MATH690 – Credit Risk Models

Homework Two

Spring, 2005 *Course Instructor: Prof. Y.K. Kwok*

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1. In the Merton model of risky debt, suppose we define

$$\sigma_V(\tau; d) = \frac{\sigma A}{V}\frac{\partial V}{\partial A},$$

which gives the volatility of the value of the risky debt. Also, we denote the credit spread by $s(\tau; d)$, where $s(\tau; d) = Y(\tau) - r$ and $d = Be^{-r\tau}/A$ is the quasi debt-firm value ratio. Show that (Merton, 1974)

(a) $\dfrac{\partial s}{\partial d} = \dfrac{1}{\tau d}\dfrac{\sigma_V(\tau; d)}{\sigma} > 0$;

(b) $\dfrac{\partial s}{\partial \sigma^2} = \dfrac{1}{2\sqrt{\tau}}\dfrac{N'(d_1)}{N(d_1)}\dfrac{\sigma_V(\tau; d)}{\sigma} > 0$, where $d_1 = \dfrac{\ln d}{\sigma\sqrt{\tau}} - \dfrac{\sigma\sqrt{\tau}}{2}$;

(c) $\dfrac{ds}{dr} = -\dfrac{\sigma_V(\tau; d)}{\sigma} < 0$.

Give financial interpretation to each of the above results.

2. Consider the defaultable bond model (Black and Cox, 1976), where the firm issues senior debt, junior debt and the remaining liabilities are equity. Both debts are assumed to have the same maturity date. The payoffs to the claimants at bond maturity are summarized in the following table:

claim	$V < P$	$P \le V \le P + Q$	$V > P + Q$
senior bond	V	P	P
junior bond	0	$V - P$	Q
stock	0	0	$V - P - Q$

Hence, P and Q are the par value of the senior and junior debts, respectively. Let the reorganization boundary be defined by $\alpha Pe^{-r(T-t)}$, and let $B(V, t; P, \alpha Pe^{-r(T-t)})$ denote the value of the defaultable bond with par value P with only one debt outstanding. It is assumed that the same seniority rule of payment is applied during inter-temporal reorganization. Show that the value of the junior debt is given by

$$\begin{array}{ll} B(V, t; P + Q, \alpha Pe^{-r(T-t)}) - B(V, t; P, \alpha Pe^{-r(T-t)}), & \alpha < 1; \\ B(V, t; P + Q, \alpha Pe^{-r(T-t)}) - Pe^{-r(T-t)}, & 1 \le \alpha \le \dfrac{P+Q}{P}; \\ Qe^{-r(T-t)} & \alpha > \dfrac{P+Q}{P}. \end{array}$$

Hint: For part (i), observe that the payoff to the junior debt at maturity $= \min(V, P+Q) - \min(V, P)$. For part (iii), the junior debt is riskless.

3. A firm is an entity consisting of its assets and let A denote the market value of total assets. Assume that the total asset value follows a stochastic process modeled by

$$\frac{dA}{A} = \mu\, dt + \sigma\, dZ,$$

where μ and σ^2 (assumed to be constant) are the instantaneous mean and variance, respectively, of the rate of return on A. Let C and D denote the market value of current liabilities and market value of debt, respectively. Let T be the maturity date of the debt with face value D_T. Suppose the current liabilities of amount C_T are also payable at time T, and it constitutes a claim senior to the debt. Also, let F denote the present value of total amount of interest and dividends paid over the term T. For simplicity, F is assumed to be prepaid at time $t = 0$.

The debt is in default if A_T is less than the total amount payable at maturity date T, that is,

$$A_T < D_T + C_T.$$

(a) Show that the probability of default is given by

$$p = N\left(\frac{\ln \frac{D_T + C_T}{A - F} - \mu T + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right).$$

(b) Explain why the expected loan loss L on the debt is given by

$$EL = \int_{C_T}^{D_T + C_T} (D_T + C_T - a) f(a)\ da + \int_0^{C_T} D_T f(a)\ da$$

where f is the density function of A_T. Give financial interpretation to each integral.

4. A *vulnerable option* is an option with exposure of default risk of the option writer. Let V denote the wealth of the option writer at time t. The terminal payoff of a vulnerable call option is

$$\min(V_T, \max(S_T - X, 0)),$$

where S_T is the asset price at expiration time T and X is the strike price. Define

$$A_T = \min(X + V_T, S_T),$$

show that

$$\max(A_T - X, 0) = \min(V_T, \max(S_T - X, 0)).$$

With the usual assumption of lognormal diffusion processes for S and V, and let ρ be the correlation coefficient between the two diffusion processes, show that the value of the European vulnerable call option is given by (Johnson and Stulz, 1987)

$$\begin{aligned} c(S, V, \tau; X) = & \frac{e^{-r\tau}}{2\pi\tau\sqrt{1-\rho^2}\sigma_V\sigma_S}\left[\int_X^\infty \frac{dS_T}{S_T}\int_0^{S_T - X} e^{-D} dV_T\right. \\ & \left. + \int_0^\infty \frac{dV_T}{V_T}\int_X^{V_T + X} e^{-D}(S_T - X)\frac{dS_T}{S_T}\right], \quad \tau = T - t, \end{aligned}$$

where

$$D = \frac{A^2 - 2\rho AB + B^2}{2(1-\rho^2)}, A = \frac{\ln\frac{S}{S_T} + \left(r - \frac{\sigma_S^2}{2}\right)\tau}{\sigma_S\sqrt{\tau}},$$

$$B = \frac{\ln\frac{V}{V_T} + \left(r - \frac{\sigma_V^2}{2}\right)\tau}{\sigma_V\sqrt{\tau}}.$$

Here, σ_V and σ_S are the volatility values of the diffusion processes for V and S, respectively, and r is the riskless interest rate.

