MATH246 — Probability and Random Processes

Solution to Homework Two

1. Given $P[X > t] = e^{-\lambda t}, t > 0$

   (a) When $x \le t$,

$$F_X(x|X > t) = P[X \le x|X > t] = 0;$$

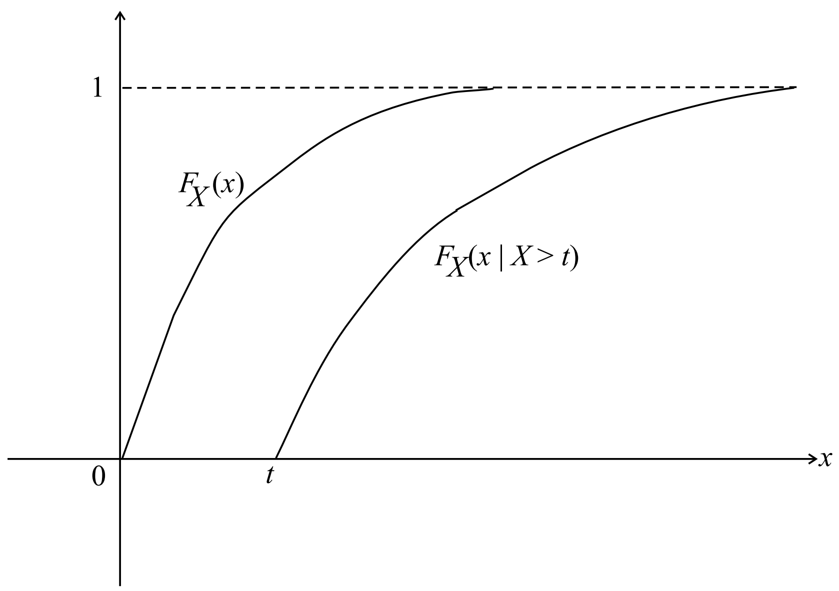
   when $x > t$,

$$\begin{aligned} F_X(x|X > t) &= P[X \le x|X > t] \\ &= 1 - P[X > t + (x - t)|X > t] \\ &= 1 - P[X > x - t] \quad \text{by memoryless property} \\ &= 1 - e^{-\lambda(x-t)} \end{aligned}$$

   so

$$F_X(x|X > t) = \begin{cases} 0, & x \le t \\ 1 - e^{-\lambda(x-t)}, & x > t \end{cases}.$$

   As shown in the figure below, $F_X(x|X > t)$ is a shift of $F_X(x)$ by t units.



   (b) Note that

$$\begin{aligned} F_X(x|X > t) &= P[X \le x|X > t] \\ &= \frac{P[t < X \le x]}{P[X > t]} \\ &= \begin{cases} 0, & x \le t \\ \frac{F_X(x) - F_X(t)}{1 - F_X(t)}, & x > t \end{cases}. \end{aligned}$$

   Hence, for $x \ne t$,

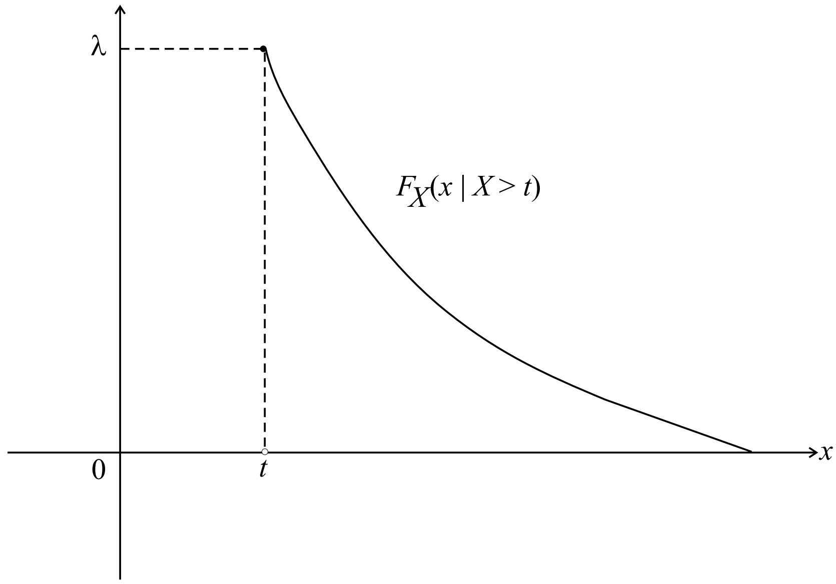
$$\begin{aligned} f_X(x|X > t) &= \frac{d}{dx} F_X(x|X > t) \\ &= \begin{cases} 0, & x < t \\ \frac{f_X(x)}{1 - F_X(t)}, & x > t \end{cases}. \end{aligned}$$

