MATH685Z — Mathematical Models in Financial Economics

Topic 5 – Investment, consumption and endowment

Key question How to choose the best strategy for transforming wealth invested at time t = 0 into wealth at t = 1, with possible endowment and a portion of wealth being consumed at time t = 0?

Measure of performance – expected utility criterion

Let $u(W, \omega)$ represent the *utility* of amount W, P be a probability measure on Ω , with $P(\omega) > 0$ for all $\omega \in \Omega$.

 $u : \mathbb{R} \times \Omega \to \mathbb{R}$ is a function such that $W \to u(W, \omega)$ is differentiable, concave and strictly increasing for each $\omega \in \Omega, \omega$ is the state and

$$Eu(V_1) = \sum_{\omega \in \Omega} P(\omega)u(V_1(\omega), \omega).$$

In most applications, it suffices for u to depend on W only.

Assumptions

- Uncertainty listing of all basic events or states that could occur and their probabilities; sample space is $\Omega = \{\omega_1, \dots, \omega_K\}$; each state $\omega \in \Omega$ occurs with a positive probability $P(\omega)$.
- Securities contracts for a future delivery of cash, contingent on the prevailing state
- Endowments cash that the traders receive from sources other than trading.
- There is a finite number N of endogenous securities (all securities are created by traders, no entities outside the model).

Payouts of the securities

$$D = \begin{pmatrix} d_1(\omega_1) & \cdots & d_N(\omega_1) \\ \cdots & \cdots \\ d_1(\omega_K) & \cdots & d_N(\omega_K) \end{pmatrix}$$

 d_n is a random variable defined on the sample space $\Omega, 1 \leq n \leq N$.

- There are a finite number I of traders. At time 0, traders know only the set of possible states Ω , and at time T they know the prevailing state $\omega \in \Omega$.
- All traders are *price takers*. They determine their demands and supplies of securities without paying attention to the impact that their actions on the ultimate market prices of securities.

Endowment process

At time 0, trader *i* receives an endowment $e^i(0)$. At time *T*, he receives the endowment $e^i(T, \omega)$ contingent on the prevailing state ω .

 $e^{i} = \{e^{i}(0), e^{i}(T)\}$ is the endowment process of trader *i*.

Consumption process

The uncertain terminal endowments and payouts of securities introduce uncertainty into the consumption at time T.

* At time 0, given security prices P_1, \dots, P_N , each trader *i* faces constraints on consumption imposed by her endowment process $e^i = \{e^i(0), e^i(T)\}.$ Budget set

For an endowment process e^i and security prices

$$P = (P_1, \cdots, P_N)$$

the budget set $B(e^i, P)$ of trader *i* is the subset of the consumption set X such that $c \in B(e^i, P)$ iff there are numbers $\theta_1, \dots, \theta_N$ such that

$$c(0) = e^{i}(0) - \sum_{n=1}^{N} \theta_{n} P_{n}$$

$$c(T) = e^{i}(T) + \sum_{n=1}^{N} \theta_{n} d_{n}.$$

 $\theta = (\theta_1 \cdots \theta_N)$ is called the trading strategy.

The consumption process $\{c(0), c(T)\}$ is said to be generated by the endowment process e^i and the trading strategy θ .

Example K = 2, N = 4

$$D = \begin{pmatrix} 100 & 40 & 60 & 120 \\ 100 & 0 & 40 & 80 \end{pmatrix}$$
$$P = (50 & 4 & 22 & 44).$$

A trader has the endowment process

$$e(0) = 9, \quad e(T, \omega_1) = 10, \quad e(T, \omega_2) = 20.$$

The consumption set $X = \mathbb{R}^3$. A consumption process $\{c(0), c(T, \omega_1), c(T, \omega_2)\}$ belongs to the trader's budget set iff the following system of equations

$$-50\theta_1 - 4\theta_2 - 22\theta_3 - 44\theta_4 = c(0) - 9$$

$$100\theta_1 + 40\theta_2 + 60\theta_3 + 120\theta_4 = c(T, \omega_1) - 10$$

$$100\theta_1 + 40\theta_3 + 80\theta_4 = c(T, \omega_2) - 20$$

has a solution $\theta_1, \theta_2, \theta_3$ and θ_4 . The solvability condition may dictate certain condition on the consumption process.

Portfolio optimization

Assume that there are N risky assets and single riskfree asset.

Let \mathbb{H} denote the set of all trading strategies, $\mathbb{H} = \mathbb{R}^{N+1}$. Let $v \in \mathbb{R}$ be a specified scalar representing the initial wealth at t = 0.

Optimal portfolio problem:

maximize $H \in \mathbb{H}$ $Eu(V_1)$ subject to $V_0 = v$

Since $V_1 = B_1 V_1^*$ and $V_1^* = V_0^* + G^*$, the above is the same as

maximize $E[u(B_1[v + H_1 \Delta S_1^* + \dots + H_N \Delta S_N^*])].$ (1)

Here, B_1 is the money market account at t = 1, with $B_0 = 1, V_1^*$ is the discount value process.

Theorem If there exists an optimal solution of the portfolio problem (2), then there are no arbitrage opportunities.

Proof We prove by contradiction. Suppose \widehat{H} is an optimal solution and H is an arbitrage opportunity. Write $\widetilde{H} = \widehat{H} + H$, then

$$v + \sum_{n=1}^{N} \widetilde{H}_n \Delta S_n^* = v + \sum_{n=1}^{N} \widehat{H}_n \Delta S_n^* + \sum_{n=1}^{N} H_n \Delta S_n^* \ge v + \sum_{n=1}^{N} \widehat{H}_n \Delta S_n^*.$$

H is an arbitrage opportunity

The inequality is strict for at least one $\omega \in \Omega$.

Since u is strictly increasing in wealth and $P(\omega) > 0$ for all $\omega \in \Omega$, the objective value in (1) is strictly greater under \widetilde{H} than under \widehat{H} . Hence, \widehat{H} cannot be an optimal solution.

Remark

If there exists an optimal solution to (1), then there exists a risk neutral probability measure.

Theorem If (H, v) is a solution of the optimal portfolio problem, then a risk neutral probability measure exists, which is related to the optimal solution $V_1(\omega)$ as follows:

$$Q(\omega) = \frac{P(\omega)B_1(\omega)u'(V_1(\omega),\omega)}{E[B_1u'(V_1)]}, \omega \in \Omega,$$
(2)

Remark

Eq. (2) gives the relation between $Q(\omega)$ and the optimal solution $V_1(\omega)$ for any utility function u. Recall that $Q(\omega)$ depends only on $S^*(0)$ and $S^*(1;\omega)$ but not u. In case when Q is not unique, this would imply multiple optimal solutions for any given u.

Proof Rewrite the objective function $E[u(V_1)]$ as

$$Eu(V_1) = \sum_{\omega \in \Omega} P(\omega)u(B_1(\omega)[v + H_1 \Delta S_1^*(\omega) + \dots + H_N \Delta S_N^*(\omega)]), \omega).$$

The first order necessary condition is

$$0 = \frac{\partial}{\partial H_n} Eu(V_1)$$

= $\sum_{\omega \in \Omega} P(\omega)u'(B_1(\omega)[v + H_1 \Delta S_1^*(\omega) + \dots + H_n \Delta S_N^*(\omega)], \omega)B_1(\omega)\Delta S_n^*(\omega)$

$$= E[B_1 u'(V_1) \Delta S_n^*], \quad n = 1, \cdots, N.$$

On the other hand, a risk neutral probability measure must satisfy

$$0 = E_Q[\Delta S_n^*] = \sum_{\omega \in \Omega} Q(\omega) \Delta S_n^*(\omega), n = 1, \cdots, N.$$

From the first order condition, we may write

$$(P(\omega_{1})B_{1}(\omega_{1})u'(V_{1}(\omega_{1}))\cdots P(\omega_{K})B_{1}(\omega_{K})u'(V_{1}(\omega_{K})))\overbrace{\left(\begin{array}{ccc}S_{1}^{*}(1;\omega_{1})&\cdots&S_{N}^{*}(1;\omega_{1})\\\vdots&\vdots&\vdots\\S_{1}^{*}(1;\omega_{K})&\cdots&S_{N}^{*}(1;\omega_{K})\end{array}\right)}^{S}\\E[B_{1}u'(V_{1})](S_{1}^{*}(0)\cdots S_{N}^{*}(0)),$$

and the risk neutral probability values satisfy

$$(Q_1(\omega_1)\cdots Q(\omega_K))S = (S_1^*(0)\cdots S_N^*(0)).$$

Assuming that a right inverse of S exists (not necessarily unique), then we can deduce the following relation between $Q(\omega)$ and $u'(V_1(\omega))$ as follow

$$Q(\omega) = \frac{P(\omega)B_1(\omega)u'(V_1(\omega))}{E[B_1u'(V_1)]}, \quad \omega \in \Omega.$$

Note that $P(\omega)B_1(\omega)u'(V_1(\omega))/E[Bu'(V_1)] > 0$ for all $\omega \in \Omega$ since u is strictly increasing and

$$\sum_{k=1}^{K} \frac{P(\omega)B_1(\omega)u'(V_1(\omega))}{E[B_1u'(V_1)]} = 1.$$

- 1. Converse of the above theorem: if there exists a risk neutral measure Q, then does the optimal portfolio problem have a solution?
- 2. The direct solution of the non-linear system

$$E[B_1 u'(V_1) \Delta S_n^*] = 0, \quad n = 1, 2, \cdots, N$$

for H is complicated. How to find some convenient mean to get around it?

