Worked examples — Multiple Random Variables

Example 1 Let X and Y be random variables that take on values from the set $\{-1, 0, 1\}$.

(a) Find a joint probability mass assignment for which X and Y are independent, and confirm that X^2 and Y^2 are then also independent.

(b) Find a joint pmf assignment for which X and Y are **not** independent, but for which X^2 and Y^2 are independent.

Solution

(a) We assign a joint probability mass function for X and Y as shown in the table below. The values are designed to observe the relations: $P_{XY}(x_k, y_j) = P_X(x_k)P_Y(y_j)$ for all k, j. Hence, the independence property of X and Y is enforced in the assignment.

$P_{XY}(x_k, y_j)$	$x_1 = -1$	$x_2 = 0$	$x_3 = 1$	$P_Y(y_j)$
$y_1 = -1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
$y_2 = 0$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$
$y_3 = 1$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{6}$
$P_X(x_k)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	

Given the above assignment for X and Y, the corresponding joint probability mass function for the pair X^2 and Y^2 is seen to be

$P_{X^2Y^2}(\widetilde{x_k}, \widetilde{y_j})$	$\widetilde{x}_1 = 1$	$\widetilde{x}_2 = 0$	$P_{Y^2}(\widetilde{y}_j)$
$\widetilde{y}_1 = 1$	$\frac{1}{12} + \frac{1}{4} + \frac{1}{36} + \frac{1}{12} = \frac{4}{9}$	$\frac{1}{6} + \frac{1}{18} = \frac{2}{9}$	$\frac{2}{3}$
$\widetilde{y}_2 = 0$	$\frac{1}{18} + \frac{1}{6} = \frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
$P_{X^2}(\widetilde{x}_k)$	$\frac{2}{3}$	$\frac{1}{3}$	

Note that $P_{X^2,Y^2}(\widetilde{x}_k, \widetilde{y}_j) = P_{X^2}(\widetilde{x}_k)P_{Y^2}(\widetilde{y}_j)$ for all k and j, so X^2 and Y^2 are also independent.

(b) Suppose we take the same joint pmf assignment for X^2 and Y^2 as in the second table, but modify the joint pmf for X and Y as shown in the following table.

$P_{XY}(x_k, y_j)$	$x_1 = -1$	$x_2 = 0$	$x_3 = 1$	$P_Y(y_j)$
$y_1 = -1$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{2}$
$y_2 = 0$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$
$y_3 = 1$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{6}$
$P_X(x_k)$	$\frac{7}{18}$	$\frac{1}{3}$	$\frac{5}{18}$	

This new joint pmf assignment for X and Y can be seen to give rise to the same joint pmf assignment for X^2 and Y^2 in the second table. However, in this new assignment, we observe that

$$\frac{1}{4} = P_{XY}(x_1, y_1) \neq P_X(x_1)P_Y(y_1) = \frac{7}{18} \cdot \frac{1}{2} = \frac{7}{36},$$

and the inequality of values can be observed also for $P_{XY}(x_1, y_3), P_{XY}(x_3, y_1)$ and $P_{XY}(x_3,\ y_3)$, etc. Hence, X and Y are **not** independent.

Remark

1. Since -1 and 1 are the two positive square roots of 1, we have

   $$P_X(1) + P_X(-1) = P_{X^2}(1) \quad \text{and} \quad P_Y(1) + P_Y(-1) = P_{Y^2}(1),$$

   therefore

   $$\begin{aligned} P_{X^2}(1)P_{Y^2}(1) &= [P_X(1) + P_X(-1)][P_Y(1) + P_Y(-1)] \\ &= P_X(1)P_Y(1) + P_X(-1)P_Y(1) + P_X(1)P_Y(-1) + P_X(-1)P_Y(-1). \end{aligned}$$