For $x = t$,

$$\begin{aligned} f_X(t|X>t) &= \lim_{h\to 0^+} \frac{F_X(t+h|X>t) - F_X(t|X>t)}{h} \\ &= \lim_{h\to 0^+} \frac{\frac{F_X(t+h)-F_X(t)}{1-F_X(t)} - 0}{h} \\ &= \frac{1}{1-F_X(t)} \lim_{h\to 0^+} \frac{F_X(t+h)-F_X(t)}{h} \\ &= \frac{f_X(t)}{1-F_X(t)}. \end{aligned}$$

In summary,

$$f_X(x|X>t) = \begin{cases} 0, & x < t \\ \frac{f_X(x)}{1-F_x(t)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda t}} = \lambda e^{-\lambda(x-t)}, & x \geq t \end{cases}.$$



(c) $$\begin{aligned} P[X>t+x|X>t] &= \frac{P[X>t+x, X>t]}{P[X>t]} \\ &= \frac{P[X>t+x]}{[X>t]} \\ &= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} \\ &= e^{-\lambda x} \\ &= P[X>x]. \end{aligned}$$

Due to the memoryless property, the probability of waiting at least an additional x second is the same regardless of how long one has already been waiting.

2. Given $P[X = k] = (1-p)^{k-1}p, k = 1, 2, 3, \cdots$. We have

$$P[X \leq x] = P[X \leq \lfloor x \rfloor]$$

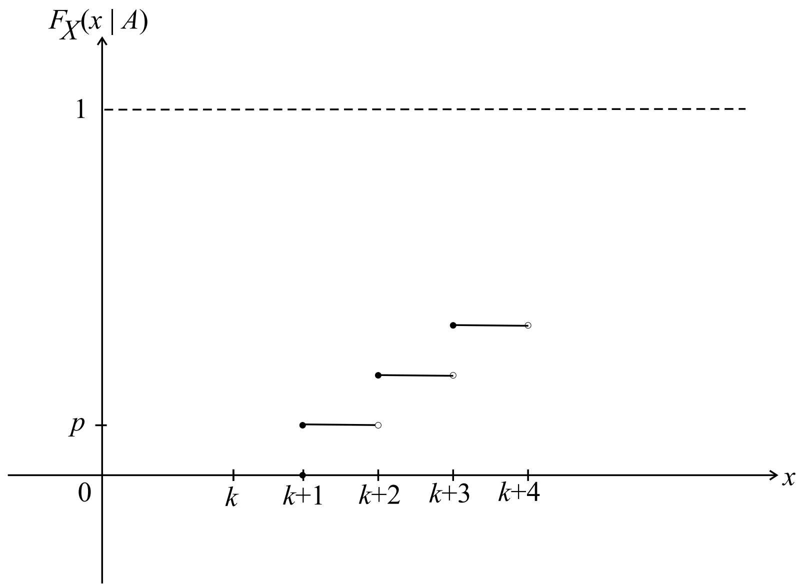
$$
\begin{aligned}
&= \sum_{k=1}^{\lfloor x \rfloor} (1-p)^{k-1} p \\
&= p \cdot \frac{1-(1-p)^{\lfloor x \rfloor}}{1-(1-p)} \\
&= 1-(1-p)^{\lfloor x \rfloor}.
\end{aligned}
$$

(a) $A = \{X > k\}$
When $\lfloor x \rfloor \le k$,

$$F_X(x|X > k) = P[X \le x|X > k] = 0.$$

When $\lfloor x \rfloor > k$,

$$
\begin{aligned}
F_X(x|X > k) &= P[X \le x|X > k] \\
&= 1 - P[X > k + (x-k)|X > k] \\
&= 1 - P[X > x - k] \quad \text{by memoryless property} \\
&= P[X \le x - k] \\
&= 1 - (1-p)^{\lfloor x-k \rfloor}.
\end{aligned}
$$



