## 3.1 Mathematical preliminaries for the construction of intensity processes

- Default occurs without warning at an exogenous default rate or intensity. The dynamics of the intensity are specified under the pricing measure.
- Instead of asking why the firm defaults, the intensity model is calibrated from market prices, typically bond prices.
- Cox process construction of a single jump time $\tau$

  A process $X$ of state variables in $\mathcal{R}^n$ is defined on a probability space $(\Omega, \mathcal{F}, Q)$. Let $\lambda : \mathcal{R}^n \to R$ be a non-negative measurable function. Construct a jump process $N_t$ with the property that

  $$N_t - \int_0^t \lambda(X_u) \mathbf{1}_{\{\tau > u\}} \, du$$

  is a martingale. The jump time $\tau$ is defined by

  $$\tau = \inf \left\{ t : \int_0^t \lambda(X_u) \, du \geq E_1 \right\}.$$

- Model the random time of arrival of default as a stopping time. In simple words, stopping times are random times that do not require knowledge about the future.

## Stopping times

Let $\tau$ denote the random time of default, $\tau \in \mathcal{R}_+ \cup \{\infty\}$. Here, $\infty$ is included in order to model events that may never occur. A *stopping time* with respect to $\mathcal{F}$ is a random variable such that

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \text{for all} \quad t \geq 0.$$

That is, at the time of event, it is known that this event has occurred or not.

*Example of a non-stopping random time*

Consider a Brownian motion $W(t)$ over a fixed time interval $[0, T]$, and let $\tau_{max}$ be the random time at which $W(t)$ attains its maximum. We need to observe the whole path of $W$ over $(0, T]$ in order to find the value taken on by $\tau_{max}$.

*Indicator process*

Define

$$N_\tau(t) = \mathbf{1}_{\{\tau \leq t\}}$$

which jumps from 0 to 1 at the stopping time.

## Hazard rate function

Let $\tau$ be a stopping time, $F(T) = P[\tau \leq T]$ be its distribution function. Assume $F(T) < 1$ for all $T, f(T) = F'(T)$ is the density function. The hazard rate function $h$ of $\tau$ is defined by

$$h(T) = \frac{f(T)}{1 - F(T)} = \frac{f(T)}{S(T)}, \text{ where } S(T) \text{ is the survival function.}$$

Now,

$$h(T) = \frac{f(T)}{S(T)} = -\frac{d}{dT} \ln S(T)$$

so that

$$S(T) = \exp\left(-\int_0^T h(u)\,du\right) \quad \text{and} \quad f(T) = h(T)\exp\left(-\int_0^T h(u)\,du\right).$$

Consider

$$\begin{aligned} P[\tau \le T + \Delta T | \tau > T] &= \frac{P[\tau \le T + \Delta T, \tau > T]}{P[\tau > T]} \\ &= \frac{[1 - S(\tau + \Delta T)] - [1 - S(T)]}{S(T)} \\ &= 1 - \exp\left( - \int_T^{T+\Delta T} h(u)\, du \right) \end{aligned}$$

so that

$$\lim_{\Delta T \to 0} \frac{1}{\Delta T} P[\tau \le T + \Delta T | \tau > T] = h(T).$$

The hazard rate $h(T)$ is the local arrival probability of the stopping time per unit time.

$h(T)\Delta T \approx$ conditional probability of a default in a small interval after $T$ given survival up to and including $T$.

It is a conditional default probability which is already known at time 0. The only reason that this conditional default probability changes with $t$ is the passing of time itself.

Suppose we have access to information at time $t$ which is not available at time 0, we would like to condition on a more general information set.

*Conditional hazard rate*

At later points in time $t > 0$ with $\tau > t$,

$$h(t,T) = \frac{f(t,T)}{1 - F(t,T)}$$

where $F(t,T) = P[\tau \leq T|\mathcal{F}_t]$ = conditional distribution of $\tau$ given the information at time $t$.

We obtain

$$F(t,T) = 1 - e^{-\int_t^T h(t,u)\,du}.$$

## Point processes

A point process is a collection of points in time

$$\{\tau_i, i \in N\} = \{\tau_1, \tau_2, \cdots\}.$$

These points in time have been indexed in an ascending order ($\tau_i < \tau_{i+1}$). They are all stopping times, distinctive from each other. Also, there is only a finite number of such points over any finite time horizon.