Definition

A securities market is said to be *viable* if there exists a function $u : \mathbb{R} \times \Omega \to \mathbb{R}$ and an initial wealth v such that $W \to u(W, \omega)$ is concave and strictly increasing for each $\omega \in \Omega$ and the corresponding optimal portfolio problem has an optimal solution H.

Theorem The securities market model is *viable* iff there exists a risk neutral probability measure Q.

Proof " \Rightarrow " part is shown by eq. (2). We only need to consider " \Leftarrow " part. It suffices to show by assuming the existence of a risk neutral probability measure, cleverly select u and v, then demonstrate the existence of the optimal solution to the portfolio problem.

Now, we choose
$$u(W, \omega) = W \frac{Q(\omega)}{P(\omega)B_1(\omega)}$$
 while v will be arbitrary.

For an arbitrary (H_1, \dots, H_N) , we have

 $E[u(B_1\{v + H_1 \Delta S_1^* + \dots + H_N \Delta S_N^*\}, \omega)]$

 $= \Sigma P(\omega)B_1(\omega)\{v + H_1 \Delta S_1^* + \dots + H_N \Delta S_N^*\}Q(\omega)/[P(\omega)B_1(\omega)]$

$$= \Sigma Q(\omega) \{ v + H_1 \Delta S_1^* + \dots + H_N \Delta S_N^* \}$$

 $= v + H_1 E_Q[\Delta S_1^*] + \dots + H_N E_Q[\Delta S_N^*] = v.$

Hence, every vector (H_1, \dots, H_N) with the same initial wealth v gives rise to the same objective function. That is, all such trading strategies are optimal. Hence, the theorem is true by this clever choice of utility function.

Example Consider the following discounted price process

| n | $S_n^*(0)$ | $S_n^*(1)$ | | |
|---|------------|------------|------------|--------------|
| | | ω_1 | ω_2 | ω_{3} |
| 1 | 6 | 6 | 8 | 4 |
| 2 | 10 | 13 | 9 | 8 |

Solve for the risk neutral probability measure

$$6 = 6Q(\omega_1) + 8Q(\omega_2) + 4Q(\omega_3)$$

$$10 = 13Q(\omega_1) + 9Q(\omega_2) + 8Q(\omega_3)$$

$$1 = Q(\omega_1) + Q(\omega_2) + Q(\omega_3)$$

we obtain the unique solution:

$$(Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3)) = (1/3 \quad 1/3 \quad 1/3).$$

Suppose we choose the utility function: $u(W) = -\exp(-W)$ so that $u'(W) = \exp(-W)$. The necessary conditions:

$$E[B_1 u'(V_1) \Delta S_n^*] = 0, \quad n = 1, 2, \cdots, N,$$

become the following system of non-linear algebraic equations

$$0 = P(\omega_1) \exp\left\{-\frac{10}{9}(v+0H_1+3H_2)\right\} \frac{10}{9} \cdot 0 \\ + P(\omega_2) \exp\left\{-\frac{10}{9}(v+2H_1-H_2)\right\} \frac{10}{9} \cdot 2 \\ + P(\omega_3) \exp\left\{-\frac{10}{9}(v-2H_2-2H_2)\right\} \frac{10}{9}(-2) \\ 0 = P(\omega_1) \exp\left\{-\frac{10}{9}(v+0H_1+3H_2)\right\} \frac{10}{9}(3) \\ + P(\omega_2) \exp\left\{-\frac{10}{9}(v+2H_1-H_2)\right\} \frac{10}{9}(-1) \\ + P(\omega_3) \exp\left\{-\frac{10}{9}(v-2H_1-2H_2)\right\} \frac{10}{9}(-2).$$

It is quite complicated to solve for H_1 and H_2 .

More efficient computational technique

The objective function $H \rightarrow Eu(V_1)$ can be viewed as the composition of two functions:

 $H \rightarrow V_1$ (maps trading strategies into random variables) $V_1 \rightarrow Eu(V_1)$ (maps random variables into real numbers)

Two-step process

- 1. Identify the optimal random variable V_1 , the value of V_1 that maximizes $Eu(V_1)$ over the subset of feasible random variables.
- 2. Compute the trading strategy H that generates this V_1 . This is related to the solution of a linear system of equations.



For step one, we need to specify the subset of feasible random variables correctly and conveniently.

If the securities model is complete (every contingent claim lies in the asset span), the subset is simply

$$\mathbb{W}_v = \{ W \in \mathbb{R}^K : E_Q[W/B_1] = v \}.$$

 \mathbb{W}_v is called the set of attainable wealths.

- (i) Under any trading strategy H with $V_0 = v$, one has $E_Q[V_1/B_1] = v$ by the risk neutral valuation principle.
- (ii) For any contingent claim $W \in W_v$, there exists a trading strategy H such that $V_0 = v$ and $V_1 = W$.

Solution of the first subproblem

maximize Eu(W)subject to $W \in W_v$

When the model is complete, the Lagrangian formulation is

maximize
$$Eu(W) - \lambda \{E_Q[W/B_1] - v\}$$
.

Introducing L = Q/P (state price density)

$$Eu(W) - \lambda \{E_Q[W/B_1] - v\}$$

= $E[u(W) - \lambda \{LW/B_1 - v\}]$
= $\sum_{\omega} P(\omega)[u(W(\omega)) - \lambda \{L(\omega)W(\omega)/B_1(\omega) - v\}].$

If W maximizes the above quantity, then the necessary conditions are (one equation for each $\omega \in \Omega$)

$$u'(W(\omega)) = \lambda L(\omega)/B_1(\omega), \quad \omega \in \Omega.$$

Recall the earlier relation between Q and $u'(V_1)$, where

$$Q(\omega) = \frac{P(\omega)B_1(\omega)u'(V_1(\omega),\omega)}{E[B_1u'(V_1)]}, \quad \omega \in \Omega,$$

so that λ is identified as $E[B_1u'(\widehat{W})]$, where \widehat{W} is the optimal solution.

Let I denote the inverse function corresponding to u', we then have

$$W(\omega) = I(\lambda L(\omega)/B_1).$$

How to compute λ ? From the constraint condition: $E_Q[W/B_1] = v$, we obtain

$$E_Q[I(\lambda L/B_1)/B_1] = v.$$

State prices

Assuming no discount effect, let

$$Q(\omega_k) = \boldsymbol{e}_k^T \boldsymbol{Q} = E_Q[\boldsymbol{s}_k] = E[\boldsymbol{L}\boldsymbol{s}_k]$$

where $L(\omega) = Q(\omega)/P(\omega)$ is called the state price density (also called pricing kernel).

Example Take $u(W) = -\exp(-W)$ so that $u'(W) = \exp(-W)$. We have u'(W) = i iff $W = -\ln i$ so that $I(i) = -\ln i$.

$$W = -\ln(\lambda L/B_1) = -\ln\lambda - \ln L/B_1$$

To solve for λ , we use

$$v = -E_Q[B_1^{-1}\ln(\lambda L/B_1)] = -(\ln \lambda)E_QB_1^{-1} - E_Q[\ln(L/B_1)/B_1].$$

Hence, the correct value of λ is

$$\lambda = \exp\left(\frac{-v - E_Q[B_1^{-1}\ln(L/B_1)]}{E_Q B_1^{-1}}\right)$$

so that

$$W = \frac{v + E_Q[B_1^{-1}\ln(L/B_1)]}{E_Q B_1^{-1}} - \ln(L/B_1).$$

It is seen that $E_Q[W/B_1] = v$ is satisfied.

Putting back into $u(W) = -\exp(-W)$, and observing u' = -u we have

$$u(W) = -\exp\left\{\frac{-v + \ln(L/B_1)E_QB_1^{-1} - E_Q[B_1^{-1}\ln(L/B_1)]}{E_QB_1^{-1}}\right\} = -\frac{\lambda L}{B_1}$$

so that the optimal value of the objective function is

$$Eu(W) = -\lambda E[L/B_1] = -\lambda E_Q B_1^{-1}.$$

The optimal wealth W is obtained and it depends on the underlying securities market model only via the probability measures P and Q. That is, the true measure and risk neutral measure comprise what can be thought of as a sufficient statistic for the optimal portfolio subproblem.

