1.3 Credit spread and bond price-based pricing

Market's assessment of the default risk of the obligor (assuming some form of market efficiency – information is aggregated in the market prices). The sources are

- market prices of bonds and other defaultable securities issued by the obligor
- prices of CDS's referencing this obligor's credit risk

How to construct a clean term structure of credit spreads from observed market prices?

⋆ Based on no-arbitrage pricing principle, a model that is based upon and calibrated to the prices of traded assets is immune to simple arbitrage strategies using these traded assets.

Market instruments used in bond price-based pricing

- At time t, the defaultable and default-free zero-coupon bond prices of all maturities $T \geq t$ are known. These defaultable zero-coupon bonds have no recovery at default.

- Information about the probability of default over all time horizons as assessed by market participants are fully reflected when market prices of default-free and defaultable bonds of all maturities are available.

Risk neutral probabilities

The financial market is modeled by a filtered probability space (Ω, $(\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, Q$), where Q is the risk neutral probability measure.

- All probabilities and expectations are taken under Q. Probabilities are considered as state prices.

  1. For constant interest rates, the discounted Q-probability of an event A at time T is the price of a security that pays off \$1 at time T if A occurs.
  2. Under stochastic interest rates, the price of the contingent claim associated with A is $E[\beta(T)\mathbf{1}_A]$, where $\beta(T)$ is the discount factor. This is based on the risk neutral valuation principle and the money market account $M(T) = \dfrac{1}{\beta(T)} = e^{\int_t^T r_u\,du}$ is used as the numeraire.

Indicator functions

For $A \in \mathcal{F}$, $\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$.

τ = random time of default; $I(t)$ = survival indicator function

$$I(t) = \mathbf{1}_{\{\tau > t\}} = \begin{cases} 1 & \text{if } \tau > t \\ 0 & \text{if } \tau \le t \end{cases}.$$

$B(t,T)$ = price at time t of zero-coupon bond paying off \$1 at T

$\overline{B}(t,T)$ = price of defaultable zero-coupon bond if $\tau > t$;

$$I(t)\overline{B}(t,T) = \begin{cases} \overline{B}(t,T) & \text{if } \tau > t \\ 0 & \text{if } \tau \le t \end{cases}.$$

Monotonicity properties on the bond prices

1. $0 \leq \overline{B}(t,T) < B(t,T), \quad \forall t < T$
2. Starting at $\overline{B}(t,t) = B(t,t) = 1$,

$$B(t,T_1) \geq B(t,T_2) > 0 \quad \text{and} \quad \overline{B}(t,T_1) \geq \overline{B}(t,T_2) \geq 0$$
$$\forall t < T_1 < T_2, \tau > t.$$

Independence assumption

$\{B(t,T)|t \leq T\}$ and τ are independent under $(\Omega, \mathcal{F}, Q)$ (not the true measure).

Implied probability of survival in $[t,T]$− based on market prices of bonds

$$B(t,T) = E\left[e^{-\int_t^T r_u\,du}\right] \quad \text{and} \quad \overline{B}(t,T) = E\left[e^{-\int_t^T r_u\,du} I(T)\right].$$

Invoking the independence between defaults and the default-free interest rates

$$\overline{B}(t,T) = E\left[e^{-\int_t^T r_u\,du}\right] E[I(T)] = B(t,T)P(t,T)$$

implied survival probability over $[t,T] = P(t,T) = \dfrac{\overline{B}(t,T)}{B(t,T)}$.

- The *implied default probability* over $[t,T]$, $P_{def}(t,T) = 1 - P(t,T)$.
- Assuming $P(t,T)$ has a right-sided derivative in T, the *implied density of the default time*

$$Q[\tau \in (T, T+dT] | \mathcal{F}_t] = -\frac{\partial}{\partial T} P(t,T)\,dT.$$

- If prices of zero-coupon bonds for all maturities are available, then we can obtain the implied survival probabilities for all maturities (complementary distribution function of the time of default).

Properties on implied survival probabilities, $P(t,T)$

1. $P(t,t) = 1$ and it is non-negative and decreasing in T. Also, $P(t,\infty) = 0$.
2. Normally $P(t,T)$ is continuous in its second argument, except that an important event *secheduled* at some time T_1 has direct influence on the survival of the obligor.
3. Viewed as a function of its first argument t, all survival probabilities for fixed maturity dates will tend to increase.

If we want to focus on the default risk over a given time interval in the future, we should consider conditional survival probabilities.

$$\begin{aligned} &\text{conditional survival probability over } [T_1, T_2] \text{ as seen from } t \\ = \; & P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)}, \quad \text{where } t \le T_1 < T_2. \end{aligned}$$

Implied hazard rate (default probabilities per unit time interval length)

