MATH 246 — Probability and Random Processes

Solution to Final Examination

Fall 2004 **Course Instructor:** *Prof. Y. K. Kwok*

Time allowed: 100 minutes

1. Now $E[X|Y=1]=3, E[X|Y=2]=5+E[X]$ and $E[X|Y=3]=7+E[X]$. We then have

$$E[X]=\frac{1}{3}(3+5+E[X]+7+E[X]) \quad \text{so that} \quad E[X]=15.$$

2.

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\begin{cases}\frac{2}{\pi}\sqrt{1-x^2}, & -1\le x\le 1\\ 0, & \text{otherwise}\end{cases}$$

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\begin{cases}\frac{2}{\pi}\sqrt{1-y^2}, & -1\le y\le 1\\ 0, & \text{otherwise.}\end{cases}$$

When $-1\le x\le 1$ and $-1\le y\le 1$,

$$f_X(x)f_Y(y)\neq f_{XY}(x,y)$$

so that X and Y cannot be independent. On the other hand

$$E[X]=\int_{-1}^{1}\frac{2x}{\pi}\sqrt{1-x^2}\,dx=0$$

$$E[Y]=\int_{-1}^{1}\frac{2y}{\pi}\sqrt{1-y^2}\,dy=0$$

$$E[XY]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)\,dxdy$$

$$=\int_{-1}^{1}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{xy}{\pi}\,dy=0$$

so that

$$\mathrm{COV}(X,Y)=E[XY]-E[X]E[Y]=0.$$

Hence, X and Y are uncorrelated.

3. Let T_S and T_L denote the times between now and the next earthquake in San Francisco and Los Angeles, respectively. To calculate $P[T_S>T_L]$, we condition on T_L:

$$P[T_S>T_L]=\int_0^{\infty}P[T_S>T_L|T_L=y]f_{T_L}(y)\,dy$$

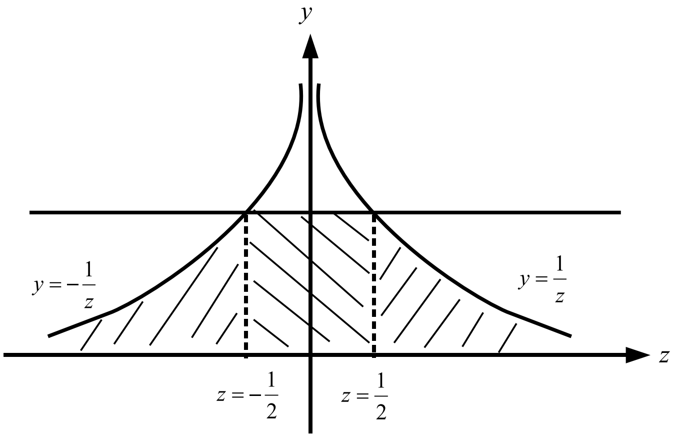
$$=\int_0^{\infty}P[T_S>y]\lambda_2e^{-\lambda_2 y}\,dy$$

$$=\int_0^{\infty}e^{-\lambda_1 y}\lambda_2e^{-\lambda_2 y}\,dy=\frac{\lambda_2}{\lambda_1+\lambda_2}.$$

4. $f_X(x) = \begin{cases} \frac{1}{2} & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$, $\qquad f_Y(y) = \begin{cases} \frac{1}{2} & 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$

$f_Z(z) = \int_{-\infty}^{\infty} |y| f_{XY}(yz, y)\, dy$ and observe that

$$f_{XY}(yz, y) = \begin{cases} \frac{1}{4} & -1 < yz < 1 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}.$$



$f_{XY}(yz, y)$ is non-zero over the above shaded region.