Continued with the numerical example

Let $P(\omega_1) = 1/2, P(\omega_2) = P(\omega_3) = 1/4$ so that

 $L(\omega_1) = 2/3, L(\omega_2) = L(\omega_3) = 4/3.$ With r = 1/9 so that $B_1 = 10/9$,

$$E_Q[\ln(L/B_1)] = \frac{1}{3} \left[\ln\left(\frac{2}{3} \cdot \frac{9}{10}\right) + 2\ln\left(\frac{4}{3} \cdot \frac{9}{10}\right) \right] = -0.04873$$

so that the optimal attainable wealth is

$$W = v(1+r) + E_Q[\ln(L/B_1)] - \ln(L/B_1)$$

=
$$\begin{cases} v(10/9) + 0.46209 & \omega = \omega_1 \\ v(10/9) - 0.23105 & \omega = \omega_2 \text{ or } \omega = \omega_3 \end{cases}$$

Now

$$\lambda = \exp\left\{-\frac{10}{9}v + 0.04873\right\}$$

so the optimal value of the objective function is

$$E[u(W)] = -\lambda E_Q B_1^{-1} = -\frac{9}{10}\lambda.$$

Note that this is consistent to $\lambda = E[B_1u'(V_1)]$. Once the optimal attainable wealth W is computed, we solve for the optimal trading strategy H by solving $W/B_1 = v + G^*$.

In state ω_1

$$W(\omega_1)/B_1 = v + \frac{9}{10}(0.46209) = v + 0.41590;$$

 $v + G^*(\omega_1) = v + H_1(6 - 6) + H_2(13 - 10) = v + 3H_2.$

Similar procedures are carried for states ω_2 and ω_3 . We obtain

$$\begin{array}{l} \omega_1 : 0.41590 = 0 \cdot H_1 + 3H_2 \\ \omega_2 : -0.20795 = 2H_1 - H_2 \\ \omega_3 : -0.20795 = -2H_1 - 2H_2 \end{array} \right\} \quad \Rightarrow \quad \begin{array}{l} H_1 = -0.03466 \\ H_2 = 0.13863 \end{array}$$

Lastly, we compute H_0 using

 $V_0 = v = H_0 + 6H_1 + 10H_2$ so that $H_0 = v - 1.17834$.

Check that this trading strategy satisfies the first order condition.

Consumption-investment problem

- * A consumption process $C = (C_0, C_1)$ consists of a non-negative scalar C_0 and a non-negative random variable C_1 .
- * A consumption-investment plan consists of a pair (C, H) where C is a consumption process and H is a trading strategy.
- 1. $C_0 = \text{time zero consumption}$

 $V_0 = H_0 + \Sigma H_n S_n(0) =$ amount invested at time zero. Amount of money available at time zero = $\nu \ge C_0 + V_0$.

2. $V_1 = H_0 B_1 + \Sigma H_n S_n(1) =$ amount of money available at t = 1so $C_1 \leq V_1$.

We assume that a sensible investor who can consume only at t = 0and t = 1 would not leave money "lying in the drawer". Throughout the subsequent analysis, we always assume the absence of arbitrage opportunities so that a risk neutral probability measure exists.

A consumption-investment plan is said to be *admissible* if

(1)
$$C_0 + V_0 = \nu$$
 and (2) $C_1 = V_1$. We always assume $\nu \ge 0$.

If (C, H) is admissible, then C_1 is an attainable contingent claim with

$$E_Q[C_1/B_1] = E_Q[V_1/B_1] = V_0$$

for every risk neutral measure Q, and correspondingly,

 $E_Q[C_0 + C_1/B_1] = \nu.$

Given $\nu \ge 0$ and some consumption process C, how to we know whether there exists some trading strategy H such that (C, H) is admissible?

1. Well, if C_1 is an attainable contingent claim, then there exists some trading strategy H such that

$$C_1 = V_1 = H_0 B_1 + \Sigma H_n S_n.$$

2. Further, if $E_Q[C_0 + C_1/B_1] = \nu$ is satisfied for some Q, then $C_0 + V_0 = \nu$, in which case (C, H) is admissible.

Summary

Let the initial amount of money $\nu \ge 0$ and the consumption process C be fixed. There exists a trading strategy H such that the consumption-investment plan (C, H) is admissible if and only if

$$C_0 + E_Q[C_1/B_1] = \nu$$

for every risk neutral probability measure Q.

Example

The securities model is complete with Q = (1/3, 1/3, 1/3). In order for (C_0, C_1) to be a part of an admissible consumption-investment plan, we must have

$$\nu - C_0 = \frac{9}{10} E_Q[C_1] = \frac{3}{10} [C_1(\omega_1) + C_1(\omega_2) + C_1(\omega_3)].$$

There is only one constraint involving C_0 and $C_1(\omega)$.

Maximization problem

An investor starts with initial wealth ν and wants to choose an admissible consumption-investment plan so as to maximize the expected value of the utility of consumption at both times zero and one.

 $u: \mathbb{R}_+ \to \mathbb{R}$ is concave, differentiable and strictly increasing. Maximize $u(C_0) + E[u(C_1)]$

subject to

$$C_{0} + H_{0}B_{0} + \sum_{n=1}^{N} H_{n}S_{n}(0) = \nu$$

$$C_{1} - H_{0}B_{1} - \sum_{n=1}^{N} H_{n}S_{n}(1) = 0 \text{ for all } \omega \in \Omega$$

$$C_{0} \ge 0, \quad C_{1} \ge 0, \quad H \in \mathbb{R}^{N+1}$$

Example Consider the above securities model:

Take $u(C) = \ln C$. Since $\lim_{C \to 0^+} \ln C$ is $-\infty$, we may *drop* the *explicit non-negativity* constraint.

Given
$$P(\omega_1) = 1/2$$
, $P(\omega_2) = P(\omega_3) = 1/4$ and $r = 1/9$:
Maximize $\ln C_0 + \frac{1}{2} \ln C_1(\omega_1) + \frac{1}{4} \ln C_1(\omega_2) + \frac{1}{4} \ln C_1(\omega_3)$

subject to $C_0 = \nu - H_0 - 6H_1 - 10H_2$

$$C_{1}(\omega_{1}) = \frac{10}{9}H_{0} + \frac{60}{9}H_{1} + \frac{130}{9}H_{2}$$

$$C_{1}(\omega_{2}) = \frac{10}{9}H_{0} + \frac{80}{9}H_{1} + \frac{90}{9}H_{2}$$

$$C_{1}(\omega_{3}) = \frac{10}{9}H_{0} + \frac{40}{9}H_{1} + \frac{80}{9}H_{2}.$$

After some simplication:

Maximize

$$\ln(\nu - H_0 - 6H_1 - 10H_2) + \frac{1}{2}\ln\left(\frac{10}{9}H_0 + \frac{60}{9}H_1 + \frac{130}{9}H_2\right) + \frac{1}{4}\ln\left(\frac{10}{9}H_0 + \frac{80}{9}H_1 + \frac{90}{9}H_2\right) + \frac{1}{4}\ln\left(\frac{10}{9}H_0 + \frac{40}{9}H_1 + \frac{80}{9}H_2\right).$$

The necessary conditions are:

$$\begin{aligned} & -\frac{1}{C_0} + \frac{1}{2}\frac{10}{9}\frac{1}{C_1(\omega_1)} + \frac{1}{4}\frac{10}{9}\frac{1}{C_1(\omega_2)} + \frac{1}{4}\frac{10}{9}\frac{1}{C_1(\omega_3)} = 0 \\ & -\frac{6}{C_0} + \frac{160}{2}\frac{1}{9}\frac{1}{C_1(\omega_1)} + \frac{180}{4}\frac{1}{9}\frac{1}{C_1(\omega_2)} + \frac{140}{4}\frac{1}{9}\frac{1}{C_1(\omega_3)} = 0 \\ & \frac{10}{C_0} + \frac{1}{2}\frac{130}{9}\frac{1}{C_1(\omega_1)} + \frac{190}{4}\frac{1}{9}\frac{1}{C_1(\omega_2)} + \frac{180}{4}\frac{1}{9}\frac{1}{C_1(\omega_3)} = 0. \end{aligned}$$

Solve for H_0, H_1 and H_2 , then obtain $C_0, C_1(\omega_1), C_1(\omega_2)$ and $C_1(\omega_3)$.