   On the other hand, $P_{X^2Y^2}(1,1) = P_{XY}(1,1) + P_{XY}(-1,1) + P_{XY}(1,-1) + P_{XY}(-1,-1)$. Given that X^2 and Y^2 are independent, we have $P_{X^2Y^2}(1,1) = P_{X^2}(1)P_{Y^2}(1)$, that is,

   $$\begin{aligned} &P_{XY}(1,1) + P_{XY}(-1,1) + P_{XY}(1,-1) + P_{XY}(-1,-1) \\ &= P_X(1)P_Y(1) + P_X(-1)P_Y(1) + P_X(1)P_Y(-1) + P_X(-1)P_Y(-1). \end{aligned}$$

   However, there is no guarantee that $P_{XY}(1,1) = P_X(1)P_Y(1), P_{XY}(1,-1) = P_X(1)P_Y(-1)$, etc., though their sums are equal.

2. Suppose X^3 and Y^3 are considered instead of X^2 and Y^2. Can we construct a pmf assignment where X^3 and Y^3 are independent but X and Y are not?

3. If the set of values assumed by X and Y is $\{0, 1, 2\}$ instead of $\{-1, 0, 1\}$, can we construct a pmf assignment for which X^2 and Y^2 are independent but X and Y are not?

Example 2 Suppose the random variables X and Y have the joint density function defined by

$$f(x,y) = \begin{cases} c(2x+y) & 2 < x < 6, \quad 0 < y < 5 \\ 0 & \text{otherwise} \end{cases}.$$

(a) To find the constant c, we use

$$\begin{aligned} 1 = \text{total probability } &= \int_2^6 \int_0^5 c(2x+y)\ dydx = \int_2^6 c\left(2xy + \frac{y^2}{2}\right)\Big|_0^5\ dx \\ &= \int_2^6 c\left(10x + \frac{25}{2}\right)\ dx = 210c, \end{aligned}$$

so $c = \dfrac{1}{210}$.

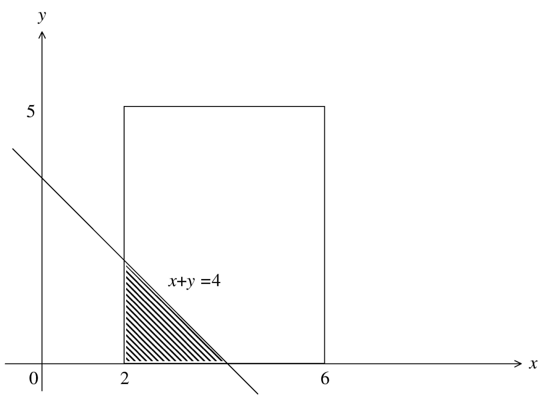
(b) The marginal cdf for X and Y are given by

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(x,y)\ dydx$$
$$= \begin{cases} 0 & x < 2 \\ \displaystyle\int_2^x \int_0^5 \frac{2x+y}{210}\ dydx = \frac{2x^2+5x-18}{84} & 2 \le x < 6 \\ \displaystyle\int_2^6 \int_0^5 \frac{2x+y}{210}\ dydx = 1 & x \ge 6 \end{cases};$$
$$F_Y(y) = P(Y \le y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} \frac{2x+y}{210}\ dydx$$
$$= \begin{cases} 0 & y < 0 \\ \displaystyle\int_2^6 \int_0^y \frac{2x+y}{210}\ dydx = \frac{y^2+16y}{105} & 0 \le y < 5 \\ \displaystyle\int_2^6 \int_0^5 \frac{2x+y}{210}\ dydx = 1 & y \ge 5 \end{cases}.$$

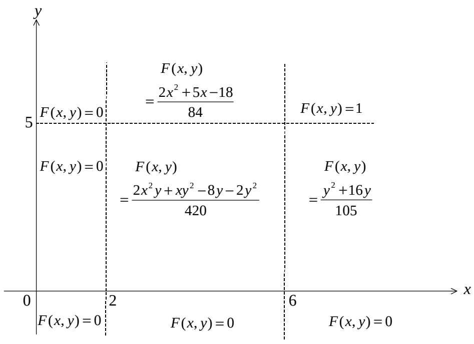
(c) Marginal cdf for X: $f_X(x) = \dfrac{d}{dx}F_X(x) = \begin{cases} \frac{4x+5}{84} & 2 < x < 6 \\ 0 & \text{otherwise} \end{cases}$.