(b) $A = \{X < k\}$
When $\lfloor x \rfloor < 1$,

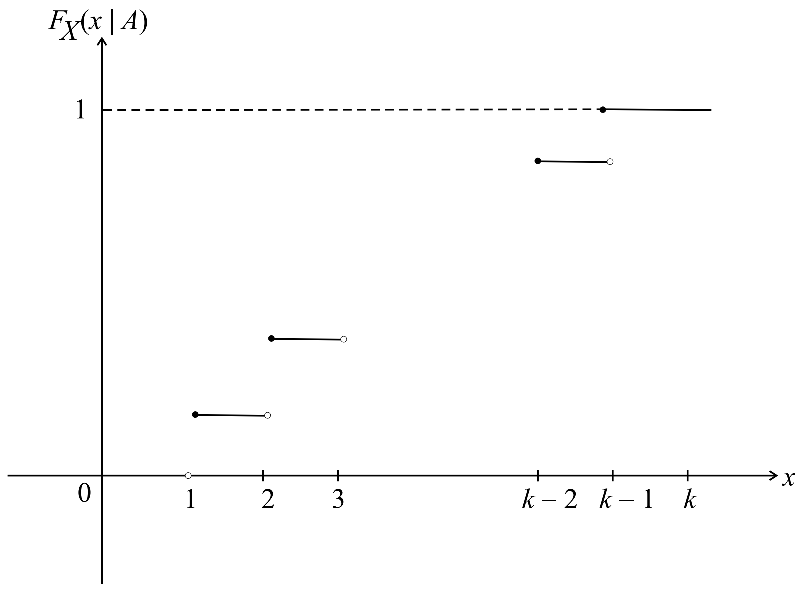
$$F_X(x|X < k) = P[X \le x|X < k] = 0.$$

When $\lfloor x \rfloor \ge k-1$,

$$F_X(x|X < k) = P[X \le x|X < k] = 1.$$

When $1 \le \lfloor x \rfloor < k-1$,

$$\begin{aligned}
F_X(x|X<k) &= P[X \le x|X<k] \\
&= \frac{P[X \le x, X<k]}{P[X<k]} \\
&= \frac{P[X \le x]}{P[X<k]} \\
&= \frac{1-(1-p)^{\lfloor x \rfloor}}{1-(1-p)^{k-1}}
\end{aligned}$$



(v) $A = \{X \text{ is an even number }\}$

$$\begin{aligned}
P[A] &= \sum_{k=1}^{\infty} P[X=2k] \\
&= \sum_{k=1}^{\infty} (1-p)^{2k-1} p \\
&= \frac{p}{1-p} \sum_{k=1}^{\infty} [(1-p)^2]^k \\
&= \frac{p}{1-p} \cdot \frac{(1-p)^2}{1-(1-p)^2} = \frac{p(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}. \qquad (1)
\end{aligned}$$

$$\begin{aligned}
p[X \le x, A] &= \sum_{k=1}^{\lfloor x/2 \rfloor} P[X=2k] \\
&= \sum_{k=1}^{\lfloor x/2 \rfloor} (1-p)^{2k-1} p \\
&= \frac{p}{1-p} \sum_{k=1}^{\lfloor x/2 \rfloor} [(1-p)^2]^k
\end{aligned}$$

$$= \frac{p}{1-p} \cdot \frac{(1-p)^2[1-(1-p)^{2\lfloor x/2 \rfloor}]}{1-(1-p)^2}$$

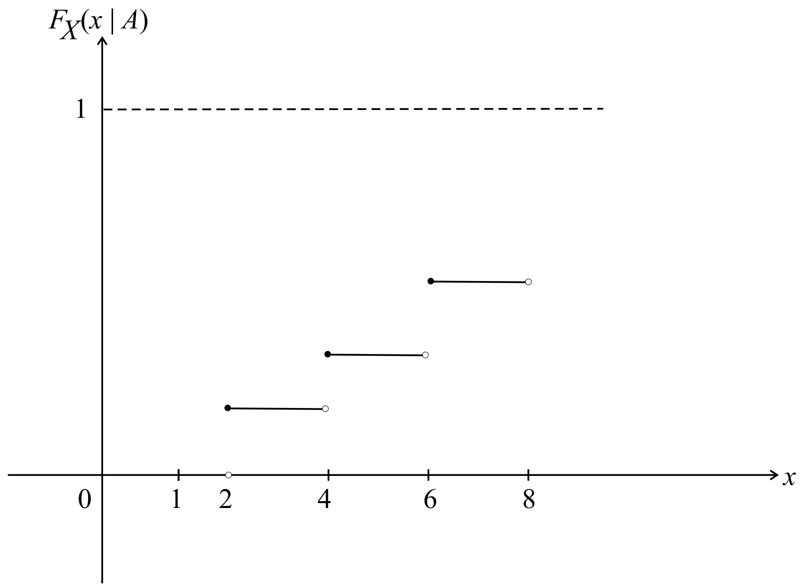
$$= \frac{1-p}{2-p}[1-(1-p)^{2\lfloor x/2 \rfloor}]. \quad (2)$$