Solution of the maximization problem

Differentiate the objective function with respect to H_0, \dots, H_N successively and substitute for C_0 and C_1 . The following N + 1 first order necessary conditions are obtained:

$$u'(C_0) = E[B_1 u'(C_1)]$$

$$u'(C_0)S_n(0) = E[u'(C_1)S_n(1)], \quad n = 1, \cdots, N$$

Recall that C_0 and C_1 must be both non-negative. If u is chosen such that $u(C) \to -\infty$ as $C \to 0^+$, then these constraints will not be binding.

Theorem

If C is a part of a solution to the optimal consumption-investment problem with $C_0 \ge 0$ and $C_1(\omega) \ge 0$ for all ω , then

$$Q(\omega) = P(\omega)B_1(\omega)\frac{u'(C_1(\omega))}{u'(C_0)}.$$

Proof

In a similar manner, we have

$$(P(\omega_1)B_1(\omega_1)u'(C_1(\omega_1))\cdots P(\omega_K)B_1(\omega_K)u'(C_1(\omega_K)))S = u'(C_0)(S_1^*(0)\cdots S_N^*(0)),$$

and upon comparing with

$$(Q_1(\omega_1)\cdots Q(\omega_K))S = (S_1^*(0)\cdots S_N^*(0)).$$

we deduce that

$$Q(\omega) = P(\omega)B_1(\omega)\frac{u'(C_1(\omega))}{u'(C_0)}.$$

Note that $P(\omega)B_1(\omega)\frac{u'(C_1(\omega))}{u'(C_0)} > 0$ and

$$\sum_{k=1}^{K} P(\omega_k) B_1(\omega_k) \frac{u'(C_1(\omega_k))}{u'(C_0)} = 1.$$

Risk neutral computational approach (consumption-investment problem)

Alternative formulation: maximize $u(C_0) + E[u(C_1)]$

subject to
$$C_0 + E_Q[C_1/B_1] = \nu$$

 $C_0 \geq 0$ and $C_1 \geq 0$.

First, we analyze the constrained problem:

Maximize $u(C_0) + E[u(C_1)] - \lambda \{C_0 + E[C_1L/B_1] - \nu\}.$

Assume that we choose an utility function such that the optimal solution features strictly positive consumption values.
Consider the satisfaction of the first order conditions:

$$u'(C_0) = \lambda$$
 and $u'(C_1(\omega)) = \lambda L/B_1$,

we have

$$C_0 = I(\lambda)$$
 and $C_1(\omega) = I(\lambda L/B_1).$

What is the governing equation for λ ?

$$I(\lambda) + E_Q[I(\lambda L/B_1)/B_1] = \nu.$$

Solution generally exists if $I(\lambda)$ is monotonic. If the corresponding values of C_0 and C_1 are non-negative, then they must be an optimal solution.

Example

Suppose $u(C) = \ln C$ so that u'(C) = 1/C and I(i) = 1/i.

$$C_0=1/\lambda$$
 and $C_1(\omega)=rac{1}{\lambda L}/B_1$

and

$$\frac{1}{\lambda} + \frac{1}{\lambda} E_Q[L^{-1}] = \frac{1}{\lambda} + \frac{1}{\lambda} E[1] = \frac{2}{\lambda} = \nu,$$

so $\lambda = 2/\nu$ and $C_0 = \nu/2, C_1(\omega) = \nu B_1(\omega) P(\omega)/[2Q(\omega)]$. Both C_0 and C_1 are non-negative if $\nu \ge 0$. The maximum value of the objective function is

$$2\ln\frac{\nu}{2} + E\left[\ln\frac{\nu B_1}{L}\right].$$

With $L(\omega_1) = 2/3, L(\omega_2) = L(\omega_3) = 4/3$ and r = 1/9

$$C_1(\omega) = \nu \frac{5}{9} L^{-1} = \begin{cases} \frac{5}{6}\nu & \text{if } \omega = \omega_1 \\ \frac{5}{12}\nu & \text{if } \omega = \omega_2 \text{ or } \omega = \omega_3 \end{cases}$$

Note that the first order conditions are satisfied. Lastly, we compute the optimal H_1 and H_2 using $\frac{C_1}{B_1} = V_0 + G^* = \frac{\nu}{2} + G^*$:

$$\frac{3}{4}\nu = \frac{1}{2}\nu + 0H_1 + 3H_2$$

$$\frac{3}{8}\nu = \frac{1}{2}\nu + 2H_1 - H_2$$

$$\frac{3}{8}\nu = \frac{1}{2}\nu - 2H_1 - 2H_2$$

2 unknowns: H_1, H_2 but 3 equations. Solution exists provided that C_1 is attainable. By enforcing $E_Q[C_1/B_1]$ to have the same value for all Q, C_1 is guaranteed to be attainable.

The solution is given by: $H_1 = -\nu/48, H_2 = \nu/12.$

From
$$\frac{\nu}{2} = H_0 + 6H_1 + 10H_2$$
, we obtain $H_0 = -\frac{5}{24}\nu$.

Generalizations

- 1. The objective function is given as $u(C_0) + \beta E[u(C_1)]$, where $0 < \beta \le 1$; here β is considered as the discount factor.
- 2. Allow the consumer to have endowment (income) \tilde{E} at time t = 1, where \tilde{E} is a specified random variable. The second constraint becomes

$$C_1 - H_0 B_1 - \sum_{n=1}^N H_n S_n(1) = \tilde{E}.$$

The pair (ν, \tilde{E}) is called the endowment process for the consumer.

* The consumption-investment plan (C, H) is *admissible* if and only if

$$C_0 + E_Q[(C_1 - \tilde{E})/B_1] = \nu.$$

Complete markets

Definitions

A contingent claim can be considered as a random variable Y that represents a terminal payoff whose value depends on the occurrence of a particular state ω_k , where $\omega_k \in \Omega$.

A securities model is complete if every contingent claim Y lies in the asset span, that is, Y can be generated by some trading strategy.

$$S(1;\Omega) = \begin{pmatrix} S_1(1;\omega_1) & \cdots & S_M(1;\omega_1) \\ \vdots & & \vdots \\ S_1(1;\omega_K) & \cdots & S_M(1;\omega_K) \end{pmatrix}.$$

- Y always lies in the asset span iff the column space of $S(1; \Omega)$ is equal to \mathbb{R}^{K} .
- A necessary condition for market completeness: $M \ge K$.

- When there is no redundant security, the trading strategy that generates Y must be unique.
- Under market completeness, if the set of risk neutral probability measures is non-empty, then it must be a singleton.

Proof

Let M denote the set of all risk neutral probability measures. We quote the following well known result without proof.

"The contingent claim Y is attainable iff $E_Q[Y/B_1]$ takes the same value for every $Q \in M$."

The above result is consistent with the law of one price.

We prove by contradiction. Assume market is complete so that every Y is attainable but M has two distinct risk neutral probability measures, Q and \hat{Q} . There must exist some state ω_k with $Q(\omega_k) \neq \hat{Q}(\omega_k)$.

Take

$$Y(\omega) = \begin{cases} B_1(\omega_k) & \omega = \omega_k \\ 0 & \text{otherwise} \end{cases},$$

then

$$E_Q[Y/B_1] = Q(\omega_k) \neq \widehat{Q}(\omega_k) = E_{\widehat{Q}}[Y/B_1].$$

This leads to a contradiction.

Incomplete markets

When the market is incomplete, then a non-attainable contingent claim cannot be priced using risk neutral valuation principle.

• We still can specify an interval $(V_{-}(Y), V_{+}(Y))$ where a reasonable price at t = 0 of the contingent claim should lie.

 $V_+(Y) = \inf\{E_Q[\widetilde{Y}/B_1] : \widetilde{Y} \ge Y \text{ and } \widetilde{Y} \text{ is attainable}\}$ $V_-(Y) = \sup\{E_Q[\widetilde{Y}/B_1] : \widetilde{Y} \le Y \text{ and } \widetilde{Y} \text{ is attainable}\}.$

• $V_+(Y)$ can be seen as the minimum value among all prices of attainable contingent claims that dominate the non-attainable claim Y.

What happens when $V(Y) > V_+(Y)$?

An arbitrageur can lock in riskless profit by selling the contingent claim to receive V(Y) and use $V_+(Y)$ to construct the replicating portfolio that generates the attainable \tilde{Y} .

- The upfront positive gain is $V(Y) V_+(Y)$.
- Since $\tilde{Y} \ge Y$, the replicating portfolio always dominates Y so that no loss at expiry is ensured.