Marginal cdf for Y: $f_Y(y) = \dfrac{d}{dy}F_Y(y) = \begin{cases} \frac{2y+16}{105} & 0 < y < 5 \\ 0 & \text{otherwise} \end{cases}$.

(d)
$$P(X > 3, Y > 2) = \frac{1}{210}\int_3^6 \int_2^5 (2x+y)\ dydx = \frac{3}{20}$$
$$P(X > 3) = \frac{1}{210}\int_3^6 \int_0^5 (2x+y)\ dydx = \frac{23}{28}$$
$$P(X + Y < 4) = \frac{1}{210}\int_2^4 \int_0^{4-x} (2x+y)\ dxdy = \frac{2}{35}$$



(e) Joint distribution function



Suppose (x, y) is located in $\{(x, y) : x > 6, 0 < y < 5\}$, then

$$F(x, y) = \int_2^6 \int_0^y \frac{2x + y}{210}\, dydx = \frac{y^2 + 16y}{105},$$

and $f(x, y) = \dfrac{2y + 16}{105}$.

Note that for this density $f(x, y)$, we have

$$f(x, y) \neq f_X(x) f_Y(y),$$

so x and Y are not independent.

Example 3 The joint density of X and Y is given by

$$f(x, y) = \begin{cases} \dfrac{15}{2} x(2 - x - y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Compute the condition density of X, given that $Y = y$, where $0 < y < 1$.

Solution For $0 < x < 1, 0 < y < 1$, we have

$$\begin{aligned} f_X(x|y) &= \frac{f(x, y)}{f_Y(y)} = \frac{f(x, y)}{\int_{-\infty}^{\infty} f(x, y)\, dx} \\ &= \frac{x(2 - x - y)}{\int_0^1 x(2 - x - y)\, dx} = \frac{x(2 - x - y)}{\frac{2}{3} - \frac{y}{2}} = \frac{6x(2 - x - y)}{4 - 3y}. \end{aligned}$$

Example 4 If X and Y have the joint density function

$$f_{XY}(x, y) = \begin{cases} \dfrac{3}{4} + xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

find (a) $f_Y(y|x)$, (b) $P\left(Y > \frac{1}{2}|\frac{1}{2} < X < \frac{1}{2} + dx\right)$.

Solution

(a) For $0 < x < 1$,

$$f_X(x) = \int_0^1 \left(\frac{3}{4} + xy\right)\, dy = \frac{3}{4} + \frac{x}{2}$$

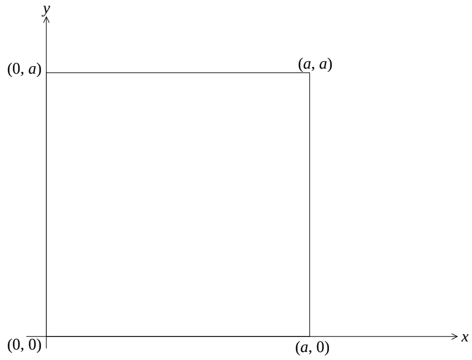
and

$$f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \begin{cases} \frac{3+4xy}{3+2x} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}.$$

For other values of x, $f(y|x)$ is not defined.

(b) $P\left(Y > \frac{1}{2}\middle|\frac{1}{2} < X < \frac{1}{2} + dx\right) = \int_{1/2}^{\infty} f_Y\left(y|\frac{1}{2}\right)\, dy = \int_{1/2}^{1} \frac{3+2y}{4}\, dy = \frac{9}{16}.$

Example 5 Let X and Y be independent exponential random variables with parameter α and β, respectively. Consider the square with corners $(0,0), (0,a), (a,a)$ and $(a,0)$, that is, the length of each side is a.