Discrete implied hazard rate of default over $(T, T + \Delta T]$ as seen from time t

$$H(t, T, T + \Delta T)\Delta T = \frac{P(t, T)}{P(t, T + \Delta T)} - 1 = \frac{P_{def}(t, T, T + \Delta T)}{P(t, T, T + \Delta T)},$$

so that

$$P(t, T) = P(t, T + \Delta T)[1 + H(t, T, T + \Delta T)\Delta T].$$

In the limit of $\Delta T \to 0$, the continuous hazard rate at time T as seen at time t is given by

$$h(t, T) = -\frac{\partial}{\partial T} \ln P(t, T).$$

Proof First, we recall

$$\frac{1}{P(t,T,T+\Delta T)} = \frac{P(t,T)}{P(t,T,T+\Delta T)}.$$

We have

$$\begin{aligned} h(t,T) &= \lim_{\Delta T \to 0} H(t,T,T+\Delta T) \\ &= \lim_{\Delta T \to 0} \frac{1 - P(t,T,T+\Delta T)}{\Delta T P(t,T,T+\Delta T)} \\ &= \lim_{\Delta T \to 0} \frac{1}{\Delta T}\left[\frac{P(t,T)}{P(t,T+\Delta T)} - 1\right] \\ &= \lim_{\Delta T \to 0} -\frac{1}{P(t,T+\Delta T)}\frac{P(t,T+\Delta T) - P(t,T)}{\Delta T} \\ &= -\frac{1}{P(t,T)}\frac{\partial}{\partial T}P(t,T) \\ &= -\frac{\partial}{\partial T}\ln P(t,T). \end{aligned}$$

Forward spreads and implied hazard rate of default

For $t \leq T_1 < T_2$, the simply compounded forward rate over the period $(T_1, T_2]$ as seen from t is given by

$$F(t, T_1, T_2) = \frac{B(t, T_1)/B(t, T_2) - 1}{T_2 - T_1}.$$

This is the price of the forward contract with expiration date T_1 on a unit-par zero-coupon bond maturing on T_2. To prove, we consider the compounding of interest rates over successive time intervals.

$$\underbrace{\frac{1}{B(t, T_2)}}_{\text{compounding over } [t, T_2]} = \underbrace{\frac{1}{B(t, T_1)}}_{\text{compounding over } [t, T_1]} \underbrace{[1 + F(t, T_1, T_2)(T_2 - T_1)]}_{\text{simply compounding over } [T_1, T_2]}$$

Defaultable simply compounded forward rate over $[T_1, T_2]$

$$\overline{F}(t, T_1, T_2) = \frac{\overline{B}(t, T_1)/\overline{B}(t, T_2) - 1}{T_2 - T_1}.$$

Instantaneous continuously compounded forward rates

$$
\begin{aligned}
f(t,T) &= \lim_{\Delta T \to 0} F(t,T,T+\Delta T) = -\frac{\partial}{\partial T} \ln B(t,T) \\
\overline{f}(t,T) &= \lim_{\Delta T \to 0} \overline{F}(t,T,T+\Delta T) = -\frac{\partial}{\partial T} \ln \overline{B}(t,T).
\end{aligned}
$$

Implied hazard rate of default

Recall

$$
\begin{aligned}
P(t,T_1,T_2) &= \frac{\overline{B}(t,T_2)}{B(t,T_2)} \frac{B(t,T_1)}{\overline{B}(t,T_1)} \\
&= \frac{1 + F(t,T_1,T_2)(T_2-T_1)}{1 + \overline{F}(t,T_1,T_2)(T_2-T_1)} = 1 - P_{def}(t,T_1,T_2),
\end{aligned}
$$

and upon expanding, we obtain

$$
P_{def}(t,T_1,T_2) \underbrace{[1 + \overline{F}(t,T_1,T_2)(T_2-T_1)]}_{\overline{B}(t,T_1)/\overline{B}(t,T_2)} = [\overline{F}(t,T_1,T_2) - F(t,T_1,T_2)](T_2-T_1).
$$

Define $H(t, T_1, T_2) = \dfrac{P_{def}(t, T_1, T_2)}{(T_2 - T_1)P(t, T_1, T_2)}$ as the discrete implied rate of default. We then have

$$\begin{aligned} H(t, T_1, T_2) &= \frac{\overline{B}(t, T_2)}{\overline{B}(t, T_1)} \frac{[\overline{F}(t, T_1, T_2) - F(t, T_1, T_2)]}{P(t, T_1, T_2)} \\ &= \frac{B(t, T_2)}{B(t, T_1)} [\overline{F}(t, T_1, T_2) - F(t, T_1, T_2)]. \end{aligned}$$

Taking the limit $T_2 \to T_1$, then the implied hazard rate of default at time $T > t$ as seen from time t is the spread between the forward rates:

$$h(t, T) = \overline{f}(t, T) - f(t, T).$$

Alternatively, we obtain the above relation using

$$\begin{aligned} \overline{f}(t, T) - f(t, T) &= -\frac{\partial}{\partial T} \ln \frac{\overline{B}(t, T)}{B(t, T)} \\ &= -\frac{\partial}{\partial T} \ln P(t, T) = h(t, T). \end{aligned}$$

The *local default probability* at time t over the next small time step Δt

$$\frac{1}{\Delta t} Q[\tau \leq t + \Delta t | \mathcal{F}_t \wedge \{\tau > t\}] \approx \overline{r}(t) - r(t) = \lambda(t)$$

where $r(t) = f(t,t)$ is the riskfree short rate and $\overline{r}(t) = \overline{f}(t,t)$ is the defaultable short rate.

Recovery value

View an asset with positive recovery as an asset with an additional positive payoff at *default*. The recovery value is the *expected* value of the recovery shortly after the occurrence of a default.