How to characterize the set of all risk neutral measures M?

$$M = W^{\perp} \cap P^{+}$$

 $W = \{X \in \mathbb{R}^K : X = G^* \text{ for some trading strategy } H\}, \text{ where }$

discount gain
$$= G^* = \sum_{m=1}^M H_m \Delta S_m^*.$$

$$W^{\perp} = \{ Y \in \mathbb{R}^{K} : X^{T}Y = 0 \text{ for all } X \in W \}$$

$$P^{+} = \{ X \in \mathbb{R}^{K} : X_{1} + \dots + X_{K} = 1, X_{1} > 0, \dots, X_{K} > 0 \}.$$

This is because

$$G^{T}Q = E_{Q}[G^{*}] = \sum_{k=1}^{K} Q(\omega_{k}) \left[\sum_{m=1}^{M} h_{m} \Delta S_{n}^{*}(\omega_{k}) \right]$$
$$= \sum_{m=1}^{M} h_{m} E_{Q}[\Delta S_{m}^{*}] = 0$$

and $Q(\omega) > 0, \Sigma Q(\omega) = 1.$

Numerical example

Consider

$$S_0^* = \begin{pmatrix} 3 & 3 \end{pmatrix} \text{ and } S^*(1;\omega) = \begin{pmatrix} 4 & 3 \\ 3 & 2 \\ 2 & 4 \end{pmatrix},$$

the discounted gains of the two risky securities are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.
$$W = \left\{ h_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + h_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \text{ where } h_1 \text{ and } h_2 \text{ are scalars} \right\}.$$
$$W^{\perp} \text{ is the line through the origin which is perpendicular to } \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

The line should assume the form $\lambda \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix}$. Together with the constraints that sum of components equals one and every component is positive, we obtain

$$Q = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Let $Q_j \in M = W^{\perp} \cap P^+$, $j = 1, \dots J$ are chosen to be independent vectors, thus forming a basis of W^{\perp} (assume to have dimension J). Then

$$W = \{X \in \mathbb{R}^K : X^T Q_j = 0, \quad j = 1, 2, \cdots, J\}.$$

How to solve for $V^+(Y)$?

Solve the following linear program

minimize λ

subject to

$$\begin{array}{rcl}
\widetilde{Y} & \geq & Y \\
U & = & \widetilde{Y}/B_1 \\
\lambda & = & U^T Q_1 \\
& \vdots \\
\lambda & = & U^T Q_J
\end{array}$$

 $\lambda \in \mathbb{R}, \tilde{Y} \in \mathbb{R}^K.$

We enforce the condition that $E_Q[\tilde{Y}/B_1]$ takes the same value for every risk neutral measure Q.

If λ and \tilde{Y} are part of the optimal solution of the above linear program problem, then $\lambda = V_+(Y)$ and \tilde{Y} is an attainable contingent claim with $\tilde{Y} \ge Y$.

Optimal portfolios in incomplete markets

Need to properly identify the set of attainable wealths.

A contingent claim (or wealth) W is attainable iff $E_Q[W/B_1]$ takes the same value for every risk neutral probability measure $Q \in M$.

$$W_v = \{ W \in \mathbb{R}^K : E_Q[W/B_1] = v, \quad Q \in M \},\$$

where W_v is the set of wealths that can be generated starting with initial capital v.

If there exists a finite number of independent vectors $Q(1), \dots, Q(J)$ such that every element of M can be expressed as a linear combination of these J vectors. We have

$$E_Q[W/B_1] = v$$
 for all $Q \in M$ iff $E_{Q(J)}[W/B_1] = v$, $j = 1, 2, \cdots, J$.

The optimal portfolio problem becomes

maximize
$$Eu(W)$$

subject to $E_{Q(j)}[W/B_1] = v, j = 1, 2, \dots, J.$
Define $L_j = Q(j)/P$, maximize $Eu(W) - \sum_{j=1}^J \lambda_j \{E[L_jW/B_1] - v\}.$

The first order conditions, one for each ω , give

$$u'(W(\omega)) = \sum_{j=1}^{J} \lambda_j L_j(\omega) / B_1(\omega), \quad \forall \omega \in \Omega,$$

or

$$W(\omega) = I\left[\sum_{j=1}^{J} \lambda_j L_j(\omega) / B_1(\omega)\right], \quad \forall \omega \in \Omega.$$

We need to solve for the Lagrangian multipliers from the following set of equations.

$$E[L_j I(\lambda_1 L_1 / B_1 + \dots + \lambda_J L_J / B_1) / B_1] = v, j = 1, \dots, J.$$

Example
$$K = 3, N = 1, r = 1/9, S_1(0) = 5$$

| ω | $S_1(\omega)$ | $S_1^*(\omega)$ | $P(\omega)$ |
|--------------|---------------|-----------------|-------------|
| ω_1 | 20/3 | 6 | 1/3 |
| ω_2 | 40/9 | 4 | 1/3 |
| ω_{3} | 30/9 | 3 | 1/3 |

$$\begin{cases} Q_1 - Q_2 - 2Q_3 = 0\\ Q_1 + Q_2 + Q_3 = 1 \end{cases}$$

The model is incomplete with M consisting of all probability measures of the form

$$Q = (heta, 2 - 3 heta, -1 + 2 heta)$$
 where $\frac{1}{2} < heta < \frac{2}{3}$

Note that dim W = 1, dim $W^{\perp} = 2$ so that the set of risk neutral measures is generated by one free parameter.

A contingent claim $X = (X_1 \ X_2 \ X_3)$ is attainable iff

 $X_1 - 3X_2 + 2X_3 = 0.$

This is the plane spanned by $(1 \ 1 \ 1)^T$ and $(6 \ 4 \ 3)^T$.

Say, take $\theta = 1/2$ and $\theta = 1/3$, we obtain

$$Q(1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \text{ and } Q(2) = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

Note that $Q(1)$ and $Q(2)$ are obtained by taking the two endpoints
in the range $\frac{1}{2} < \theta < \frac{2}{3}$.
$$L_1 = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} & 0 \end{pmatrix} \text{ and } L_2 = (2 \quad 0 \quad 1).$$

Remark

We have relaxed the positivity condition on values of Q to nonnegativity condition instead of the usual positivity condition. In this case, the probability measure is called a pricing measure.

Recall
$$W(\omega) = I\left[\sum_{j=1}^{2} \lambda_j L_j(\omega) / B_1(\omega)\right]$$
. Taking $u(W) = \ln W, u'(W) = 1/W$ and $I(i) = 1/i$, so that

$$\frac{1}{3\lambda_1 + 4\lambda_2} + \frac{1}{3\lambda_1} = v$$
$$\frac{1}{9\lambda_1 + 12\lambda_2} + \frac{1}{3\lambda_2} = v.$$

The unique (non-negative) solution is

$$\lambda_1 = \frac{0.46482}{v}$$
 and $\lambda_2 = \frac{0.53519}{v}$.

$$W(\omega) = \frac{v}{0.46482 \left(\frac{9}{10}\right) L_1(\omega) + 0.53519 \left(\frac{9}{10}\right) L_2(\omega)} \\ = \begin{cases} 0.62860v, & \omega = \omega_1 \\ 1.59360v, & \omega = \omega_2 \\ 2.07611v, & \omega = \omega_3 \end{cases}$$

Note that $W(\omega)$ satisfies $X_1 - 3X_2 + 2X_3 = 0$.

Solve
$$H_1$$
 and H_0 from
$$\begin{cases} H_0 + 6H_1 = \left(\frac{9}{10}\right) (0.6286)v \\ H_0 + 4H_1 = \left(\frac{9}{10}\right) (1.5936)v \end{cases}$$

yielding $H_0 = 3.17124v$ and $H_1 = -0.43425v$.

The optimal objective value is $0.24409 + \ln v = E[\ln W]$.

Alternative approach

- Add one or more securities to the model such that it is made to be complete. The computation is easier since it is done for a complete market.
- Solve the optimal portfolio problem with the constraint that no position can be taken in any of the added fictitious securities.

Be careful that when adding new fictitious securities, one has to avoid adding arbitrage opportunities.

Equilibrium models

- How equilibriums of prices are related to the attributes of the agents in the economy, such as the endowments, beliefs, and preferences, as well as to the type and structure of the traded securities? The characteristics of these securities are fixed at the outset.
- Result from the optimizing action of all the agents in the market. Equilibrium is reached when the prices are such that each agent's expected utility is maximized.
- No-arbitrage pricing fails when a new security cannot be replicated by existing securities. The equilibrium approach is more general as it tries to relate the prices of securities to more fundamental economic concepts (where the prices come from). This is why it needs to impose more structure than in the no-arbitrage approach.

Remarks

- If a market is in equilibrium, it should *not permit arbitrage opportunities*. If otherwise, agents are able to improve their welfare at zero cost. This contradicts the property of an equilibrium that the agent's utilities are already maximized.
- Again, the price of a security can be expressed as the discounted expectation of its payoff, but the actual payoff is adjusted by a risk factor. The expectation is taken with respect to a probability measure that corresponds to realistic probabilities.
- Under certain conditions, we can unite the characteristics of each individual agent into one representative agent. We then derive the equilibrium prices in terms of the attributes of this representative agent. This technique can lead to the derivation of the CAPM.