(a) Find the value of a for which the probability that (X,Y) falls inside a square of side a is $1/2$.

(b) Find the conditional pdf of (X,Y) given that $X \geq a$ and $Y \geq a$.

Solution

(a) The density function of X and Y are given by

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0 \\ 0 & x < 0 \end{cases}, \quad f_Y(y) = \begin{cases} \beta e^{-\beta y}, & y \geq 0 \\ 0 & y < 0 \end{cases}.$$

Since X and Y are independent, so $f_{XY}(x,y) = f_X(x) f_Y(y)$. Next, we compute

$$P[0 \leq X \leq a, 0 \leq Y \leq a] = \int_0^a \int_0^a \alpha\beta e^{-\alpha x} e^{-\beta y}\, dxdy = (1 - e^{-a\alpha})(1 - e^{-a\beta}),$$

and solve for a such that $(1-e^{-a\alpha})(1-e^{-a\beta}) = 1/2$.

(b) Consider the following conditional pdf of (X, Y)

$$
\begin{aligned}
& F_{XY}(x, y|X \geq a, Y \geq a) \\
&= P[X \leq x, Y \leq y | X \geq a, Y \geq a] \\
&= \frac{P[a \leq X \leq x, a \leq Y \leq y]}{P[X \geq a, Y \geq a]} \\
&= \frac{P[a \leq X \leq x] P[a \leq Y \leq y]}{P[X \geq a] P[Y \geq a]} \quad \text{since } X \text{ and } Y \text{ are independent} \\
&= \begin{cases} \dfrac{\int_a^y \int_a^x \alpha\beta e^{-\alpha x} e^{-\beta y}\, dxdy}{\int_a^\infty \int_a^\infty \alpha\beta e^{-\alpha x} e^{-\beta y}\, dxdy} = \dfrac{(e^{-a\alpha} - e^{-\alpha x})(e^{-a\beta} - e^{-\beta y})}{e^{-a\alpha} e^{-a\beta}}, & x > a, y > a \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

$$
\begin{aligned}
f_{XY}(x, y|X \geq a, Y \geq a) &= \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y|X \geq a, Y \geq a) \\
&= \begin{cases} \alpha\beta e^{-\alpha x} e^{-\beta y} / e^{-\alpha a} e^{-\beta a} & \text{for } x > a, y > a \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

Example 6 A point is chosen uniformly at random from the triangle that is formed by joining the three points $(0, 0), (0, 1)$ and $(1, 0)$ (units measured in centimetre). Let X and Y be the co-ordinates of a randomly chosen point.

(i) What is the joint density of X and Y?

(ii) Calculate the expected value of X and Y, i.e., expected co-ordinates of a randomly chosen point.

(iii) Find the correlation between X and Y. Would the correlation change if the units are measured in inches?

Solution

(i) $f_{X,Y}(x, y) = \dfrac{1}{\text{Area } \triangle} = 2, \quad (x, y)$ lies in the triangle.

(ii) $f_X(x) = \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x, y')\, dy' = \int_0^{1-x} 2\, dy = 2(1-x).$

$f_Y(y) = \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x', y)\, dx' = \int_0^{1-y} 2\, dx = 2(1-y).$

Hence, $E[X] = 2\displaystyle\int_0^1 x(1-x)\, dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

and $E[Y] = 2\int_0^1 y(1-y)\ dy = \frac{1}{3}$.