When $\lfloor x \rfloor < 2$

$$F_X(x|A) = P[X \leq x | X \text{ is even}] = 0.$$

When $\lfloor x \rfloor \geq 2$,

$$\begin{aligned} F_X(x|A) &= P[X \leq x|A] \\ &= \frac{P[X \leq x, A]}{P[A]} \\ &= \frac{1-p}{2-p}[1-(1-p)^{2\lfloor x/2 \rfloor}] \Big/ \frac{1-p}{2-p} \quad \text{by (1) \& (2)} \\ &= 1-(1-p)^{2\lfloor x/2 \rfloor}. \end{aligned}$$



3. Let N = number of messages arrived in 1 second. Given that N is a Poisson random variable with $\alpha = 15$, i.e.,

$$P[N = k] = \frac{15^k}{k!}e^{-15}, \quad k = 0, 1, 2, \cdots.$$

(a) The required probability $= P[N = 0] = e^{-15} = 3.0590 \times 10^{-7}$.

(b) The required probability

$$\begin{aligned} &= P[N > 10] = 1 - P[N \leq 10] \\ &= 1 - e^{-15} - \frac{15^1}{1!}e^{-15} - \frac{15^2}{2!}e^{-15} - \cdots - \frac{15^{10}}{10!}e^{-15} \\ &= 1 - e^{15}\left(1 + 15 + \frac{15^2}{2!} + \cdots + \frac{15^{10}}{10!}\right) \\ &= 1 - 0.118 = 0.882. \end{aligned}$$

4. Let N = number of silent error during the given week, where N is binomial random variable with $n = 20$ and $p = 0.1$, that is,

$$P[N = k] = C_k^n (1-p)^{n-k} p^k.$$

(a) The required probability $= P[N = 0] = C_0^{20}(1-0.1)^{20} = 0.1216$.

(b) The required probability $= P[N \geq 1] = 1 - P[N = 0] = 0.8784$.

(c) The required probability

$$\begin{aligned} &= P[N > 4] = P[N \geq 1] - P[N = 1] - P[N = 2] - P[N = 3] - P[N = 4] \\ &= 0.8784 - \sum_{k=1}^{4} C_k^{20}(0.9)^{20-k}(0.1)^k \\ &= 0.8784 - 0.2702 - 0.2852 - 0.1901 - 0.0898 \\ &= 0.0431. \end{aligned}$$

Now, $P[N > 4] = 0.0431$ is small but in general "a small probability event should not happen in one experiment". Hence it would be unusual.

5. Given that X is Gaussian with mean m and variance σ^2, then $Z = \dfrac{X-m}{\sigma}$ is the standard normal random variable with mean 0 and variance 1. That is,

$$P[Z \leq x] = \phi(x).$$

(a) $P[X < m] = \dfrac{1}{2}$ by symmetry.

(b)
$$\begin{aligned} P[|X-m| > k\sigma] &= P\left[\left|\frac{x-m}{\sigma}\right| > k\right] \\ &= P[|Z| > k] \\ &= P[Z > k \quad \text{or} \quad Z < -k] \\ &= P[Z > k] + P[Z < -k] \\ &= 2P[Z > k] \quad \text{by symmetry} \\ &= 2[1 - \phi(k)]. \end{aligned}$$

From the table, we have

$P[\lvert X-m\rvert > k\sigma]$	0.318	0.0456	0.0027
k	1	2	3

(c)
$$\begin{aligned} P[X > m + k\sigma] &= P\left[\frac{X-m}{\sigma} > k\right] \\ &= P[Z > k] \\ &= 1 - \phi(k) \\ &= \begin{cases} 0.1 & k = 1.28 \\ 0.001 & k = 3.09 \end{cases}. \end{aligned}$$

6. (a)
$$\begin{aligned} F_Y(y) &= P[Y \leq y] = P[|x| \leq y] \\ &= P[-y \leq X \leq y] \\ &= F_X(y) - F_X(-y). \end{aligned}$$

$$f_Y(y) = \frac{d}{dy}F_y(Y) = \frac{d}{dy}[F_X(y) - F_X(-y)]$$
$$= f_X(y) + f_X(-y).$$

(b) $$\begin{aligned} f_Y(y) &= \lim_{dy\to 0} \frac{P[y < Y \leq y + dy]}{dy} \\ &= \lim_{dy\to 0} \frac{P[y < |X| \leq y + dy]}{dy} \\ &= \lim_{dy\to 0} \frac{P[y < X \leq y + dy]}{dy} + \lim_{dy\to 0} \frac{P[-y - dy \leq X < -y]}{dy} \\ &= f_X(y) + f_X(-y). \end{aligned}$$

The answer agrees with part (a).