- Useful to analyze timing risk of several events, for example, rating transitions, multiple defaults, etc.
- Counting process

$$N(t) = \sum_i \mathbf{1}_{\{\tau_i \le t\}}$$

  A sample path of $N(t)$ would be a step function that starts at zero and increases by one at each $\tau_i$. Now, $N(t)$ is a stochastic process.

## Intensity function

- Over small time steps, the local implied default probability is proportional to the length of the time step. The proportionality factor is the short-term credit spread under zero recovery.
- The local probability of a jump of a Poisson process over a small time step is approximately proportional to the length of this time interval.

We would like to build models in which we can condition on a more general information set $\mathcal{F}_t$. In loose sense

$$P[\tau \leq t + \Delta t | \mathcal{F}_t] \approx \mathbf{1}_{\{\tau > t\}} \lambda(t) \Delta t,$$

where $\mathcal{F}_t$ contains information on the survival up to time $t$ and $\lambda$ is a stochastic process which is adapted to the filtration $\mathcal{F}$.

## Model for default arrival risk

- A counting process is a non-decreasing, integer-valued process $N(t)$ with $N(0) = 0$.

  Let $N(t)$ be a counting process with (possibly stochastic) intensity $\lambda(t)$. The time of default $\tau$ is the time of the first jump of $N$, that is,

  $$\tau = \inf\{t \in R_+ | N(t) > 0\}.$$

  The survival probability is given by

  $$P(0, T) = P[N(T) = 0 | \mathcal{F}_0].$$

*Poisson process*

A Poisson process with intensity $\lambda > 0$ is a non-decreasing, integer-valued process with initial value $N(0) = 0$ whose increments are independent and satisfy

$$P[N(T) - N(t) = n] = \frac{1}{n!}(T-t)^n \lambda^n e^{-\lambda(T-t)}, \quad \text{for all} \quad 0 \leq t \leq T.$$

*Intuitive construction of a Poisson process*

We look at the times of the jumps $\tau_1, \tau_2, \cdots$ and the probability of a jump in the next instant.

- The Poisson process has no memory

  The probability of $n$ jumps in $[t, t+s]$ is independent of $N(t)$ and the history of $N$ before $t$.
- Two or more jumps at exactly the same time have probability zero.

  $P[N(t+\Delta t)-N(t) = 1] = \lambda \Delta t$ (in the continuous limit, $E[dN] = \lambda\, dt$)

  or

$$P[N(t+\Delta t) - N(t) = 0] = 1 - \lambda \Delta t.$$

Jumps in disjoint time intervals happen independently of each other.

$$\begin{aligned} & P[N(t+2\Delta t) - N(t) = 0] \\ = \; & P[N(t+\Delta t) - N(t) = 0] P[N(t+2\Delta t) - N(t+\Delta t) = 0] \\ = \; & (1 - \lambda \Delta t)^2. \end{aligned}$$

Subdivide the interval $[t, T]$ into $n$ subintervals of length $\Delta t = \dfrac{T-t}{n}$.

1 $P[N(T) = N(t)] = (1 - \lambda \Delta t)^n = \left[1 - \frac{1}{n}\lambda(T-t)\right]^n$

$$\longrightarrow e^{-\lambda(T-t)} \text{ as } n \to \infty.$$

2 $$\begin{aligned} P[N(T) - N(t) = 1] &= n\lambda\Delta t(1-\lambda\Delta t)^{n-1} \\ &= n\lambda\frac{T-t}{n}\left[1 - \frac{\lambda(T-t)}{n}\right]^n \Big/ \left[1 - \frac{\lambda(T-t)}{n}\right] \\ &= \frac{\lambda(T-t)}{1 - \frac{\lambda(T-t)}{n}}\left[1 - \frac{\lambda(T-t)}{n}\right]^n \end{aligned}$$

$$\longrightarrow \lambda(T-t)e^{-\lambda(T-t)} \text{ as } n \to \infty.$$

In general,

$$P[N(T) - N(t) = n] = \frac{(T-t)^n}{n!}\lambda^n e^{-\lambda(T-t)}.$$

The inter-arrival times of a Poisson process $\tau_{n+1} - \tau_n$ are exponentially distributed with density

$$P[(\tau_{n+1} - \tau_n) \in dt] = \lambda e^{-\lambda t}\, dt.$$

*Large portfolio approximation*

We have a large portfolio of defaultable securities that are all driven by independent Poisson processes. Then we can assume that Poisson events happen almost continuously at a rate of $\lambda\, dt$ to the whole portfolio.