(iii) To find the correlation between X and Y, we consider

$$\begin{aligned}
E[XY] &= 2\int_0^1 \int_0^{1-y} xy\ dxdy = 2\int_0^1 y\left[\frac{x^2}{2}\right]_0^{1-y}\ dy \\
&= \int_0^1 y(1-2y+y^2)\ dy \\
&= \left[\frac{y^2}{2} - \frac{2}{3}y^3 + \frac{y^4}{4}\right]_0^1 = \frac{1}{12}. \\
\text{COV}(X,Y) &= E[XY] - E[X]E[Y] \\
&= \frac{1}{12} - \left(\frac{1}{3}\right)^2 = -\frac{1}{36}. \\
E[X^2] &= 2\int_0^1 x^2(1-x)\ dx = 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{6}
\end{aligned}$$

so

$$\text{VAR}(X) = E[X^2] - [E[X]]^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}.$$

Similarly, we obtain $\text{VAR}(Y) = \frac{1}{18}$.

$$\rho_{XY} = \frac{\text{COV}(X,Y)}{\sqrt{\text{VAR}(X)}\sqrt{\text{VAR}(Y)}} = \frac{-\frac{1}{36}}{\frac{1}{18}} = -\frac{1}{2}.$$

Since $\rho(aX,bY) = \frac{\text{COV}(aX,bY)}{\sigma(aX)\sigma(bY)} = \frac{ab\text{COV}(X,Y)}{a\sigma(X)b\sigma(Y)} = \rho(X,Y)$, for any scalar multiples a and b. Therefore, the correlation would not change if the units are measured in inches.

Example 7 Let X, Y, Z be independent and uniformly distributed over $(0,1)$. Compute $P\{X \geq YZ\}$.

Solution Since X, Y, Z are independent, we have

$$f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z) = 1, \quad 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1.$$

Therefore,

$$
\begin{aligned}
P[X \geq YZ] &= \iiint\limits_{x \geq yz} f_{X,Y,Z}(x, y, z)\ dxdydz \\
&= \int_0^1 \int_0^1 \int_{yz}^1 \ dxdydz = \int_0^1 \int_0^1 (1 - yz)\ dydz \\
&= \int_0^1 \left(1 - \frac{z}{2}\right)\ dz = \frac{3}{4}.
\end{aligned}
$$

Example 8 The joint density of X and Y is given by

$$
f(x, y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}.
$$

Find the density function of the random variable X/Y.

Solution We start by computing the distribution function of X/Y. For $a > 0$,

$$
\begin{aligned}
F_{X/Y}(a) &= P\left[\frac{X}{Y} \leq a\right] \\
&= \iint\limits_{x/y \leq a} e^{-(x+y)}\ dxdy = \int_0^\infty \int_0^{ay} e^{-(x+y)}\ dxdy \\
&= \int_0^\infty (1 - e^{-ay})e^{-y}\ dy = \left[-e^{-y} + \frac{e^{-(a+1)y}}{a+1}\right]\Bigg|_0^\infty \\
&= 1 - \frac{1}{a+1} = \frac{a}{a+1}.
\end{aligned}
$$

By differentiating $F_{X/Y}(a)$ with respect to a, the density function X/Y is given by

$$
f_{X/Y}(a) = 1/(a+1)^2, 0 < a < \infty.
$$

Example 9 Let X and Y be a pair of independent random variables, where X is uniformly distributed in the interval $(-1, 1)$ and Y is uniformly distributed in the interval $(-4, -1)$. Find the pdf of $Z = XY$.

Solution Assume $Y = y$, then $Z = XY$ is a scaled version of X. Suppose $U = \alpha W + \beta$, then $f_U(u) = \frac{1}{|\alpha|} f_W\left(\frac{u-\beta}{\alpha}\right)$. Now, $f_Z(z|y) = \frac{1}{|y|} f_X\left(\frac{z}{y}\middle| y\right)$; the pdf of z is given by

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_X\left(\frac{z}{y}\middle| y\right) f_Y(y)\, dy = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{XY}\left(\frac{z}{y}, y\right)\, dy.$$