Assumptions of one-period securities model

- 1. No production. Such models are known as exchange economies.
- 2. Single perishable good that cannot be stored. This good is used as the unit of measurement.
- 3. A security is a contract that specifies the amount of the consumption good in each future date.
- 4. Securities have no risk of default.
- 5. Agents maximize their expected utility of consumption by trading in the available marketed securities.
- 6. The equilibrium prices in turn are set by the actions of risk-averse agents in the model.

There is a given set of traded securities. We would like to show how the prices of these securities can be related to more fundamental economic variables.

- At time 1, the state of the economy is represented by a set of outcomes $\omega \in \Omega$.
- The time 1 payout on security j in state ω , is $X_j(\omega), j = 1, 2, \cdots, N$.

Three sets of variables to be solved

- 1. Initial price x_j
- 2. Optimal consumption allocations $\{C_{i0}, C_{i1}\}$
- 3. Trading strategies θ_i^n

Agents

Each agent receives an initial endowment of consumption good: amount e_0 at time 0 and $e_1(\omega)$ at time 1 in state ω . Endowment process $\{e_0, e_1(\omega)\}$ is non-negative.

Consumption process

- Consumes an amount C_0 at time 0 and $C_1(\omega)$ at time 1 in state ω . Suppose we assume $e_1(\omega) = 0$ for all ω :
- If the agent makes no trade, the consumption is $\{e_0, 0\}$.
- If the agent buys one unit of security j for a price of x_j , the consumption process is

$$\{e_0-x_j,X_j(\omega)\}.$$

Trading strategy

- consists of an *N*-dim vector showing the net trades that the agent makes in each security;
- different trading strategies lead to different consumption patterns;
- preferable to conduct the optimization over the feasible set of trading strategies rather than over consumption allocations.

If the market is incomplete, not all consumption patterns are attainable. When the market is complete, all consumptions are attainable, and we may use consumptions as control variables in optimization procedure. Probability measure

• Agents make decisions to maximize their individual expected utilities, using the agent's own subjective probabilities.

Heterogenous beliefs: agent *i* assigns his subjective probability $P_i(\omega)$ to state ω .

Homogeneous beliefs: each agent assigns the same probability $P(\omega)$ to a given state.

The expected utility for agent i is

$$U_{i0}(C_{i0}) + \sum_{\omega} P_i(\omega) U_{i1}(C_{i1}(\omega)).$$

Each agent strives to maximize his own expected utility by trading in the marketed securities. Utility function is increasing, concave and twice differentiable.

Equilibrium prices

- Assume that each agent has made his optimal trading decisions so that the *consumption allocations are optimal* for each agent.
 Equilibrium prices are prices that *support* this allocation.
- When the system is in equilibrium, the prices and the consumption allocations are such that each agent's expected utility is maximized at these consumption allocations and prices.
- The initial endowment acts as a constraint in the consumption pattern.

Equilibrium condition

Assume that the agent has already made the optimal investment decisions, that is, any deviation from this position is no longer optimal. What is the relation between x_i and consumption pattern?

- Denote the optimal consumption process in equilibrium by $\{C_{i0}^*, C_{i1}^*(\omega)\}$.
- Assume that the agent purchases α units of security j at time 0. Agent's consumption is $C_{i0}^* - \alpha x_j$ at time 0 and $C_{i1}^*(\omega) + \alpha X_j(\omega)$ at time 1 in state ω . The optimal choice would be $\alpha = 0$. This is the standard procedure of applying the variational principle.
- With this *revised* investment and consumption plans, the agent's expected utility becomes

$$U_{i0}(C_{i0}^* - \alpha x_j) + \sum_{\omega} P_i(\omega) U_{i1}(C_{i1}^*(\omega) + \alpha X_j(\omega)).$$

Differentiating with respect to α

$$-x_j U_{i0}'(C_{i0}^* - \alpha x_j) + \sum_{\omega} P_i(\omega) U_{i1}'[C_{i1}^*(\omega) + \alpha X_j(\omega)] X_j(\omega),$$

and this vanishes at $\alpha = 0$. Hence

$$x_{j} = \sum_{\omega} P_{i}(\omega) \frac{U_{i1}'(C_{i1}^{*}(\omega))}{U_{i0}'(C_{i0}^{*})} X_{j}(\omega) = E^{P_{i}}[Z_{i}X_{j}], \text{ where}$$
$$Z_{i} = \frac{U_{i1}'(C_{i1}^{*}(\omega))}{U_{i0}'(C_{i0}^{*})},$$

which is a random variable called a pricing kernel.

The expectation is taken with respect to the subjective probabilities of the individual agent.

Arrow-Debreu security

- Pays exactly one unit of the consumption good in state ω at time 1, and zero in all other states.
- Let ψ_{ω} be the price of the Arrow-Debreu security in ω , then

$$\psi_{\omega} = P_i(\omega) \frac{U'_{i1}(C^*_{i1}(\omega))}{U'_{i0}(C^*_{i0})},$$

as from the perspective of agent i. But this must be equalized across all agents in the economy.

The price of the portfolio containing all Arrow-Debreu securities and that of unit discount bond must be equal since both pay precisely 1 unit at time 1 in every state. Hence,

$$\frac{1}{1+r_f} = \sum_{\omega} \psi_{\omega} = \sum_{\omega} P_i(\omega) \frac{U'_{i1}(C^*_{i1}(\omega))}{U'_{i0}(C^*_{i0})} \quad \text{for every agent } i.$$

Alternative approach – individual agent's consumption optimization under market completeness (all investment choices are attainable)

Assume that there is a complete market of Arrow-Debreu securities and the prices are taken as given. The agent's problem is

$$\max_{C_{i,0},C_{i,1}} \left\{ U_{i0}(C_{i0}) + \sum_{\omega \in \Omega} P_i(\omega) U_i(C_{i1}(\omega)) \right\}$$

subject to the constraint

$$e_{i_0} - C_{i_0} = \sum_{\omega \in \Omega} \psi_{\omega} [C_{i_1}(\omega) - e_{i_1}(\omega)] =$$
 initial value of portfolio.

Form the Lagrangian

$$U_{i0}(C_{i0}) + \sum_{\omega \in \Omega} P_i(\omega) U_{i1}(C_{i1}(\omega))$$

+ $\gamma_i \left[e_{i0} + \sum_{\omega \in \Omega} \psi_{\omega} e_{i1}(\omega) - C_{i0} - \sum_{\omega \in \Omega} \psi_{\omega} C_{i1}(\omega) \right]$

We differentiate this Lagrangian with respect to the consumption variables since they can be used as control variables when the market is complete.

The following first-order conditions are both necessary and sufficient for the maximum since the utility functions are assumed to be concave:

$$U_{i0}'(C_{i0}) = \gamma_i, \quad \text{for all } i,$$

 $P_i(\omega)U'_{i1}(C_{i1}(\omega)) = \gamma_i\psi_\omega$, for all *i* and ω .

Taking the ratio of the two equations, we recover

$$\psi_{\omega} = P_i(\omega) \frac{U'_{i1}(C^*_{i1}(\omega))}{U'_{i0}(C^*_{i0})}.$$

Equilibrium concept

In equilibrium, the prices that are resulted from the individual optimizing action of the agent in the economy. Assume that the agents in the market optimize directly over trading strategies. We do not assume that the securities market is complete. The trading strategies can be used to alter the agent's consumption patterns.

Assumptions

1. There are I agents in the economy. Each agent has a utility function of the form

$$U_{i0}(C_{i0}) + U_{i1}(C_{i1}),$$

where U's are increasing concave and twice differentiable.

2. There are N securities in the market. Security j has a payoff of $X_j(\omega)$ at time 1 in state ω . The time-0 price of security j is x_j .

Let Θ denote the entire set of trading strategies for the group of *I* traders. An equilibrium is a set of prices and trading strategies $\{x, \Theta\}$ such that

(i) Given these prices and trading strategies, each agent's expected utility is maximized subject to the constraint imposed by the agent's initial endowment. This furnishes $N \times I$ first order conditions.

e.g. the first order condition for agent i with respect to θ_i^j is

$$U_{i0}' \underbrace{\left(e_{i0} - \sum_{n=1}^{N} \theta_i^n x_n\right)}_{i = 1, \cdots, I, j = 1, \cdots, N} x_j = \sum_{\omega} P_i(\omega) U_{i1}' \underbrace{\left(e_{i1} + \sum_{n=1}^{N} \theta_i^n X_n(\omega)\right)}_{i = 1, \cdots, N} X_j(\omega)$$

(ii) The market for each security clears, that is,

$$\sum_{i=1}^{I} \theta_i^n = 0, \quad n = 1, 2, \cdots, N.$$

The solution of $N \times I$ first order equations and N market-clearing conditions gives the optimal trading strategies and the equilibrium prices. There are N(I+1) equations for N(I+1) unknowns.