*Spreads with Poisson processes*

Survival probability: $P(0, T) = e^{-\lambda T}$.

Assuming independence of defaults and interest rate fluctuations,

$$\begin{aligned} H(t, T, T + \Delta t) &= \frac{1}{\Delta t}\left[\frac{P(t, T)}{P(t, T + \Delta t)} - 1\right] \\ &= \frac{1}{\Delta t}(e^{\lambda \Delta t} - 1) \end{aligned}$$

so that

$$h(t, T) = \lambda.$$

Note that neither default hazard rates $H$ nor $h$ depend on the current time $t$ or the future time $T$. In this case, the term structure of spreads will be flat.

## Inhomogeneous Poisson processes

Starting from the local jump probability

$$P[N(t+\Delta t) - N(t) = 1] = \lambda(t)\Delta t,$$

we have

$$P[N(T) - N(t) = 0] = \prod_{i=1}^{n} [1 - \lambda(t + i\Delta t)\Delta t]$$

so that

$$\begin{aligned}
\ln P[N(T) - N(t) = 0] &= \sum_{i=1}^{n} \ln[1 - \lambda(t + i\Delta t)\Delta t] \\
&\approx \sum_{i=1}^{n} -\lambda(t + i\Delta t)\Delta t \\
&\to -\int_t^T \lambda(s)\,ds \quad \text{as} \quad \Delta t \to 0.
\end{aligned}$$

Hence,

$$P[N(T) - N(t) = 0] \longrightarrow \quad e^{-\int_t^T \lambda(s)\,ds} \quad \text{as} \quad \Delta t \to 0.$$

In general,

$$P[N(T) - N(t) = n] = \frac{1}{n!}\left(\int_t^T \lambda(s)\,ds\right)^n e^{-\int_t^T \lambda(s)\,ds}.$$

For the implied hazard rate over $[T, T+\Delta t]$ as seen at time $t$

$$H(t,T,T+\Delta t) = \frac{1}{\Delta t}\left[e^{\int_T^{T+\Delta t}\lambda(s)\,ds} - 1\right] \text{ and } h(t,T) = \lambda(T)$$

so that default hazard rates do depend on $T$.

Assuming $r(t)$ to be independent of the arrival of default

$$\begin{aligned}\overline{B}(0,T) &= E\left[e^{-\int_0^T r(s)\,ds}\right] E\left[\mathbf{1}_{\{N(T)=0\}}\right] \\ &= B(0,T)e^{-\int_0^T \lambda(s)\,ds}.\end{aligned}$$

Using $\overline{B}(0,T)e^{YT} = B(0,T)$, the continuously compounded yield spread $Y$ of the defaultable bond over the equivalent default-free bond is

$$Y = \frac{1}{T}\int_0^T \lambda(s)\,ds,$$

which is not stochastic.

Under the same assumption of independence, we find the value of a contingent claim, denoted by $e(0, t_k, t_{k+1})$, which pays \$1 at $t_{k+1}$ if and only if a default occurs in $(t_k, t_{k+1}]$. We have

$$\begin{aligned} e(0, t_k, t_{k+1}) &= E\left[e^{-\int_0^{t_{k+1}} r(X_s)\,ds}\right]\left\{E\left[\mathbf{1}_{\{\tau > t_k\}}\right] - E\left[\mathbf{1}_{\{\tau > t_{k+1}\}}\right]\right\} \\ &= B(0, t_{k+1})e^{-\int_0^{t_{k+1}} \lambda(s)\,ds}\left[e^{\int_{t_k}^{k_{k+1}} \lambda(s)\,ds} - 1\right] \\ &= \overline{B}(0, t_{k+1})\left[e^{\int_{t_k}^{t_{k+1}} \lambda(s)\,ds} - 1\right]. \end{aligned}$$

In the continuous limit,

$$e(0, t) = \lim_{\Delta t \to 0} \frac{e(0, t, t + \Delta t)}{\Delta t} = \overline{B}(0, t)\lambda(t).$$

## Stochastic dynamics of intensity

- Stochastic dynamics in the credit spreads are necessary for pricing credit derivatives whose payoff is directly affected by volatility e.g. credit spread options, or payoff that is correlated with the spread movements.
- *Cox processes* are Poisson processes with stochastic intensity

$$d\lambda(t) = \mu_\lambda(t)\,dt + \sigma_\lambda(t)\,dZ(t).$$

*Background driving processes*

- All default-free processes and $\lambda(t)$ are adapted to $(G_t)_{t\geq 0}$, where $(G_t)_{t\geq 0}$ is the filtration generated by background driving process $X(t)$.
- The full filtration is obtained by combining $(G_t)_{t\geq 0}$ and the filtration $(\mathcal{H}_t)_{t\geq 0}$ generated by $N_t$.