Since X is uniformly distributed over $(-1, 1)$, $f_X(x) = \begin{cases} \frac{1}{2} & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$. Similarly, Y is uniformly distributed over $(-4, -1)$, $f_Y(y) = \begin{cases} \frac{1}{3} & -4 < y < -1 \\ 0 & \text{otherwise} \end{cases}$. As X and Y are independent,

$$f_{XY}\left(\frac{z}{y}, y\right) = f_X\left(\frac{z}{y}\right) f_Y(y) = \begin{cases} \frac{1}{6} & -1 < \dfrac{z}{y} < 1 \text{ and } -4 < y < -1 \\ 0 & \text{otherwise} \end{cases}.$$

We need to observe $-1 < z/y < 1$, which is equivalent to $|z| < |y|$. Consider the following cases:

(i) $|z| > 4$; now $-1 < z/y < 1$ is never satisfied so that $f_{XY}\left(\frac{z}{y}, y\right) = 0$.

(ii) $|z| < 1$; in this case, $-1 < z/y < 1$ is automatically satisfied so that

$$f_Z(z) = \int_{-4}^{-1} \frac{1}{|y|} \frac{1}{6}\, dy = \int_{-4}^{-1} -\frac{1}{6y}\, dy = -\frac{1}{6} \ln|y| \Big]_{-4}^{-1} = \frac{\ln 4}{6}.$$

(iii) $1 < |z| < 4$; note that $f_{XY}\left(\frac{z}{y}, y\right) = \frac{1}{6}$ only for $-4 < y < -|z|$, so that

$$f_Z(z) = \int_{-4}^{-|z|} \frac{1}{|y|} \frac{1}{6}\, dy = -\frac{1}{6} \ln|y| \Big]_{-4}^{-|z|} = \frac{1}{6}[\ln 4 - \ln|z|].$$

In summary, $f_Z(z) = \begin{cases} \dfrac{\ln 4}{6} & \text{if } |z| < 1 \\ \dfrac{1}{6}[\ln 4 - \ln|z|] & \text{if } 1 < |z| < 4 \\ 0 & \text{otherwise} \end{cases}$.

Remark Check that

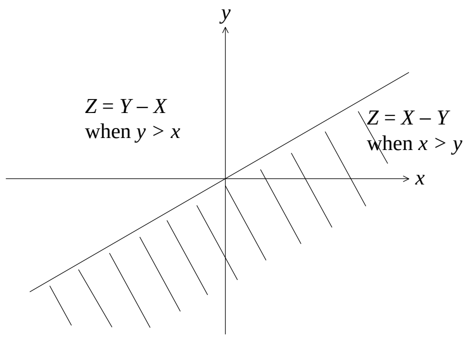
$$\int_{-\infty}^{\infty} f_Z(z)\, dz = \int_{-4}^{-1} \frac{1}{6}[\ln 4 - \ln|z|]\, dz + \int_{-1}^{1} \frac{\ln 4}{6}\, dz + \int_{1}^{4} \frac{1}{6}[\ln 4 - \ln|z|]\, dz$$

$$= \int_{-4}^{4} \frac{\ln 4}{6}\, dz - 2\int_{1}^{4} \frac{\ln|z|}{6}\, dz$$

$$= \frac{8}{6}\ln 4 - \frac{1}{3}[z\ln z - z]_1^4 = 1.$$

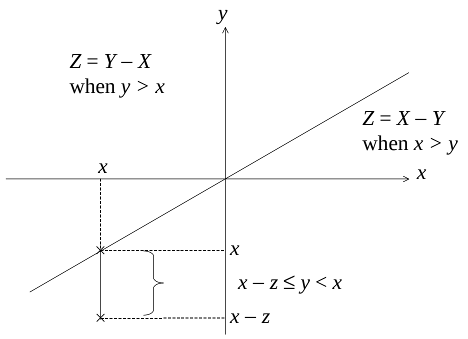
Example 10 Let X and Y be two independent Gaussian random variables with zero mean and unit variance. Find the pdf of $Z = |X - Y|$.

