MATH 246 — Probability and Random Processes

Final Examination

Fall 2004 **Course Instructor:** *Prof. Y. K. Kwok*

Time allowed: 100 minutes

[points]

1. A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety? **[4]**

   Hint: Let X be the amount of time (in hours) until the miner reaches safety, and let Y denote the door he initially chooses. By the rule of conditional expectation

$$E[X] = E[X|Y=1]P[Y=1] + E[X|Y=2]P[Y=2] + E[X|Y=3]P[Y=3].$$

2. Consider the pair of random variables X and Y whose joint density function is given by

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi} & x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

   Show that X and Y are uncorrelated. Are they independent? **[6]**

3. The time between consecutive earthquakes in San Francisco and the time between consecutive earthquakes in Los Angeles are independent and exponentially distributed with means $\dfrac{1}{\lambda_1}$ and $\dfrac{1}{\lambda_2}$, respectively. What is the probability that the next earthquake occurs in Los Angeles? **[7]**

4. Let X and Y be a pair of independent random variables, where X is uniformly distributed over $(-1,1)$ and Y is uniformly distributed over $(0,2)$. Find the probability density of $Z = X/Y$. **[8]**

   Hint: Explain why

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f_{XY}(yz, y)\, dy.$$

   The region $\{(y,z) : -1 < yz < 1 \text{ and } 0 < y < 2\}$ can be divided into 3 regions, according to (i) $z < -\dfrac{1}{2}$, (ii) $-\dfrac{1}{2} \le z \le \dfrac{1}{2}$ and (iii) $z > \dfrac{1}{2}$.

5. Let $N(t), t \geq 0$, be a Poisson process with parameter $\lambda > 0$.

(a) Show that the autocovariance $C_N(t_1, t_2)$ of $N(t)$ is given by

$$C_N(t_1, t_2) = \lambda \min(t_1, t_2).$$

In your derivation steps, explain clearly how you use the stationary increments and independent increments properties of a Poisson process. **[5]**

(b) Suppose a Poisson event is known to have occurred over the time period $[0, 1]$, show that the probability of the event occurring before time $t, 0 < t < 1$, is equal to t. **[5]**

Hint: Consider

$$P[N(t) = 1 | N(1) = 1], \quad \text{where} \quad 0 < t < 1.$$

6. Let $X(t) = A \cos \omega t + B \sin \omega t$, where A and B are independent and identically distributed Gaussian random variables with zero mean and variance σ^2. Find the mean and autocovariance of $X(t)$. **[6]**

7. A machine consists of two parts that fail and are repaired independently. A working part fails during any given day with probability α. A part that is not working is repaired by the next day with probability β. Let X_n be the number of working parts in day n. The sample space of X_n is $\{0, 1, 2\}$. We write

$$\pi_{n,j} = P[X_n = j], \quad j = 0, 1, 2.$$

(a) Find the one-step transition probability matrix P, expressed in terms of α and β. If the initial state pmf vector is

$$\boldsymbol{\pi}_0 = (\pi_{0,0} \quad \pi_{0,1} \quad \pi_{0,2}) = (0.3 \quad 0.3 \quad 0.4),$$

find $P[X_2 = 1, X_1 = 2, X_0 = 0]$. **[5]**

(b) Let $\boldsymbol{\pi}_n$ be the state pmf vector after n steps and $\boldsymbol{\pi}_\infty$ be the steady state pmf vector. Find $\boldsymbol{\pi}_n$ in terms of P and $\boldsymbol{\pi}_0$. Show that $\boldsymbol{\pi}_\infty$ is governed by

$$\boldsymbol{\pi}_\infty = \boldsymbol{\pi}_\infty P.$$

Explain why all the rows of the matrix $\lim_{n \to \infty} P^n$ are equal to $\boldsymbol{\pi}_\infty$. **[4]**

— *End* —

Attached: List of useful formulae

List of useful formulae

Binomial Random Variable

$S_X = \{0, 1, \ldots, n\}$ $\quad\quad$ $p_k = C_k^n p^k (1-p)^{n-k} \quad k = 0, 1, \ldots, n$

$E[X] = np \qquad \text{VAR}[X] = np(1-p)$

Poisson Random Variable

$S_X = \{0, 1, 2, \ldots\}$ $\quad\quad$ $p_k = \dfrac{\alpha^k}{k!} e^{-\alpha} \qquad k = 0, 1, \ldots$ and $\alpha > 0$

$E[X] = \alpha \qquad \text{VAR}[X] = \alpha$

Uniform Random Variable

$S_X = [a, b]$ $\quad\quad$ $f_X(x) = \dfrac{1}{b-a} \quad a \leq x \leq b$

$E[X] = \dfrac{a+b}{2} \qquad \text{VAR}[X] = \dfrac{(b-a)^2}{12}$

Exponential Random Variable

$S_X = [0, \infty)$ $\quad\quad$ $f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$ and $\lambda > 0$

$E[X] = \dfrac{1}{\lambda} \qquad \text{VAR}[X] = \dfrac{1}{\lambda^2}$

Gaussian (Normal) Random Variable

$S_X = (-\infty, \infty)$ $\quad\quad$ $f_X(x) = \dfrac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} \quad -\infty < x < \infty \quad$ and $\quad \sigma > 0$

$E[X] = m \qquad \text{VAR}[X] = \sigma^2$

Relations between pdf's when $Y = g(X)$

(i) $Y = aX + b$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

(ii) a non-linear function $Y = g(X)$

$$f_Y(y) = \sum_k \left.\frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\right|_{x=x_k}$$

Marginal pdf's

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y')\, dy' \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x', y)\, dx'$$

Independence of X and Y

X and Y are independent if and only if $f_{XY}(x, y) = f_X(x) f_Y(y)$, for all x, y

Conditional pdf of Y given $X = x$

$$f_Y(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

Conditional expectation of Y given $X = x$

Continuous $\quad E[Y|x] = \int_{-\infty}^{\infty} y f_Y(y|x)\ dy$

discrete $\quad F[Y|x] = \sum_{y_j} y_j P_Y(y_j|x)$

Functions of several random variables

(i) $Z = X + Y$, $\quad F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x'} f_{XY}(x', y')\ dy' dx'$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x', z - x')\ dx'$$

If X and Y are independent, then $f_Z(z) = \int_{-\infty}^{\infty} f_X(x') f_Y(z - x')\ dx'$

(ii) $Z = X/Y$, $\quad f_Z(z|y) = |y| f_X(yz|y)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_Z(z|y') f_Y(y')\ dy'$$

Correlation and covariance of two random variables

$\mathrm{COV}(X, Y) = E[(X - m_X)(Y - m_Y)]$, where m_X and m_Y are $E[X]$ and $E[Y]$, resp.

$\rho_{XY} = \frac{\mathrm{COV}(X, Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$

autocovariance $C_X(t_1, t_2)$ of a random process $X(t)$

$$C_X(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}]$$