Define $\mathcal{F}_t = G_t \bigvee \mathcal{H}_t$, where $\mathcal{H}_t = \sigma\{N_s : 0 \leq s \leq t\}$, $\mathcal{F}_t$ represents the smallest $\sigma$-field containing both $G_t$ and $\mathcal{H}_t$ and so it contains the information on both $X$ and the jump process.

*Cox process (doubly stochastic Poisson process) construction of a single jump time*

Let $G_t$ denote the filtration generated by a process $X$ of state variables with values in $\mathcal{R}^n$ defined on $(\Omega, \mathcal{F}, Q)$, where $Q$ is a pricing measure and

$$G_t = \sigma\{X_s; 0 \leq s \leq t\}.$$

Let $E_1$ be the exponential random variable with mean one, which is independent of $(G_t)_{t \geq 0}$.

We construct a jump process $N_t$ such that $\lambda(X_t)$ is the $\mathcal{F}_t$-intensity of $N$, here $\lambda$ is the intensity function. Define

$$\begin{aligned} M_t &= N_t - \int_0^t \lambda(X_u) \mathbf{1}_{\{\tau > u\}} \, du, \\ E[M_t | \mathcal{F}_s] &= M_s \quad s < t. \end{aligned}$$

Define the jump time by

$$\tau = \text{inf}\left\{t : \int_0^t \lambda(X_s)\,ds \geq E_1\right\}.$$

*Motivation*

First, recall $P[E_1 \leq x] = 1 - e^{-x}$. Now, consider

$$\begin{aligned} E[\mathbf{1}_{\{\tau > T\}}|G_T] &= Q[\tau > T|G_T] \\ &= Q\left[\int_0^T \lambda(X_s)\,ds < E_1|G_T\right], \end{aligned}$$

and as $G_T$ is known, so is $\int_0^T \lambda(X_s)\,ds$. Further, $E_1$ is independent of $G_T$ so that

$$Q\left[\int_0^T \lambda(X_s)\,ds < E_1|G_T\right] = \exp\left(-\int_0^T \lambda(X_s)\,ds\right).$$

*Pricing of risky bond at time 0, assuming zero recovery*

Assume that the default time $\tau$ of the issuing firm has an intensity $\lambda(X_t)$. Also, there is a short-rate process $r(X_s)$ such that the riskfree discount bond price

$$B(0,t) = E\left[\exp\left(-\int_0^t r(X_s)\,ds\right)\right].$$

Assuming zero recovery, consider the price of a risky bond

$$\begin{aligned}\overline{B}(0,t) &= E\left[\exp\left(-\int_0^t r(X_s)\,ds\right)\mathbf{1}_{\{\tau>t\}}\right]\\ &= E\left[E\left[\exp\left(-\int_0^t r(X_s)\,ds\right)\mathbf{1}_{\{\tau>t\}}|G_t\right]\right]\\ &= E\left[\exp\left(-\int_0^t r(X_s)\,ds\right)E\left[\mathbf{1}_{\{\tau>t\}}|G_t\right]\right]\end{aligned}$$

since $\exp\left(-\int_0^t r(X_s)\,ds\right) \in G_t$ where $G_t = \sigma\{X_s; 0 \le s \le t\}$. Furthermore,

$$\begin{aligned}\overline{B}(0,t) &= E\left[\exp\left(-\int_0^t r(X_s)\,ds\right)\exp\left(-\int_0^t \lambda(X_s)\,ds\right)\right]\\ &= E\left[\exp\left(-\int_0^t (r+\lambda)(X_s)\,ds\right)\right].\end{aligned}$$

The short rate has been replaced by the intensity-adjusted short rate $(r+\lambda)(X_s)$. It can be extended to cover a contingent claim with an actual payment of $f(X_t)\mathbf{1}_{\{\tau>t\}}$.