5. Recall that in the Black-Cox model, default is triggered when firm value hits the book value of debt, $K(t)$. Modeling the latter as a multiple of past average firm values, that is, $K(t) = e^{-\nu+Q_t}$, where

$$Q_t = \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} y_s \, ds, \quad y_t = \ln V_t,$$

show that the log-default threshold k_t satisfies

$$dk_t = \lambda(y_t - \nu - k_t) \, dt.$$

6. (optional)
Assume that the firm has a *stationary debt structure*, in the following sense: at any time t, the debt outstanding is composed of coupon bonds of maturities from the interval $[t, t+T]$ with the constant coupon rate $c = C/T$ and the face value uniformly distributed over this interval. This implies that the total coupon paid by all outstanding bonds is C per year. To preserve the debt structure as time elapses, new bonds are issued at a rate $\ell = L/T$ per year, where L is the total face value of all outstanding bonds. The same amount of principal is retired when the previously issued bonds mature. Therefore, as long as the firm remains solvent, the total face value remains constant at any date t, and the outstanding bonds have a uniform distribution of face value in the interval $[t, t+T]$.
   a. Show that at any time t before default the value of the firm's debt equals

$$D(V_t) = \int_{t}^{t+T} D_c(t, u) \, du,$$

   where the price $D_c(t, u)$ is given by the formula for the defaultable coupon bond, with L substituted with $L/T, c$ substituted with C/T, and β_2 substituted with β_2/T.
   b. Let us set $t = 0$. Show that the price $D_c(0, u)$ equals

$$D_c(0, u) = \frac{C}{rT} + \frac{e^{-ru}}{T} \left(L - \frac{C}{r} \right) g(u) + \frac{1}{T} \left(\beta_2 \overline{v} - \frac{C}{r} \right) h(u),$$

   where the functions $g, h : [0, T] \to \mathbb{R}$ are given by (as usual, $R_0 = \overline{v}/V_0$)

$$g(u) = N(k_1(V_0, u)) - R_0^{2\widetilde{a}} N(k_2(V_0, u))$$

   and

$$h(u) = R_0^{\widetilde{a}+\widetilde{\zeta}} N(g_1(V_0, u)) + R_0^{\widetilde{a}-\widetilde{\zeta}} N(g_2(V_0, u)).$$

   c. Show that the value of the firm's debt at time 0 equals

$$D(V_0) = \frac{C}{r} + \frac{1}{rT} \left(L - \frac{C}{r} \right) G(T) + \frac{1}{T} \left(\beta_2 \overline{v} - \frac{C}{r} \right) H(T),$$

   where

$$G(T) = \int_{0}^{T} e^{-ru} g(u) \, du = r^{-1}(1 - h(T) - e^{-rT} g(T)),$$

   and $H(T) := \displaystyle\int_{0}^{T} h(u) \, du = \widetilde{H}(T)$, where for every $T \geq 0$

$$\widetilde{H}(T) := \frac{\sqrt{T}}{\zeta\sigma} \left(R_0^{\widetilde{a}+\widetilde{\zeta}} g_1(V_0, T) N(g_1(V_0, T)) - R_0^{\widetilde{a}-\widetilde{\zeta}} g_2(V_0, T) N(g_2(V_0, T)) \right).$$

7. (optional)

The CreditGrades model is specified as follows:

- The firm's value follows the usual lognormal random walk.
- At $t = 0$, a *random default threshold* L is drawn. L is lognormally distributed with

$$LD = \overline{L}De^{\lambda Z - \frac{1}{2}\lambda^2},$$

  where $\overline{L}$ is the mean of L, λ the standard deviation of log L and Z is a $\Phi(0, 1)$ standard normal random variable. D is the debt per share of the firm (all quantities are set up in per share units).
- Defaults occur as soon as $V(\tau) \leq LD$.

The stochastic barrier L is interpreted as a *stochastic recovery*, and it is assumed that the recovery of a defaultable bond is indeed L. Accordingly, $\overline{L}$ is interpreted as *average recovery* and λ as *recovery rate volatility*. Prove the following approximation for the survival probability:

$$P(t) = \Phi\left(-\frac{A_t}{2} + \frac{\log(d)}{A_t}\right) - d \cdot \Phi\left(-\frac{A_t}{2} - \frac{\log(d)}{A_t}\right),$$

where

$$A_t^2 = \sigma^2 t + \lambda^2, \quad d = \frac{V_0 e^{\lambda^2}}{\overline{L}D}.$$