(c) $f_X(x)$ is an even function $\Rightarrow f_X(-x) = f_X(x)$. Hence,

$$f_Y(y) = f_X(y) + f_X(-y) = 2f_X(y).$$

7. (a) $$\begin{aligned} F_Y(y) &= P[Y \leq y] = P[e^X \leq y] \\ &= P[X \leq \log y] = F_X(\log y) \\ f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(\log y) \\ &= f_X(\log y)\frac{d}{dy}\log y = \frac{1}{y}f_X(\log y). \end{aligned}$$

(b) When X is a Gaussian random variable with mean μ and variance σ^2, we have $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-(x-\mu)^2 \Big/ 2\sigma^2\right)$. Hence

$$f_Y(y) = \frac{1}{y}f_X(\log y) = \frac{1}{\sigma y\sqrt{2\pi}}\exp\left\{-(\log y - \mu)^2 \Big/ 2\sigma^2\right\}.$$

8. (a) Suppose we want to sort the array in ascending order. Let a be the largest element.

$$\begin{aligned} P[I = 0] &= P[\text{The largest element is in the correct position}] \\ &= P[\text{the last element is } a] \\ &= \frac{1}{n} \\ P[I = 1] &= P[\text{the last element is not equal to } a] \\ &= 1 - P[I = 0] = 1 - \frac{1}{n}. \end{aligned}$$

(b) $E[I] = 0 \cdot P[I = 0] + 1 \cdot P[I = 1] = 1 - \frac{1}{n}$

(c) Since $X_{\widetilde{n}} = X_{\widetilde{n}-1} + I$

$$\text{so} \qquad \begin{aligned} E[X_{\widetilde{n}}] &= E[X_{\widetilde{n}-1}] + E[I] \\ &= E[X_{\widetilde{n}-1}] + 1 - \frac{1}{\widetilde{n}}. \end{aligned} \tag{$*$}$$

Note that (*) is valid for $\widetilde{n} = n, n-1, \cdots, 3, 2$, so

$$\begin{aligned} E[X_n] &= E[X_{n-1}] + 1 - \frac{1}{n} && (1) \\ E[X_{n-1}] &= E[X_{n-2}] + 1 - \frac{1}{n-1} && (2) \\ &\vdots \\ E[X_3] &= E[X_2] + 1 - \frac{1}{3} && \text{(n-2)} \\ E[X_2] &= E[X_1] + 1 - \frac{1}{2}. && \text{(n-1)} \end{aligned}$$

(d) Summing Equations (1) to $(n-1)$ in part (c), we have

$$\begin{aligned} E[X_n] + \sum_{i=2}^{n-1} E[X_i] &= E[X_1] + \sum_{i=2}^{n-1} E[X_i] + (n-1) - \sum_{i=2}^{n} \frac{1}{i} \\ \Rightarrow \quad E[X_n] &= E[X_1] + (n-1) - \sum_{i=1}^{n} \frac{1}{i} \\ &= n - 1 - \sum_{i=2}^{n} \frac{1}{i} \quad \text{as} \quad E[X_1] = 0. \end{aligned}$$

(e) $E[X_5] = (5-1) - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{163}{60} = 2.7167.$

(f) When $n = 5$, $\sum_{i=2}^{5} \frac{1}{i} \approx \int_{1.5}^{5.5} \frac{1}{t}\, dt = \log \frac{5.5}{1.5}$, so $E[X_5] = (5-1) - \log \frac{5.5}{1.5} = 2.7007.$

The result is very close to the exact solution found in part (e) with roughly 0.6% error.

(g) The ideal average number of interchanges $= E[X_{100}]$

$$\begin{aligned} &= (100-1) - \int_{1.5}^{100.5} \frac{1}{t}\, dt \\ &= 99 - \log \frac{100.5}{1.5} \\ &= 94.7953. \end{aligned}$$