Payment upon default

Define $e(t, T, T+\Delta T)$ to be the value at time $t < T$ of a deterministic payoff of \$1 paid at $T + \Delta T$ if and only if a default happens in $[T, T + \Delta T]$.

$$e(t, T, T + \Delta T) = E_Q \left[\beta(t, T + \Delta T)[I(T) - I(T + \Delta T)] | \mathcal{F}_t \right].$$

Note that

$$I(T) - I(T + \Delta T) = \begin{cases} 1 & \text{if default occurs in } [T, T + \Delta T] \\ 0 & \text{otherwise} \end{cases},$$

$$\begin{aligned} E_Q[\beta(t, T + \Delta T) I(T)] &= E_Q[\beta(t, T + \Delta T)] E_Q[I(T)] \\ &= B(t, T + \Delta T) P(t, T), \end{aligned}$$

$$E_Q[\beta(t, T + \Delta T) I(T + \Delta T)] = \overline{B}(t, T + \Delta T),$$

and

$$B(t, T + \Delta T) = \overline{B}(t, T + \Delta T)/P(t, T + \Delta T).$$

It is seen that

$$
\begin{aligned}
e(t,T,T+\Delta T) &= B(t,T+\Delta T)P(t,T) - \overline{B}(t,T+\Delta T) \\
&= \overline{B}(t,T+\Delta T)\left[\frac{P(t,T)}{P(t,T+\Delta T)} - 1\right] \\
&= \Delta T \overline{B}(t,T+\Delta T)H(t,T,T+\Delta T)
\end{aligned}
$$

On taking the limit $\Delta T \to 0$, we obtain

$$
\begin{aligned}
\text{rate of default compensation} &= e(t,T) = \lim_{\Delta T \to 0} \frac{e(t,T,T+\Delta T)}{\Delta T} \\
&= \overline{B}(t,T)h(t,T) = B(t,T)P(t,T)h(t,T).
\end{aligned}
$$

The value of a security that pays $\pi(s)$ if a default occurs at time s for all $t < s < T$ is given by

$$
\int_t^T \pi(s)e(t,s)\,ds = \int_t^T \pi(s)\overline{B}(t,s)h(t,s)\,ds.
$$

This result holds for deterministic recovery rates.

Random recovery value

- Suppose the payoff at default is not a deterministic function $\pi(\tau)$ but a random variable π' which is drawn at the time of default τ. π' is called a *marked point process*. Define

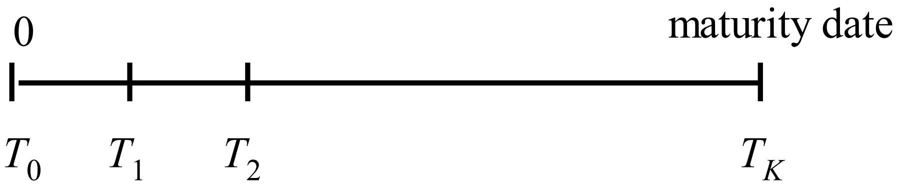
$$\pi_e(t,T) = E_Q[\pi' | \mathcal{F}_t \wedge \{\tau = T\}].$$

which is the expected value of π' conditional on default at T and information at t.
- Conditional on a default occurring at time T, the price of a security that pays π' at default is $B(t,T)\pi_e(t,T)$.
- Since the time of default is not known, we have to integrate these values over all possible default times and weight them with the respective probability of default occurring.
- The price at time t of a payoff of π' at τ if $\tau \in [t,T]$ is given by

$$\int_t^T \pi_e(t,s) \underbrace{B(t,s)P(t,s)}_{\overline{B}(t,s)} h(t,s)\, ds.$$

Building blocks for credit derivatives pricing

Tenor structure



$$\delta_k = T_{k+1} - T_k, 0 \leq k \leq K - 1$$

Coupon and repayment dates for bonds, fixing dates for rates, payment and settlement dates for credit derivatives all fall on $T_k, 0 \leq k \leq K$.

Fundamental quantities of the model

- Term structure of default-free interest rates $F(0,T)$
- Term structure of implied hazard rates $H(0,T)$
- Expected recovery rate π (rate of recovery as percentage of par)

From $B(0,T_i) = \dfrac{B(0,T_{i-1})}{1+\delta_{i-1}F(0,T_{i-1}T_i)}, i = 1,2,\cdots,k$, and $B(0,T_0) = B(0,0) = 1$, we obtain

$$B(0,T_k) = \prod_{i=1}^{k} \frac{1}{1+\delta_{i-1}F(0,T_{i-1},T_i)}.$$

Similarly, from $P(0,T_i) = \dfrac{P(0,T_{i-1})}{1+\delta_{i-1}H(0,T_{i-1},T_i)}$, we deduce that

$$\begin{aligned}
\overline{B}(0,T_k) &= B(0,T_k)P(0,T_k) = B(0,T_k)\prod_{i=1}^{k} \frac{1}{1+\delta_{i-1}H(0,T_{i-1},T_i)}. \\
e(0,T_k,T_{k+1}) &= \delta_k H(0,T_k,T_{k+1})\overline{B}(0,T_{k+1}) \\
&= \text{value of \$1 at } T_{k+1} \text{ if a default} \\
&\quad\ \text{has occurred in } (T_k,T_{k+1}].
\end{aligned}$$

Taking the limit $\delta_i \to 0$, for all $i = 0, 1, \cdots, k$

$$
\begin{aligned}
B(0, T_k) &= \exp\left(-\int_0^{T_k} f(0, s)\, ds\right) \\
\overline{B}(0, T_k) &= \exp\left(-\int_0^{T_k} [h(0, s) + f(0, s)]\, ds\right) \\
e(0, T_k) &= h(0, T_k)\overline{B}(0, T_k).
\end{aligned}
$$

