MATH 685Z – Mathematical Models in Finance Economics

Homework Six

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1. (Wild cats) Suppose there are n assets which are uncorrelated. (They might be n different "wild cat" oil well prospects.) You may invest in any one, or in any combination of them. The mean rate of return $\overline{r}$ is the same for each asset, but the variances are different. The return on asset i has a variance of σ_i^2 for $i = 1, 2, \cdots, n$.

   (a) Show the situation on an $\sigma - \overline{r}$ diagram. Describe the efficient set.
   (b) Find the minimum-variance point. Express your result in terms of

   $$\overline{\sigma}^2 = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1}.$$

2. (Markowitz fun) There are just three assets with rates of return r_1, r_2 and r_3, respectively. The covariance matrix and the expected rates of return are

   $$\Omega = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \overline{\boldsymbol{r}} = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.8 \end{pmatrix}.$$

   (a) Find the minimum-variance portfolio. [*Hint*: By symmetry $w_1 = w_3$.]
   (b) Find another efficient portfolio by setting $\lambda_1 = 0$ and $\lambda_2 = 1$.
   (c) Let the risk-free rate r_f of the risk free asset be 0.2, find the efficient portfolio consisting of the 3 risky assets and the risk free asset with target expected rate of return of the portfolio set at 0.4.

3. (Betting wheel) Consider a general betting wheel with n segments. The payoff for a \$1 bet on a segment i is A_i. Suppose you bet an amount $B_i = 1/A_i$ on segment i for each i. Show that the amount you win is independent of the outcome of the wheel. What is the risk-free rate of return for the wheel?

4. Suppose that it is impractical to use all the assets that are incorporated into a specified portfolio (such as a given efficient portfolio). One alternative is to find the portfolio, made up a given set of n stocks, that tracks the specified portfolio most closely — in the sense of minimizing the variance of the difference in returns.

   Specifically, suppose that the target portfolio has (random) rate of return r_M. Suppose that there are n assets with (random) rate of return $r_1, r_2, \cdots, r_n$. We wish to find the portfolio rate of return

   $$r = \alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n$$

   $\left(\text{with } \sum_{i=1}^{n} \alpha_i = 1 \right)$ minimizing $\text{var}(r - r_M)$.

   (a) Find a set of equations for the α_i's. Solve for α_i's.

   (b) Although this portfolio tracks the desired portfolio most closely in terms of variance, it may sacrifice the mean. Hence a logical approach is to minimize the variance of the tracking error subject to achieving a given mean return. As the mean is varied, this results in a family of portfolios that are efficient in a new sense — say, tracking efficient. Find the equation for the α_i's that are tracking efficient. Solve for α_i's.

5. Consider the special case of $r = b/a$ in the discussion of the One-fund Theorem, where r is the rate of return of the risk free asset, $a = \mathbf{1}^T\Omega^{-1}\mathbf{1}$ and $b = \mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}$. Apparently, the optimal weight vector $\boldsymbol{w}^*$ of the risky assets is given by

$$\boldsymbol{w}^* = \lambda\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}),$$

where λ is any scalar multiple. Since $r = b/a$, it can be shown that $\mathbf{1}^T\boldsymbol{w}^* = 0$. Explore the properties on the set of minimum-variance portfolios as λ varies in value, say, whether the minimum-variance portfolio corresponding to a given value of λ lies on the upper or lower half line.

6. Let $\boldsymbol{w}_0$ be the portfolio (weights) of risky assets corresponding to the global minimum-variance point in the feasible region. Let $\boldsymbol{w}_1$ be any other portfolio on the efficient frontier. Define r_0 and r_1 to be the corresponding rate of returns.

   (a) There is a formula of the form $\text{cov}(r_0, r_1) = A\sigma_0^2$. Find A.

   Hint: Consider the set of portfolios $(1-\alpha)\boldsymbol{w}_0 + \alpha\boldsymbol{w}_1$ with varying values of α, and consider small variations of the variance of such portfolios near $\alpha = 0$.

   (b) Corresponding to the portfolio $\boldsymbol{w}_1$ there is a portfolio $\boldsymbol{w}_z$ on the minimum-variance set such that $\text{cov}(r_1, r_z) = 0$, where r_z is the rate of return of portfolio $\boldsymbol{w}_z$. This portfolio can be expressed as $\boldsymbol{w}_z = (1-\alpha)\boldsymbol{w}_0 + \alpha\boldsymbol{w}_1$. Find the proper value of α.

   (c) Show the relation of the three portfolios on a diagram that includes the feasible region.

7. Solve the optimal portfolio selection model with the inclusion of the risk free asset under the risk tolerance framework. Let r be the riskfree rate, w_0 be the optimal weight of the risk free asset, $\boldsymbol{w} = (w_1 \cdots w_N)^T$ and $\boldsymbol{\mu} = (\mu_1 \cdots \mu_N)^T$. Recall that $\sum_{i=0}^{N} w_i = 1$. The portfolio expected return μ_P and σ_P^2 are given by

$$\mu_P = w_0 r + \boldsymbol{\mu}^T\mathbf{1} \quad \text{and} \quad \sigma_P^2 = \boldsymbol{w}^T\Omega\boldsymbol{w}.$$

The formulation of the model is to maximize

$$\tau\mu_P - \frac{\sigma_P^2}{2} \quad \text{subject to } \boldsymbol{w}^T\mathbf{1} + w_0 = 1,$$

where τ is the risk tolerance parameter.

8. In the asset-liability model, show the steps that lead to the formula for the weight of the optimal portfolio:

$$\boldsymbol{x}^* = \boldsymbol{x}^{min} + \boldsymbol{z}^L + \tau\boldsymbol{z}^*, \quad \tau \geq 0,$$

where

$$\begin{aligned} \boldsymbol{x}^{min} &= \frac{1}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}\Omega^{-1}\mathbf{1}, \\ \boldsymbol{z}^L &= \Omega^{-1}\boldsymbol{\gamma} - \frac{\mathbf{1}^T\Omega^{-1}\boldsymbol{\gamma}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}\Omega^{-1}\mathbf{1}, \\ \boldsymbol{z}^* &= \Omega^{-1}\boldsymbol{\mu} - \frac{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}\Omega^{-1}\mathbf{1}. \end{aligned}$$

Here, $\boldsymbol{\mu} = (\mu_1 \cdots \mu_N)^T, \mu_i = E[r_i]$,

$$\boldsymbol{\gamma} = (\gamma_1 \cdots \gamma_N)^T, \gamma_i = \frac{1}{f_0}\text{cov}(r_i, r_L), \quad i = 1, 2, \cdots, N.$$