*Remarks*

1. If the intensity $\lambda(t)$ of the process is a deterministic function of time, then the future path of the intensity is given by the forward hazard rates, that is,

$$\lambda(t) = h(0,t).$$

2. If the short-rate process $r(X_s)$ is independent of the arrival of default, then

$$\begin{aligned}\overline{B}(0,t) &= E\left[\exp\left(-\int_0^t r(X_s)\,ds\right)\right] E\left[\mathbf{1}_{\{\tau>t\}}\right] \\ &= B(0,t)e^{-\int_0^t \lambda(X_s)\,ds}.\end{aligned}$$

## Dynamic survival probabilities

We quote the following result without proof:

$$\mathbf{1}_{\{\tau>t\}} E[Z|\mathcal{F}_t] = \mathbf{1}_{\{\tau>t\}} \frac{E[Z\mathbf{1}_{\{\tau>t\}}|G_t]}{E[\mathbf{1}_{\{\tau>t\}}|G_t]},$$

replacing the total history $\mathcal{F}$ with the history of the state variable process.

Note that

$$Q[\tau > T|\mathcal{F}_t] = \mathbf{1}_{\{\tau>t\}} E\left[\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t\right] = \mathbf{1}_{\{\tau>t\}} \frac{E\left[\mathbf{1}_{\{\tau>T\}}|G_t\right]}{E\left[\mathbf{1}_{\{\tau>t\}}|G_t\right]}.$$

Furthermore,

$$\begin{aligned} E\left[\mathbf{1}_{\{\tau>T\}}|G_t\right] &= E\left[E\left[\mathbf{1}_{\{\tau>T\}}|G_T\right]|G_t\right] \\ &= E\left[\text{exp}\left(-\int_0^T \lambda(X_s)\,ds\right)|G_t\right] \\ &= \text{exp}\left(-\int_0^t \lambda(X_s)\,ds\right) E\left[\text{exp}\left(-\int_t^T \lambda(X_s)\,ds\right)|G_t\right] \end{aligned}$$

hence

$$Q[\tau > T|\mathcal{F}_t] = \mathbf{1}_{\{\tau>t\}} E\left[\text{exp}\left(-\int_t^T \lambda(X_s)\,ds\right)|G_t\right].$$

If we let $N_t = \mathbf{1}_{\{\tau\le t\}}$, then $E[1 - N_t|\mathcal{F}_s] = Q[\tau > t|\mathcal{F}_s]$ and so

$$E[N_t - N_s|\mathcal{F}_s] = \mathbf{1}_{\{\tau>s\}} \left\{1 - E\left[\text{exp}\left(-\int_s^t \lambda(X_u)\,du|G_s\right)\right]\right\}.$$

## Martingale property

We would like to show the martingale property of

$$M_t = N_t - \int_0^t \lambda_u \mathbf{1}_{\{\tau>u\}}\, du,$$

that is, $E[M_t - M_s|\mathcal{F}_s] = 0$. We consider

$$E\left[\int_0^t \lambda_u \mathbf{1}_{\{\tau>u\}}\, du - \int_0^s \lambda_u \mathbf{1}_{\{\tau>u\}}\, du|\mathcal{F}_s\right] = E\left[\int_s^t \lambda_u \mathbf{1}_{\{\tau>u\}}\, du|\mathcal{F}_s\right].$$

Noting that

$$\mathbf{1}_{\{\tau>s\}} \int_s^t \lambda_u \mathbf{1}_{\{\tau>u\}}\, du = \int_s^t \lambda_u \mathbf{1}_{\{\tau>u\}}\, du$$

and

$$E\left[\int_s^t \lambda_u \mathbf{1}_{\{\tau>u\}}\, du|\mathcal{F}_s\right] = \mathbf{1}_{\{\tau>s\}} \frac{E\left[\int_s^t \lambda_u \mathbf{1}_{\{\tau>u\}}\, du|G_s\right]}{E\left[\mathbf{1}_{\{\tau>s\}}|G_s\right]}. \qquad (1)$$