Alternatively, the above relations can be obtained by integrating

$$
\begin{aligned}
f(0, T) &= -\frac{\partial}{\partial T} \ln B(0, T) \quad \text{with} \quad B(0, 0) = 1 \\
\overline{f}(0, T) &= h(0, T) + f(0, T) = -\frac{\partial}{\partial T} \ln \overline{B}(0, T) \quad \text{with} \quad \overline{B}(0, 0) = 1.
\end{aligned}
$$

Defaultable fixed coupon bond

$$
\begin{aligned}
\overline{c}(0) = \sum_{n=1}^{K} \overline{c}_n \overline{B}(0, T_n) & \qquad \text{(coupon)} \qquad \overline{c}_n = \overline{c}\delta_{n-1} \\
+ \overline{B}(0, T_K) & \qquad \text{(principal)} \\
+ \pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k) & \qquad \text{(recovery)}
\end{aligned}
$$

The recovery payment can be written as

$$
\pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k) = \sum_{k=1}^{K} \pi \delta_{k-1} H(0, T_{k-1}, T_k) \overline{B}(0, T_k).
$$

The recovery payments can be considered as an additional coupon payment stream of $\pi\delta_{k-1}H(0, T_{k-1}, T_k)$.

Defaultable floater

Recall that $L(T_{n-1}, T_n)$ is the reference LIBOR rate applied over $[T_{n-1}, T_n]$ at T_{n-1} so that $1 + L(T_{n-1}, T_n)\delta_{n-1}$ is the growth factor over $[T_{n-1}, T_n]$. Application of no-arbitrage argument gives

$$B(T_{n-1}, T_n) = \frac{1}{1 + L(T_{n-1}, T_n)\delta_{n-1}}.$$

- The coupon payment at T_n equals LIBOR plus a spread

$$\delta_{n-1}\left[L(T_{n-1}, T_n) + s^{par}\right] = \left[\frac{1}{B(T_{n-1}, T_n)} - 1\right] + s^{par}\delta_{n-1}.$$

- Consider the payment of $\dfrac{1}{B(T_{n-1}, T_n)}$ at T_n, its value at T_{n-1} is $\dfrac{\overline{B}(T_{n-1}, T_n)}{B(T_{n-1}, T_n)} = P(T_{n-1}, T_n)$. Why? We use the defaultable discount factor $\overline{B}(T_{n-1}, T_n)$ since the coupon payment may be defaultable over $[T_{n-1}, T_n]$.

- Seen at $t = 0$, the value becomes

$$\begin{aligned} & \overline{B}(0, T_{n-1}) P(0, T_{n-1}, T_n) \\ = \; & B(0, T_{n-1}) P(0, T_{n-1}) P(0, T_{n-1}, T_n) \\ = \; & B(0, T_{n-1}) P(0, T_n). \end{aligned}$$

Combining with the fixed part of the coupon payment and observing the relation

$$\begin{aligned} [B(0, T_{n-1}) - B(0, T_n)] P(0, T_n) &= \left[\frac{B(0, T_{n-1})}{B(0, T_n)} - 1 \right] \overline{B}(0, T_n) \\ &= \delta_{n-1} F(0, T_{n-1}, T_n) \overline{B}(0, T_n), \end{aligned}$$

the model price of the defaultable floating rate bond is

$$\begin{aligned} \overline{c}(0) &= \sum_{n=1}^{K} \delta_{n-1} F(0, T_{n-1}, T_n) \overline{B}(0, T_n) + s^{par} \sum_{n=1}^{K} \delta_{n-1} \overline{B}(0, T_n) \\ &+ \overline{B}(0, T_K) + \pi \sum_{k=1}^{K} e(0, T_{k-1}, T_k). \end{aligned}$$

Credit default swap

Fixed leg Payment of $\delta_{n-1}\overline{s}$ at T_n if no default until T_n.

The value of the fixed leg is

$$\overline{s}\sum_{n=1}^{N}\delta_{n-1}\overline{B}(0,T_n).$$

Floating leg Payment of $1-\pi$ at T_n if default in $(T_{n-1},T_n]$ occurs. The value of the floating leg is

$$\begin{aligned}&(1-\pi)\sum_{n=1}^{N}e(0,T_{n-1},T_n)\\ = \ &(1-\pi)\sum_{n=1}^{N}\delta_{n-1}H(0,T_{n-1},T_n)\overline{B}(0,T_n).\end{aligned}$$

The market CDS spread is chosen such that the fixed leg and floating leg of the CDS have the same value. Hence

$$\overline{s}=(1-\pi)\frac{\sum_{n=1}^{N}\delta_{n-1}H(0,T_{n-1},T_n)\overline{B}(0,T_n)}{\sum_{n=1}^{N}\delta_{n-1}\overline{B}(0,T_n)}.$$

Define the weights

$$w_n = \frac{\delta_{n-1}\overline{B}(0,T_n)}{\sum_{k=1}^{N}\delta_{k-1}\overline{B}(0,T_k)}, \quad n = 1, 2, \cdots, N, \quad \text{and} \quad \sum_{n=1}^{N} w_n = 1,$$

then the fair swap premium rate is given by

$$\overline{s} = (1-\pi)\sum_{n=1}^{N} w_n H(0, T_{n-1}, T_n).$$

1. $\overline{s}$ depends only on the defaultable and default free discount rates, which are given by the market bond prices. CDS is an example of a cash product.
2. It is similar to the calculation of fixed rate in the interest rate swap

$$s = \sum_{n=1}^{N} w'_n F(0, T_{n-1}, T_n)$$

where

$$w'_n = \frac{\delta_{n-1}B(0,T_n)}{\sum_{k=1}^{N}\delta_{k-1}B(0,T_k)}, \quad n = 1, 2, \cdots, N.$$