Remark

The securities are all non-redundant.
Remark

The market-clearing conditions are consistent with the requirement that total consumption = total endowment in each state.

* Total consumption at time
$$0 = \sum_{i=1}^{I} \left(e_{i0} - \sum_{n=1}^{N} \theta_i^n x_n \right)$$
$$= \sum_{i=1}^{I} e_{i0} = \text{total endowment}$$

 \star Total consumption at time 1 in state ω

$$=\sum_{i=1}^{I}\left(e_{i1}(\omega)-\sum_{n=1}^{N}\theta_{i}^{n}X_{n}(\omega)\right)=\sum_{i=1}^{I}e_{i1}(\omega).$$

Summary of concepts of equilibrium

- For each set of prices, the agents determine the trading strategies that optimize their preferences. How about the process by which the prices get to equilibrium?
- The equilibrium prices are those for which the optimal trading strategies equalize the supply of and demand for each security.
- In equilibrium [characterized by price and optimized trading strategies]
 - (i) each agent uses trading strategies that optimize the agent's preferences for the given market structure and prices;
 - (ii) the market clears so that the purchases and sales of each security are in balance.

• Inputs that generate the equilibrium prices and trading strategies

- (i) initial endowments,
- (ii) preferences and beliefs of the agents,
- (iii) security market structure.

The equilibrium prices emerge from a constrained optimization problem that uses these inputs. Say, if we add a new security that cannot be replicated by existing securities, then the resulting prices and equilibrium allocations also change. Example - 3 states and 3 agents

- * Each agent has an initial endowment of 16 units at time 0 and 50 units at time 1 in each state.
- * Each agent has the same utility function: $\sqrt{C_0}$ and $\sqrt{C_1}$.
- * The agents have different subjective probabilities of the states:

| <u>Agent</u> | <u>State 1</u> | <u>State 2</u> | <u>State 3</u> |
|--------------|----------------|----------------|----------------|
| One | 0.50 | 0.25 | 0.25 |
| Two | 0.25 | 0.50 | 0.25 |
| Three | 0.25 | 0.25 | 0.50 |

* Specification of the securities available for trading

Assume that the 3 Arrow-Debreu securities are traded so that the securities market is complete.

Under market completeness, all consumption allocations are attainable.

15 unknowns — 12 consumption amounts (4 for each agent) and 3 securities prices

15 equations, namely,

(i) 9 first-order conditions:
$$\psi_{\omega} = P_i(\omega) \frac{U'_{i1}(C^*_{i1}(\omega))}{U'_{i0}(C^*_{i0})}$$
, $i = 1, 2, 3$ and $\omega = \omega_1, \omega_2, \omega_3$.

- (ii) 3 equations that equate market value of each agent's optimal consumption allocation with the market value of the agent's original endowment (one equation for each agent).
- (iii) 3 equations that equate total aggregate consumption with total aggregate endowment in each state.

The equilibrium allocations for the three agents are found to be

| <u>agent</u> | <u>time 0</u> | <u>State 1</u> | <u>State 2</u> | <u>State 3</u> |
|--------------|---------------|----------------|----------------|----------------|
| One | 16 | 100 | 25 | 25 |
| Two | 16 | 25 | 100 | 25 |
| Three | 16 | 25 | 25 | 100 |

The price of the three Arrow-Debreu securities are 0.2. It can be checked that they satisfy

$$\psi_{\omega} = P_i(\omega) \frac{U'_{i1}(C^*_{i1}(\omega))}{U'_{i0}(C^*_{i0})}.$$

• The allocations satisfy the 9 first-order conditions (one equation of each agent and each state). Hence, each agent's expected utility is maximized.

Consider agent one, he has an endowment at time 1 of 50 units in all 3 states.

 Sell the Arrow-Debreu securities associated with state 2 and 3, by selling 25 units of each security. This produces an inflow of 10 units. The amount is used to buy 50 units of the state-1 Arrow-Debreu security.

| <u>Agent</u> | <u>Security 1</u> | <u>Security 2</u> | Security 3 |
|--------------|-------------------|-------------------|------------|
| one | 50 | -25 | -25 |
| two | -25 | 50 | -25 |
| three | -25 | -25 | 50 |

- Check the market clearing conditions.
- The total amount of consumption in each state is 150, which equals the total endowment.

• Sum of prices of the Arrow-Debreu securities is $\frac{1}{1+r_f} = 0.6$ so that the one-period risk-free interest rate is 66.67%.

The ability to trade in the market improves overall welfare.

(i) expected utility of agent one if initial endowment were retained $= \sqrt{16} + \frac{1}{2}\sqrt{50} + \frac{1}{4}\sqrt{50} + \frac{1}{4}\sqrt{50} = 11.0711.$

(ii) expected utility assuming optimal consumption

$$=\sqrt{16} + \frac{1}{2}\sqrt{100} + \frac{1}{4}\sqrt{25} + \frac{1}{4}\sqrt{25} = 11.50.$$

Connection between P measure and Q measure

• The risk neutral measure Q represents probability measure that is generated using the prices of the existing traded assets.

Equilibrium price of security $j = x_j = \sum_{\omega} P(\omega) X_j(\omega) Z(\omega)$, where Z is the ratio of marginal utilities between time 1 and time 0.

Since the riskfree bond pays 1 unit in each state at time 1,

$$\frac{1}{1+i_f} = e^{-r} = \sum_{\omega} P(\omega) Z(\omega), Z(\omega) > 0.$$

Actually, the riskfree interest rate is determined as derived solution of the equilibrium model.

Define $q(\omega) = (1 + i_f)P(\omega)Z(\omega), \omega \in \Omega$. Note that

$$q(\omega) > 0$$
 and $\sum_{\omega \in \Omega} q(\omega) = 1$

so that $q(\omega)$ defines a probability measure. Under this measure

$$x_j = \frac{1}{1+i_f} \sum_{\omega} q(\omega) X_j(\omega) = \frac{E^Q[X]}{1+i_f}.$$

This is just the risk neutral valuation formula.

These q probabilities relate prices x_j and $X_j(\omega)$ and they are in general different from those of P measure.

Two expectation representations of x_j in terms of $X_j(\omega)$:

$$x_{j} = E^{P}[Z_{i}X_{j}] = \sum_{\omega} \frac{P(\omega)U_{i1}'(C_{i1}^{*}(\omega))}{U_{i0}'(C_{i0}^{*})}X_{j}(\omega)$$

and

$$x_j = \frac{E^Q[X]}{1+i_f} = \frac{1}{1+i_f} \sum_{\omega} q(\omega) X_j(\omega)?$$

Suppose a particular agent chooses the risk neutral utility function $C_0 + e^{-\rho}C_1$, where ρ is the rate of time preference. We then have $Z(\omega) = e^{-\rho}$. If we set $e^{-\rho} = e^{-r}$, then Q and P coincide:

$$q(\omega) = e^{-r} e^{\rho} P(\omega) = P(\omega)$$
 if $\rho = r$.

This explains the term "risk-neutral measure".

Current price is given by the discounted expectation under the subjective measure of the terminal payoff when the risk neutral utility $C_0 + e^{-r}C_1$ is chosen.

Equilibrium solution and Pareto efficiency

The *i*th trader has consumption process $C^i = (C_0^i, C_1^i)$ and trading strategy: $H^i = (H_0^i, H_1^i, \dots, H_N^i)$.

Recall the formulation of an equilibrium model:

The variables $S_n(0), n = 1, \dots, N$ and $\{C^i, H^i\}, i = 1, \dots, I$, are said to be an *equilibrium solution* if for each *i* the consumption investment plan (C^i, H^i) is optimal for investor *i*, that is, (C^i, H^i) is a solution of the following maximization problem. maximize

$$U_i(C_0^i) + E[U_i(C_1^i)]$$

subject to

$$C_{0}^{i} + H_{0}^{i}B_{0} + \sum_{n=1}^{N} H_{n}^{i}S_{n}(0) = v_{i}$$
$$C_{1}^{i} - H_{0}^{i}B_{1} - \sum_{n=1}^{N} H_{n}^{i}S_{n}(1) = E_{i}$$
$$H^{i} \in \mathbb{R}^{N+1}$$

and the security market clears, that is, the aggregate demand for each security is zero

$$\sum_{i=1}^{I} H_{n}^{i} = 0 \quad \text{for} \quad n = 0, 1, \cdots, N.$$

Remark

There is no explicit constraint that requires the consumption to be non-negative. One can specify utility functions that would force the consumption to be non-negative. Aggregate consumptions = aggregate endowments

Adding the time-0 budget constraint across i

$$B_{0}\sum_{\substack{i=1\\ \text{zero}}}^{I}H_{0}^{i} + \sum_{n=1}^{N}S_{n}(0)\sum_{\substack{i=1\\ \text{zero}}}^{I}H_{n}^{i} = \sum_{i=1}^{I}v_{i} - \sum_{i=1}^{I}C_{0}^{i}.$$

If this is an equilibrium solution, then LHS = 0 = RHS. We can deduce similar result from the t = 1 budget constraints. Therefore, suppose the consumption processes $C^i, i = 1, \dots, I$ are part of an equilibrium solution, then

$$\sum_{i=1}^{I} C_{0}^{i} = \sum_{i=1}^{I} v_{i} \text{ and } \sum_{i=1}^{I} C_{1}^{i} = \sum_{i=1}^{I} E_{i}$$

Feasible processes

The processes $\{C_0^i, C_1^i\}$ are said to be *feasible* if they satisfy the above two constraints.

