frac{\partial V_\infty}{\partial M}(MF,I,t)=F\left[W_\infty(y,\tau)-\frac{\partial W_\infty}{\partial y}(y,\tau)\right],$$

where $y=\ln(I/MF)$. When $M=I/F$, we have $y=0$ so that

$$\begin{aligned}\frac{\partial V_{\text{mid}}}{\partial M}(F,I,M,t)\Big|_{M=I/F} &= \frac{\partial V_\infty}{\partial M}(MF,I,t)\Big|_{M=I/F}\\ &= F\left[W_\infty(0,\tau)-\frac{\partial W_\infty}{\partial y}(0,\tau)\right]=0\end{aligned}$$

by virtue of the Robin boundary condition.

Cost to the sponsor

$U_{\text{grant}}(F, I, t) =$ cost to the sponsor at the grant date
$U_{\text{mid}}(F, I, M, t) =$ cost to the sponsor at mid-contract time

Note that the terminal payoff $U_{\text{mid}}(F, I, M, T)$ is given by

$$U_{\text{mid}}(F, I, M, T) = V_{\text{mid}}(F, I, M, T) - F = \max(M - 1, 0)F.$$

Relation between U_{mid} and V_{mid}

$$U_{\text{mid}}(F, I, M, t) = V_{\text{mid}}(F, I, M, t) - Fe^{-q_p(T-t)}$$

since both $V_{\text{mid}}(F, I, M, t)$ and $Fe^{-q_p(T-t)}$ satisfy the Black-Scholes equation and the terminal payoff condition.

The factor $e^{-q_p(T-t)}$ appears in front of F since the holder of the protected fund does not receive the dividends paid by the primary fund.

At the grant-date, we have $F \geq I$ so that $M = I/F \leq 1$. We then have

$$U_{\text{grant}}(F, I, t) = U_{\text{mid}}(F, I, M, t) = V_\infty(F, I, t) - Fe^{-q_p(T-t)}.$$

The two cost functions $U_{\text{mid}}(F, I, M, t)$ and $U_{\text{grant}}(F, I, t)$ are related by

$$U_{\text{mid}}(F, I, M, t) = U_{\text{grant}}(\max(M, 1)F, I, t) + \max(M-1, 0)Fe^{-q_p(T-t)}.$$

The last term in the above equation gives the present value of additional units of primary fund supplied by the sponsor due to the protection clause. The sponsor has to add $M - 1$ units of primary fund when $M > 1$, but supplements nothing when $M \leq 1$.

Integral representation of the price function $V_{\textbf{mid}}(F, I, M, t)$

Let $\widehat{M}_t^{t'} = \max\left(\max_{t \le u \le t'} \frac{I(u)}{F(u)}, 1\right)$

and at the current time t, the following quantity

$$\widehat{M}_0^t = \max\left(\max_{0 \le u \le t} \frac{I(u)}{F(u)}, 1\right) = \max(M, 1)$$

is known. The terminal payoff of the protected fund can be expressed as $\max(\widehat{M}_0^t, \widehat{M}_t^T)F_T$, and F is the current value of the primary fund.

Rollover hedging strategy

We increase the number of units of fund to $\widehat{M}_t^u e^{-q_p(T-u)}$ whenever a higher realized maximum value of $\widehat{M}_t^u$ occurs at time u, where $t \leq u \leq T$. This rollover strategy would guarantee that the number of units of fund at maturity T is $\max(\widehat{M}_0^t, \widehat{M}_t^T)$. The corresponding present value of the replenishment premium is

$$e^{-q_p(T-t)} F E[\max(\widehat{M}_t^T - \widehat{M}_0^t, 0)] = e^{-q_p(T-t)} F \int_{\widehat{M}_0^t}^{\infty} P[\widehat{M}_t^T \geq \xi] \, d\xi.$$

The value of the protected fund is the sum of the value of the sub-replicating portfolio and the replenishment premium. An alternative analytic representation of the mid-contract price function is given by

$$\begin{aligned} V_{\text{mid}}(F, I, M, t) &= \max(M, 1) e^{-q_p(T-t)} F \\ &\quad + e^{-q_p(T-t)} F \int_{\max(M,1)}^{\infty} P[\widehat{M}_t^T \geq \xi] \, d\xi. \end{aligned}$$

Note that $\max(M, 1)$ is the number of units of primary fund to be held at time t, given the current value of M. In the remaining life of the contract, the holder is not entitled to receive the dividend yield, so the discount factor $e^{-q_p(T-t)}$ should be appended.

Distribution function

For $\xi \geq 1$, we have the following distribution function under the Geometric Brownian motion:

$$
\begin{aligned}
P[\widehat{M}_t^T \geq \xi] &= P\left[\max_{t \leq u \leq T} \frac{I(u)}{F(u)} \geq \xi\right] \\
&= e^{2\mu\xi/\sigma^2} N\left(-\frac{\xi + \mu\tau}{\sigma\sqrt{\tau}}\right) + N\left(-\frac{\xi - \mu\tau}{\sigma\sqrt{\tau}}\right),
\end{aligned}
$$

where $\mu = q_p - q_i - \dfrac{\sigma^2}{2}$ and $\sigma^2 = \sigma_p^2 - 2\rho\sigma_p\sigma_i + \sigma_i^2$.