Marked-to-market value

original CDS spread $= \overline{s}'$; new CDS spread $= \overline{s}$

Let $\Pi = \text{CDS}_{old} - \text{CDS}_{new}$, and observe that $\text{CDS}_{new} = 0$, then

$$\text{marked-to-market value} = \text{CDS}_{old} = \Pi = (\overline{s} - \overline{s}') \sum_{n=1}^{N} \overline{B}(0, T_n)\delta_{n-1}.$$

Why? If an offsetting trade is entered at the current CDS rate $\overline{s}$, only the fee difference $(\overline{s} - \overline{s}')$ will be received over the life of the CDS. Should a default occurs, the protection payments will cancel out, and the fee difference payment will be cancelled, too. The fee difference stream is defaultable and must be discounted with $\overline{B}(0, T_n)$.

- CDS's are useful instruments to gain exposure against spread movements, not just against default arrival risk.

Valuing credit default swap I: No counterparty default risk

by John Hull and Alan White, *Journal of Derivatives* (Fall 2000) p.29-40.

- Estimation of the risk neutral probability that the reference bond will default at different times in the future. The market prices of bonds issued by the same obligor and Treasury bonds are used to provide the market information of the expected default loss of the reference entity.

– Choose a set of N bonds issued by the obligor with maturity dates $t_j, j = 1, 2, \cdots, N$, where $t_{j-1} < t_j$ and $t = 0$. The life of CDS is $[0, t_N]$.

$B_j =$ market price of Treasury bond with maturity date t_j

$\overline{B}_j =$ market price of defaultable bond with maturity date t_j

$B_j - \overline{B}_j$ gives the market estimation of the present value of expected default loss of the j^{th} defaultable bond over the period $[0, t_j], j = 1, 2, \cdots, N$.

Assume the risk neutral default probability density function $q(t)$ to be piecewise constant over $[0, t_N], q(t) = q_i, t \in (t_{i-1}, t_i], i = 1, 2, \cdots, N$. Define β_{ij} be the present value of the expected default loss of the j^{th} risky bond defaulting within $(t_{i-1}, t_i], i \leq j$. We deduce that

$$\sum_{i=1}^{j} q_i \beta_{ij} = B_j - \overline{B}_j, \quad j = 1, 2, \cdots, N. \qquad (i)$$

Try to determine q_i by estimating β_{ij}.

Assumptions

1. We estimate the expected recovery rate from historical data.
2. We assume that all risky bonds have the same seniority and the expected recovery rate is time independent. Let $\widehat{R}$ be this expected recovery rate, which is independent of j and t.
3. Let $C_j(t)$ denote the claim amount on the j^{th} bond defaulting at time t, then

$$C_j(t) = L[1 + A(t)],$$

   $L =$ face value, $A(t) =$ accrued interest at time t as percentage of its face value.

4. The protection buyer has to pay at default the accrued payment covering the period between the default time and the last payment date.
5. The default event, Treasury interest rates and recovery rates are mutually independent (under the risk neutral measure).
6. From the riskless Treasury interest rate, we can compute the discount factor $v(t)$, which is the present value of \$1 received at time t with certainty.

Let $F_j(t)$ be the forward price of the j^{th} default-free bond for a forward contract maturing at time t. Assuming deterministic interest rate, then the price at time t of the no-default value of the j^{th} bond is $F_j(t)$. We then have

$$\beta_{ij} = \int_{t_{i-1}}^{t_i} v(t) \left[F_j(t) - \widehat{R} C_j(t) \right] dt \quad i \le j, j = 1, 2, \cdots, N. \qquad (ii)$$

From Eq. (i), we can deduce

$$q_j = \frac{B_j - \overline{B}_j - \sum_{i=1}^{j-1} q_i \beta_{ij}}{\beta_{jj}}, \quad j = 1, 2, \cdots, N. \qquad (iii)$$

The risk neutral expected payoff paid by the protection seller upon default at time t is $L\{1 - \widehat{R}[1 + A(t)]\}$. Therefore, the present value of the expected payoff is

$$\int_0^T L\{1 - \widehat{R}[1 + A(t)]\} q(t) v(t)\, dt.$$

Greatest weakness of the model The computation of β_{ij} requires the information of the bond prices at time t. By imposing non-stochastic property of the interest rate, the bond prices at future time t equals the current traded bond forward price $F_j(t)$. Since CDS is a cash product, it should be priced purely based on defaultable and default free discount rates without any assumption on the stochastic nature of the interest rate.

How to compute the annuity premium rate w paid by the protection buyer?

$u(t) =$ present value of payments at the rate of \$1 per year on payment dates between time zero and t

$e(t) =$ present value of an accrual payment at time t of the time interval $t - t^*$, where t^* is the last payment date.

If there is no default prior to CDS maturity, the present value of payments is $wu(T)$.

$$\text{Expected value of payments} = w \int_0^T Lq(t)[u(t) + e(t)]\, dt + wLu(T) \left[1 - \int_0^T q(t)\, dt\right].$$

Lastly, w is determined such that the present value of payments equals the present value of expected default loss. Hence

$$w = \frac{\int_0^T \{1 - \widehat{R}[1 + A(t)]\} q(t) v(t)\, dt}{\int_0^T q(t)[u(t) + e(t)]\, dt + u(T) \left[1 - \int_0^T q(t)\, dt\right]}.$$

Hedge based pricing – *approximate hedge and replication strategies*

Provide hedge strategies that cover much of the risks involved in credit derivatives – independent of any specific pricing model.