Pareto efficient

The collection $\{\hat{C}^1, \dots, \hat{C}^I\}$ of consumption processes is said to be Pareto efficient if they are feasible and there is no other collection $\{C^1, \dots, C^I\}$ of feasible consumption processes such that

$$U_i(C_0^i) + EU_i(C_1^i) \ge U_i(\hat{C}_0^i) + EU_i(\hat{C}_1^i), \quad i = 1, \cdots, I$$
 (1)

with inequality being strict for at least one i.

This is a very weak from efficiency. For example, the allocation of all consumptions to one single agent is Pareto efficient.

Theorem

If the model is complete and $\{\hat{C}^1, \dots, \hat{C}^I\}$ is part of an equilibrium solution, then $\{\hat{C}^1, \dots, \hat{C}^I\}$ is Pareto efficient.

Proof by contradiction

Assume there exists a process $\{C^1, \dots, C^I\}$ which is feasible and satisfies (1), where $\{\hat{C}^{,} \dots, \hat{C}^I\}$ is part of an equilibrium solution.

Since the model is complete, for each investor i, there exists a trading strategy H^i satisfying

$$H_0^i B_1 + \sum_{n=1}^N H_n^i S_n(1) = C_1^i - E_i.$$
 (a)

Define the scalars:

$$\psi_i = C_0^i - v_i + H_0^i B_0 + \sum_{n=1}^N H_n^i S_n(0), i = 1, \cdots, I.$$
 (b)

Think of $C_0^i - \psi_i$ as the time t = 0 consumption for investor *i*.

The consumption process $\{C_0^i - \psi_i, C_1^i\}$ is seen to be attainable by virtue of eqs (a) and (b).

Next step: we try to argue $\psi_i \ge 0, i = 1, \dots, I$. How?

If $\psi_i < 0$, then investor *i* prefers (strictly) the consumption process $\{C_0^i - \psi_i, C_1^i\}$ to $\{C_0^i, C_1^i\}$. Also, $\{C_i^0 - \psi_i, C_1^i\}$ satisfies the budget constraints in investor *i*'s optimization problem, and so $\{C_0^i, C_1^i\}$ is an optimal solution rather than $\{\hat{C}_0^i, \hat{C}_1^i\}$.

By similar logic, $\psi_i = 0$ leads to a contradiction since $U_i(C_0^i) + EU_i(C_1^i) > U_i(\hat{C}_0^i) + EU_i(\hat{C}_1^i)$ at least for one *i*. We then have $\psi_i > 0$.

If we sum

$$0 < \sum_{i=1}^{I} \psi_i = \sum_{i=1}^{I} C_0^i - \sum_{i=1}^{I} v_i + \underbrace{\left(\sum_{i=1}^{I} H_0^i\right) B_0 + \sum_{n=1}^{N} \left(\sum_{i=1}^{I} H_n^i\right) S_n(0)}_{\text{Zero}},$$

so that

$$\sum_{i=1}^{I} C_0^i > \sum_{i=1}^{I} v_i.$$

But this contradicts with the feasibility condition for the consumption process $\{C^1, C^2, \dots, C^I\}$. Hence, we conclude that the collection $\{\hat{C}^1, \hat{C}^2, \dots, \hat{C}^I\}$ must be Pareto efficient.

Numerical example

Suppose $N = 2, K = 3, P(\omega) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ and constant interest rate.

There are I identical investors with $u_i(\omega) = \ln \omega$, and $v_i = v$. All investors share common beliefs about time t = 1 prices in each of the states.

| n | $S_{n}^{*}(1)$ | | |
|---|----------------|------------|--------------|
| | ω_1 | ω_2 | ω_{3} |
| 1 | 6 | 8 | 4 |
| 2 | 13 | 9 | 8 |

We assume that $C_0^i = 0$ and $E_i = 0$ for all *i* and ignore the utility of time t = 0 consumption.

The securities prices at time t = 0 are to be determined such that

$$\sum_{i=1}^{I} H_{n}^{i} = s_{n} \text{ for } n = 1, 2, \cdots, N,$$

where $s_n > 0$ is the total supply of shares of security n.

The N companies are making initial public offerings of their securities. They can assess the correct t = 1 prices and they want to set t = 0 offering prices properly.

First, we compute the risk neutral probability measure as a function of the unknown time t = 0 prices.

From $E_Q[S_n^*(1)] = S_n(0), n = 1, 2$; and $\Sigma Q(\omega) = 1$, we obtain

$$Q(\omega) = \begin{cases} \frac{-28 - S_1(0) + 4S_2(0)}{18} & \omega = \omega_1, \\ \frac{-4 + 5S_1(0) - 2S_2(0)}{18} & \omega = \omega_2, \\ \frac{50 - 4S_1(0) - 2S_2(0)}{18} & \omega = \omega_3. \end{cases}$$

In order that $Q(\omega)$ is strictly positive and less that 1, $S_1(0)$ and $S_2(0)$ must lie within the triangle with vertices at (4,8), (6,13) and (8,9). These serve as constraints on the time t = 0 prices if there is a risk neutral probability measure.

• Optimal problem

Maximize $Eu(W) - \lambda \{E_Q[W/B_1] - v\} = E[u(\omega) - \lambda (LW/B_1 - v)]$, where L = Q/P with $E_Q[W/B_1] = v$.

Recall the solution procedure:-

first order condition: $u'(W(\omega)) = \lambda L(\omega)/B_1(\omega), \quad \omega \in \Omega.$

Write \widehat{I} as the inverse function for u', then

 $W(\omega) = \widehat{I}[\lambda L(\omega)/B_1(\omega)].$

 λ is solved by $E_Q[\widehat{I}[\lambda L(\omega)/B_1(\omega)]/B_1(\omega)] = v.$

Suppose we choose $u = \ln w$ so that $I = \frac{1}{w}$,

$$E_Q\left[\frac{1}{\lambda L(\omega)}\right] = v \quad \text{or} \quad E_P\left[\frac{1}{\lambda}\right] = v \text{ so that } \lambda = \frac{1}{v}.$$
$$W(\omega) = I\left[\frac{1}{v}L(\omega)/B_1\right] = vB_1\frac{P(\omega)}{Q(\omega)}.$$

Now, $W(\omega)/B_1 = H_0 + H_1 S_1^*(1; \omega) + H_2 S_2^*(1; \omega)$, we obtain

$$H_{1}^{i}(S_{0}) = \frac{-\frac{1}{3}v}{-28 - S_{1}(0) + 4S_{2}(0)} - \frac{\frac{4}{3}v}{50 - 4S_{1}(0) - 2S_{2}(0)} + \frac{\frac{5}{3}v}{-4 + 5S_{1}(0) - 2S_{2}(0)}.$$

$$H_{2}^{i}(S_{0}) = \frac{\frac{4}{3}v}{-28 - S_{1}(0) + 4S_{2}(0)} - \frac{\frac{2}{3}v}{50 - 4S_{1}(0) - 2S_{2}(0)} - \frac{\left(\frac{2}{3}\right)v}{-4 + 5S_{1}(0) - 2S_{2}(0)}.$$

The dependence of H_n^i on S_0 can be visualized as the demand function for security n for agent i. The existence of H_n^i is consistent with the absence of arbitrage, otherwise the investors may long or short an infinite number of units of some of the securities. Suppose $s_1 = 4000, s_2 = 2000$ shares of securities 1 and 2 are available, v = \$6,000, and I = 2.

We solve for $S_1(0)$ and $S_2(0)$ and obtain

 $S_1(0) = 5$ and $S_2(0) = 9$. Once $S_1(0)$ and $S_2(0)$ are known, we obtain $H_1^i = 2000$ and $H_2^i = 1000$. Lastly, $H_0 = v - H_1S_1(0) - H_2S_2(0) = -13,000$.

All investors borrow \$13,000 in order to finance the trades.