Furthermore,

$$\begin{aligned} E\left[\int_s^t \lambda_u \mathbf{1}_{\{\tau>u\}}\, du|G_s\right] &= \int_s^t E\left[\lambda_u \mathbf{1}_{\{\tau>u\}}|G_s\right]\, du \\ &= \int_s^t E\left[E\left[\lambda_u \mathbf{1}_{\{\tau>u\}}|G_u\right]\right]|G_s\, du \end{aligned}$$

$$
\begin{aligned}
&= \int_s^t E\left[\lambda_u \exp\left(-\int_0^u \lambda_v\, dv\right) |G_s\right]\, du \\
&= E\left[\int_s^t \lambda_u \exp\left(-\int_0^u \lambda_v\, dv\right)\, du|G_s\right] \\
&= E\left[\int_s^t -\frac{\partial}{\partial u} \exp\left(-\int_0^u \lambda_v\, dv\right)\, du|G_s\right] \\
&= E\left[\exp\left(-\int_0^s \lambda_v\, dv\right) - \exp\left(-\int_0^t \lambda_u\, du\right) |G_s\right].
\end{aligned}
$$

Lastly, we obtain the RHS of Eq. (1) as

$$
\begin{aligned}
&\mathbf{1}_{\{\tau>s\}} \frac{E\left[\exp\left(-\int_0^s \lambda_v\, dv\right) - \exp\left(-\int_0^t \lambda_v\, dv\right) |G_s\right]}{\exp\left(-\int_0^s \lambda_v\, dv\right)} \\
= \; &\mathbf{1}_{\{\tau>s\}} \left\{1 - E\left[\exp\left(-\int_s^t \lambda_v\, dv\right) |G_s\right]\right\} = E[N_t - N_s|\mathcal{F}_s]
\end{aligned}
$$

and hence the martingale result.

## Interacting intensities

Consider two firms $A$ and $B$, and define $\tau^i$ as the default time of issuer $i$ and let $N_t^i = \mathbf{1}_{\{\tau^i \leq t\}}$. Assume that the pre-default intensities of $A$ and $B$ are

$$\begin{aligned} \lambda_t^A &= a_1 + a_2 \mathbf{1}_{\{\tau^B \leq t\}} \\ \lambda_t^B &= b_1 + b_2 \mathbf{1}_{\{\tau^A \leq t\}}, \end{aligned}$$

where $a_1, a_2, b_1$ and $b_2$ are all different. If $a_2 = b_2 = 0$, we are back to the case where the default times are independent exponential distributions.

*Markov chain approach*

Four-state Markov chain in continuous time whose state space is

$$\{(N,N),(D,N),(N,D),(D,D)\}.$$

The generator of the Markov process is

$$\Lambda = \begin{pmatrix} -(a_1+b_1) & a_1 & b_1 & 0 \\ 0 & -(b_1+b_2) & 0 & b_1+b_2 \\ 0 & 0 & -(a_1+a_2) & a_1+a_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $(D,D)$ is an absorbing state so that the entries in the last row is zero.

Since $\Lambda$ is an upper triangular matrix, so its eigenvalues are just its diagonal elements. Performing the spectral decomposition of $\Lambda$, we obtain

$$\Lambda = BDB^{-1}$$

where

$$B = \begin{pmatrix} 1 & \frac{a_1}{a_1-b_2} & \frac{b_1}{b_1-a_2} & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} -(a_1+b_1) & 0 & 0 & 0 \\ 0 & -(b_1+b_2) & 0 & 0 \\ 0 & 0 & -(a_1+b_2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$B^{-1} = \begin{pmatrix} 1 & -\frac{a_1}{b_2-a_2} & \frac{b_1}{a_2-b_1} & 1+\frac{b_1}{b_1-a_2}+\frac{a_1}{a_1-b_2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The transition probability matrix

$$P(t) = Be^{Dt}B^{-1}$$

so that

$$P(t) = B \begin{pmatrix} e^{-(a_1+b_1)t} & 0 & 0 & 0 \\ 0 & e^{-(b_1+b_2)t} & 0 & 0 \\ 0 & 0 & e^{-(a_1+a_2)t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} B^{-1}.$$

$$\begin{aligned} Q[\tau^A \leq t] &= P_{12}(t) + P_{14}(t) \\ &= \frac{a_2 - a_2 e^{-(a_1+b_1)t} + b_1 \left[ e^{-(a_1+a_2)t} - 1 \right]}{a_2 - b_1}. \end{aligned}$$

Note that $Q[\tau^A \leq t]$ does not depend on $b_2$; $b_2$ only takes effect after the default of $A$ and from that time it controls the waiting time for $B$ to follow $A$ in default. Changing $b_2$ only serves to move probability mass between $P_{12}(t)$ and $P_{14}(t)$ but will not alter the sum.