Solution We try to find $F_Z(z) = P[Z \leq z]$. Note that $z \geq 0$ since Z is a non-negative random variable.



Consider the two separate cases: $x > y$ and $x < y$. When $X = Y$, Z is identically zero.

(i) $x > y, Z \leq z \Leftrightarrow x - y \leq z, z \geq 0$; that is, $x - z \leq y < x$.



$$F_Z(z) = \int_{-\infty}^{\infty}\int_{x-z}^{x} f_{XY}(x,y)\ dydx$$

$$\begin{aligned} f_Z(z) = \frac{d}{dz}F_Z(z) &= \int_{-\infty}^{\infty} f_{XY}(x, x-z)\ dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{-[x^2+(x-z)^2]/2}\ dx \\ &= \frac{1}{2\sqrt{\pi}}e^{-z^2/4}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-\left(x-\frac{z}{2}\right)^2}\ dx = \frac{1}{2\sqrt{\pi}}e^{-z^2/4}. \end{aligned}$$

(ii) $x < y, Z \le z \Leftrightarrow y - x \le z, z \ge 0$; that is $x < y \le x + z$.

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{x}^{x+z} f_{XY}(x, y)\ dydx$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, x+z)\ dx = \frac{1}{2\sqrt{\pi}} e^{-z^2/4} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(x+z)^2}\ dx = \frac{1}{2\sqrt{\pi}} e^{-z^2/4}.$$

Example 11 Suppose two persons A and B come to two separate counters for service. Let their service times be independent exponential random variables with parameters λ_A and λ_B, respectively. Find the probability that B leaves before A?

Solution Let T_A and T_B denote the continuous random service time of A and B, respectively. Recall that the expected value of the service times are: $E[T_A] = \dfrac{1}{\lambda_A}$ and $E[T_B] = \dfrac{1}{\lambda_B}$. That is, a higher value of λ implies a shorter average service time. One would expect

$$P[T_A > T_B] : P[T_B > T_A] = \frac{1}{\lambda_A} : \frac{1}{\lambda_B};$$

and together with $P[T_A > T_B] + P[T_B > T_A] = 1$, we obtain

$$P[T_A > T_B] = \frac{\lambda_B}{\lambda_A + \lambda_B} \quad \text{and} \quad P[T_B > T_A] = \frac{\lambda_A}{\lambda_A + \lambda_B}.$$

Justification:- Since T_A and T_B are independent exponential random variables, their joint density $f_{T_A,T_B}(t_A, t_B)$ is given by

$$\begin{aligned} & f_{T_A,T_B}(t_A, t_B)\ dt_A dt_B \\ &= P[t_A < T_A < t_A + dt_A, t_B < T_B < t_B + dt_B] \\ &= P[t_A < T_A < t_A + dt_A] P[t_B < T_B < t_B + dt_B] \\ &= (\lambda_A e^{-\lambda_A t_A} dt_A)(\lambda_B e^{-\lambda_B t_B} dt_B). \end{aligned}$$

$$\begin{aligned} P[T_A > T_B] &= \int_0^{\infty} \int_0^{t_A} \lambda_A \lambda_B e^{-\lambda_A t_A} e^{-\lambda_B t_B}\ dt_B dt_A \\ &= \int_0^{\infty} \lambda_A e^{-\lambda_A t_A} (1 - e^{-\lambda_B t_A})\ dt_A \\ &= \int_0^{\infty} \lambda_A e^{-\lambda_A t_A}\ dt_A - \int_0^{\infty} \lambda_A e^{-(\lambda_A + \lambda_B) t_A}\ dt_A \\ &= 1 - \frac{\lambda_A}{\lambda_A + \lambda_B} = \frac{\lambda_B}{\lambda_A + \lambda_B}. \end{aligned}$$

Integrate wrt t_B from $t_B = 0$ to $t_B = t_A$, then integrate wrt t_A from 0 to ∞.