Basic instruments

1. Default free bond

$$C(t) = \text{time-}t \text{ price of default-free bond with fixed-coupon } C$$
$$B(t,T) = \text{time-}t \text{ price of default-free zero-coupon bond}$$

2. Defaultable bond

$$\overline{C}(t) = \text{time-}t \text{ price of defaultable bond with fixed-coupon } \overline{c}$$
$$\overline{C}'(t) = \text{time-}t \text{ price of defaultable bond with floating coupon LIBOR } + s^{par}$$

3. Interest rate swap

$$\begin{aligned} S(t) &= \text{swap rate at time } t \text{ of a standard fixed-for-floating} \\ &= \frac{B(t,t_n) - B(t,t_N)}{A(t;t_n,t_N)}, \quad t \le t_n \end{aligned}$$

where $A(t;t_n,t_N) = \sum_{i=n+1}^{N} \delta_i B(t,t_i) =$ value of the payment stream paying δ_i on each date t_i.

Proof of the swap rate formula

The floating rate coupon payments can be generated by putting \$1 at t_n and taking away the floating interests immediately. At t_N, \$1 remains. The sum of the present value of the floating interests $= B(t,t_n) - B(t,t_N)$.

Intuition behind cash-and-carry arbitrage pricing of CDSs

A combined position of a CDS with a defaultable bond C is very well hedged against default risk.

Hedge strategy using fixed-coupon bonds

Portfolio 1

- One defaultable coupon bond $\overline{C}$; coupon $\overline{c}$, maturity t_N.
- One CDS on this bond, with CDS spread $\overline{s}$

The portfolio is unwound after a default.

Portfolio 2

- One default-free coupon bond C: with the same payment dates as the defaultable coupon bond and coupon size $\overline{c} - \overline{s}$.

The bond is sold after default.

Observations

1. In survival, the cash flows of both portfolio are identical.

	Portfolio 1	*Portfolio 2*
$t = 0$	$-\overline{C}(0)$	$-C(0)$
$t = t_i$	$\overline{c} - \overline{s}$	$\overline{c} - \overline{s}$
$t = t_N$	$1 + \overline{c} - \overline{s}$	$1 + \overline{c} - \overline{s}$

2. At default, portfolio 1's value = par = 1 (full compensation by CDS); that of portfolio 2 is $C(\tau)$, τ is the time of default.

   The price difference at default $= 1 - C(\tau)$. This difference is very small when the default-free bond is a par bond. "The issuer can choose $\overline{c}$ to make the bond be a par bond."

This is an approximate replication. Neglecting the price difference at default, the no-arbitrage principle dictates

$$\overline{C}(0) = C(0) = B(0, t_N) + \overline{c}A(0) - \overline{s}A(0).$$

The equilibrium CDS rate $\overline{s}$ can be solved. $\overline{C}(0)$ and $C(0)$ have almost the same value when the defaultable bond pays less coupon of amount equals the spread $\overline{s}$.

Cash-and carry arbitrage with par floater

A par floater $\overline{C}'$ is a defaultable bond with a floating-rate coupon of $\bar{c}_i = L_{i-1} + s^{par}$, where the par spread s^{par} is adjusted such that at issuance the par floater is valued at par.

Portfolio 1

- One defaultable par floater $\overline{C}'$ with spread s^{par} over LIBOR.
- One CDS on this bond: CDS spread is $\overline{s}$.

The portfolio is unwound after default.

Portfolio 2

- One default-free floating-coupon bond C': with the same payment dates as the defaultable par floater and coupon at LIBOR, $c_i = L_{i-1}$.

The bond is sold after default.

Time	*Portfolio 1*	*Portfolio 2*
$t = 0$	-1	-1
$t = t_i$	$L_{i-1} + s^{par} - \overline{s}$	L_{i-1}
$t = t_N$	$1 + L_{N-1} + s^{par} - \overline{s}$	$1 + L_{N-1}$
τ (default)	1	$C'(\tau) = 1 + L_i(\tau - t_i)$

The hedge error in the payoff at default is caused by accrued interest. If we neglect the *small* hedge error at default, then

$$s^{par} = \overline{s}.$$

Asset swap packages

An asset swap package consists of a defaultable coupon bond $\overline{C}$ with coupon $\overline{c}$ and an interest rate swap. The bond's coupon is swapped into LIBOR plus the asset swap rate s^A. Asset swap package is sold at par.

Remark Asset swap transactions are driven by the desire to strip out unwanted structured features from the underlying asset.

Payoff streams to the buyer of the asset swap package

time	defaultable bond	swap	net
$t = 0$	$-\overline{C}(0)$	$-1 + \overline{C}(0)$	-1
$t = t_i$	$\overline{c}^*$	$-\overline{c} + L_{i-1} + s^A$	$L_{i-1} + s^A + (\overline{c}^* - \overline{c})$
$t = t_N$	$(1 + \overline{c})^*$	$-\overline{c} + L_{N-1} + s^A$	$1^* + L_{N-1} + s^A + (\overline{c}^* - \overline{c})$
default	recovery	unaffected	recovery

* denotes payment contingent on survival.

$s(0)$ = fixed-for-floating swap rate (market quote)

$A(0)$ = value of an annuity paying at the \$1 (calculated based on observable default free bond prices)

The value of asset swap package is set at par at $t = 0$, so that

$$\overline{C}(0) + \underbrace{A(0)s(0) + A(0)s^A(0) - A(0)\overline{c}}_{\text{swap arrangement}} = 1.$$

The present value of the floating coupons is given by $A(0)s(0)$. The swap continues even after default so that $A(0)$ appears in all terms associated with the swap arrangement. Solving for $s^A(0)$

$$s^A(0) = \frac{1}{A(0)}[1 - \overline{C}(0)] + \overline{c} - s(0).$$

Rearranging the terms,

$$\overline{C}(0) + A(0)s^A(0) = \underbrace{[1 - A(0)s(0)] + A(0)\overline{c}}_{\text{default-free bond}} \equiv C(0)$$

where the right-hand side gives the value of a default-free bond with coupon $\overline{c}$. Note that $1 - A(0)s(0)$ is the present value of receiving \$1 at maturity t_N. We obtain

$$s^A(0) = \frac{1}{A(0)}[C(0) - \overline{C}(0)].$$

Credit spread options

The terminal payoff is given by

$$P_{sp}(r, s, T) = \max(s - K, 0)$$

where $r =$ riskless interest rate

$s =$ credit spread

$K =$ strike spread

Discrete-time Heath-Jarrow-Morton (HJM) method (Das and Sundaram, 2000)

- Follows the HJM term structure approach that models the forward rate process and forward spread process for riskless and risky bonds.
- The model takes the observed term structures of riskfree forward rates and credit spreads as input information.
- Find the risk neutral drifts of the stochastic processes such that all discounted security prices are martingales.

Example Price a one-year put spread option on a two-year risky zero-coupon bond struck at the strike spread $K = 0.01$.

Let the current observed term structure of riskless interest rates as obtained from the spot rate curve for Treasury bonds be

$$\boldsymbol{r} = \begin{pmatrix} 0.07 \\ 0.08 \end{pmatrix}.$$

The riskless forward rate between year one and year two is

$$f_{12} = \frac{1.08^2}{1.07} - 1 \approx 0.09.$$

The market one-year and two-year spot spreads are

$$\boldsymbol{s} = \begin{pmatrix} 0.010 \\ 0.012 \end{pmatrix}.$$

The two-year risky rate is $0.08 + 0.012 = 0.092$. The current price of a risky two-year zero coupon bond with face value \$100 is

$$\overline{B}(0) = \$100/(1.092)^2 = \$83.86.$$

- The discrete stochastic process for the spread under the true measure is assumed to take the form of a square-root process where the volatility depends on $\sqrt{s(0)}$

$$s(\Delta t) = s(0) + k[\theta - s(0)]\Delta t \pm \sigma\sqrt{s(0)\Delta t}$$

where $k = 0.3, \theta = 0.02$ and $\sigma = 0.04, \Delta t = 1, s(0) = 0.01$.

- We need to add an adjustment term γ in the drift term in order to risk-adjust the stochastic forward spread process

$$s(t) = s(0) + k[\theta - s(0)]\Delta t + \gamma \pm \sigma\sqrt{s(0)\Delta t}.$$

The adjustment term γ is determined by requiring the discounted bond prices to be martingales.

- Let $\overline{B}(1)$ denote the price at $t = 1$ of the risky bond maturing at $t = 2$. The forward defaultable discount factor over year one and year two is $\dfrac{1}{1 + f_{12} + s(1)}$, where $s(1)$ is the forward spread over the period.

$$s(1) = \begin{cases} \gamma + 0.017 \\ \gamma + 0.009 \end{cases} \quad \text{so that } \overline{B}(1) = \begin{cases} \frac{100}{1+f_{12}+\gamma+0.017} \\[2ex] \frac{100}{1+f_{12}+\gamma+0.009}, \end{cases}$$

with equal probabilities for assuming the high and low values.

We determine γ such that the bond price is a martingale.

$$\overline{B}(0) = 83.86 = \frac{1}{1 + 0.07 + 0.01} \times \frac{1}{2}\left(\frac{100}{1.107 + \gamma} + \frac{100}{1.099 + \gamma}\right).$$

The first term is the risky defaultable discount factor and the last term is the expected value of $\overline{B}(1)$. We obtain $\gamma = 0.0012$ so that $s(1) = \begin{cases} 0.0182 \\ 0.0102 \end{cases}$.

The current value of put spread option is

$$\frac{1}{1.07} \times \frac{1}{2}[(0.0182 - 0.01) + (0.0102 - 0.01)]L = 0.00393L,$$

where L is the notional value of the put spread option. Note that the default free discount factor $1/1.07$ is used in the option value calculation.

Two-factor Hull-White "no-arbitrage" model

Under the risk neutral measure Q, the stochastic processes followed by the short rate r and the short spread rate s are

$$dr = [\phi(t) - \alpha r]\ dt + \sigma_r\ dZ_r$$

$$ds = [\theta(t) - \beta s]\ dt + \sigma_s\ dZ_s,$$

where $\phi(t)$ and $\theta(t)$ are time dependent parameter functions, α and β are mean reversion parameters, $dZ_r\, dZ_s = \rho\, dt$. The sum $r(t)+s(t)$ is considered as the risky short rate.

- "No-arbitrage" refers to the determination of the time dependent drift terms in the mean reversion stochastic processes of r and s by fitting the current term structures of default free and defaultable bond prices. This is a *calibration* procedure.

$$\begin{aligned} B(t,T) &= E_Q\left[\exp\left(-\int_t^T r(u)\,du\right)\right] \\ \overline{B}(t,T) &= E_Q\left[\exp\left(-\int_t^T [r(u)+s(u)]\,du\right)\right]. \end{aligned}$$

$B(r,t;T)$ admits solution of the form $a(r,t;T)e^{-rb(r,t;T)}$. The process followed by $B(t,T)$ is given by

$$\frac{dB}{B} = r\,dt + \sigma_B(t,T)\,dZ_r$$

where

$$\sigma_B(t,T) = -\frac{\sigma_r}{\alpha}[1 - e^{-\alpha(T-t)}].$$

In terms of the forward measure Q^T where $B(t,T)$ is used as the numeraire, the defaultable bond price is

$$\overline{B}(t,T) = B(t,T)E_{Q^T}\left[\exp\left(-\int_t^T s(u)\,du\right)\right].$$

In general, $\dfrac{V(t)}{B(t,T)} = E_{Q^T}\left[\dfrac{V(T)}{B(T,T)}\right]$ so that $V(t) = B(t,T)E_{Q^T}[V(T)]$.

The process followed by the credit spread s under Q^T is given by

$$ds = [\theta(t) - \beta s + \rho\sigma_s\sigma_B(t,T)]\,dt + \sigma_s\,dZ^T$$

where Z^T is the standard Wiener process under Q^T.

The stochastic quantity $\int_t^T s(u)\,du$ has

$$\begin{aligned}
\text{mean } = E_{Q^T}\left[\int_t^T s(u)\,du\right] &= \frac{s(t)}{\beta}[1-e^{-\beta(T-t)}] \\
&\quad + \int_t^T [\theta(u)+\rho\sigma_s\sigma_B(u,T)]\frac{1-e^{-\beta(T-u)}}{\beta}\,du \\
\text{variance } = \text{var }\left(\int_t^T s(u)\,du\right) &= \int_t^T \frac{\sigma_s^2}{\beta^2}\left[1-e^{-\beta(T-u)}\right]^2\,du.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\overline{B}(t,T)}{B(t,T)} &= E_{Q^T}\left[\exp\left(-\int_t^T s(u)\,du\right)\right] \\
&= \exp\left(-\frac{s(t)\sigma_s}{\beta}[1-e^{-\beta(T-t)}]\right) \\
&\quad -\int_t^T [\theta(u)+\rho\sigma_s\sigma_B(u,T)]\frac{1-e^{-\beta(T-u)}}{\beta}\,du \\
&\quad +\int_t^T \frac{\sigma_s^2}{2\beta^2}[1-e^{-\beta(T-u)}]^2\,du.
\end{aligned}$$

By solving the integral equation

$$\theta(T) = \frac{\sigma_s^2}{2\beta}[1 - e^{-2\beta(T-t)}] - \beta\frac{\partial}{\partial T}\left[\ln\frac{\overline{B}(t,T)}{B(t,T)}\right] - \frac{\partial^2}{\partial T^2}\left[\ln\frac{\overline{B}(t,T)}{B(t,T)}\right]$$
$$+ \rho\sigma_s\sigma_r\left[\frac{1-e^{-\alpha(T-t)}}{\alpha} + e^{-\alpha(T-t)}\frac{1-e^{-\beta(T-t)}}{\beta}\right].$$

Trick to solve for $\theta(T)$

Differentiate the integral involving $\theta(u)$ with respect to T and subtract those terms involving $\int_t^T \theta(u)e^{-\beta(T-u)}\,du$ so as to obtain an explicit expression for $\int_t^T \theta(u)\,du$.

Reference

C.C. Chu and Y.K. Kwok, "No arbitrage approach for pricing credit spread derivatives," *Journal of Derivatives*, Spring issue (2003) p.51-64.

Pricing of the credit spread option

$$p_{sp}(r,s,t) = B(t,T)E_{Q^T}[\max(s-K,0)].$$

Note the presence of the mean reversion term $-\beta s$ in the drift term. Define

$$\xi(t) = e^{-\beta(T-t)}s(t)$$

so that

$$d\xi = [\theta(t) + \rho\sigma_s\sigma_B(t,T)][-\beta(T-t)]\,dt + \sigma_s[-\beta(T-t)]\,dZ^T.$$

Now, ξ_T is normally distributed with mean μ_ξ and σ_ξ^2, where

$$\begin{aligned}\mu_\xi &= s(t)e^{-\beta(T-t)} + \frac{\rho\sigma_s\sigma_r}{\alpha}\left[\frac{1-e^{-(\alpha+\beta)(T-t)}}{\alpha+\beta} - \frac{1-e^{-\beta(T-t)}}{\beta}\right] \\ &\quad + \int_t^T \theta(u)e^{-\beta(T-u)}\,du\end{aligned}$$

and

$$\sigma_\xi^2 = \frac{\sigma_s^2}{2\beta}[1-e^{2\beta(T-t)}].$$

Spread option value

$$p_{sp}(r,s,t) = B(r,t)\left[\frac{\sigma_\xi}{\sqrt{2\pi}}e^{-(K-\mu_\xi)^2/2\sigma_s^2} + (\mu_\xi - K)N\left(\frac{\mu_\xi - K}{\sigma_\xi}\right)\right].$$