(i) $z > \frac{1}{2}$

$$f_Z(z) = \int_0^{\frac{1}{z}} \frac{y}{4}\, dy = \frac{1}{8z^2}$$

(ii) $-\frac{1}{2} \leq z \leq \frac{1}{2}$

$$f_Z(z) = \int_0^{2} \frac{y}{4}\, dy = \frac{1}{2}$$

(iii) $z < -\frac{1}{2}$

$$f_Z(z) = \int_0^{-\frac{1}{z}} \frac{y}{4}\, dy = \frac{1}{8z^2}.$$

In summary

$$f_z(z) = \begin{cases} \frac{1}{2} & |z| \leq \frac{1}{2} \\ \frac{1}{8z^2} & |z| > \frac{1}{2} \end{cases}.$$

As a check

$$\int_{-\infty}^{\infty} f_Z(z)\, dz = \int_{-\infty}^{-\frac{1}{2}} \frac{1}{8z^2}\, dz + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2}\, dz + \int_{\frac{1}{2}}^{\infty} \frac{1}{8z^2}\, dz$$
$$= -\frac{1}{8z}\Big|_{-\infty}^{-\frac{1}{2}} + \frac{1}{2} - \frac{1}{8z}\Big|_{\frac{1}{2}}^{\infty} = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.$$

5. (a) Assume $t_1 < t_2$

$$\begin{aligned} C_N(t_1,t_2) &= E[(N(t_1)-\lambda t_1)(N(t_2)-\lambda t_2)] \\ &= E[(N(t_1)-\lambda t_1)\{N(t_2)-N(t_1)-\lambda(t_2-t_1)] + N(t_1)-\lambda t_1\}] \\ &= E[[N(t_1)-\lambda t_1][N(t_2)-N(t_1)-\lambda(t_2-t_1)]] + \text{var}\,(N(t_1)). \end{aligned}$$

By the independent increments property, $N(t_1)-\lambda(t_1)$ and $[N(t_2)-N(t_1)-\lambda(t_2-t_1)]$ are independent so that

$$\begin{aligned} &E[[N(t_1)-\lambda t_1][N(t_2)-N(t_1)-\lambda(t_2-t_1)]] \\ &= E[N(t_1)-\lambda t_1]E[N(t_2)-N(t_1)-\lambda(t_2-t_1)]. \end{aligned}$$

Furthermore, using the stationary increments property, we have

$$E[N(t_2)-N(t_1)] = E[N(t_2-t_1)] = \lambda(t_2-t_1)$$

so that

$$E[N(t_2)-N(t_1)-\lambda(t_2-t_1)] = 0.$$

Hence,

$$C_N(t_1,t_2) = \lambda t_1 = \text{var}\,(N(t_1)) = \lambda \min(t_1,t_2).$$

When $t_2 \leq t_1$, we can show similarly that

$$C_N(t_1,t_2) = \lambda t_2 = \lambda \min(t_1,t_2).$$

(b) Consider

$$\begin{aligned} &P[N(t)=1|N(1)=1] \\ &= \frac{P[N(t)=1, N(1)-N(t)=0]}{P[N(1)=1]} \\ &= \frac{P[N(t)=1]P[N(1-t)=0]}{P[N(1)=1]} \quad \text{(independent increments and stationary increments properties)} \\ &= \frac{\lambda t e^{-\lambda t} e^{-\lambda(1-t)}}{\lambda e^{-\lambda}} = t. \end{aligned}$$

6. mean $= E[X(t)] = E[A\cos\omega t + B\sin\omega t] = \cos\omega t E[A] + \sin\omega t E[B] = 0$

$$\begin{aligned} \text{autocovariance} &= E[\{X(t_1)-m_X(t_1)\}\{X(t_2)-m_X(t_2)\}] \\ &= E[X(t_1)X(t_2)] \\ &= E[(A\cos\omega t_1 + B\sin\omega t_1)(A\cos\omega t_2 + B\sin\omega t_2)] \\ &= E[A^2]\cos\omega t_1\cos\omega t_2 + E[AB](\cos\omega t_1\sin\omega t_2 + \sin\omega t_1\cos\omega t_2) + E[B^2]\sin\omega t_1\sin\omega t_2 \\ &= (E[A^2]-E[A]^2)\cos\omega t_1\cos\omega t_2 + (E[B^2]-E[B]^2)\sin\omega t_1\sin\omega t_2 \\ &= \sigma^2\cos\omega(t_1-t_2). \\ &\quad \text{(since } A \text{ and } B \text{ are independent, } E[AB]=E[A]E[B] \text{ and } E[A]=E[B]=0) \end{aligned}$$

7. (a)

$$P = \begin{pmatrix} (1-\beta)^2 & 2\beta(1-\beta) & \beta^2 \\ \alpha(1-\beta) & \alpha\beta + (1-\alpha)(1-\beta) & (1-\alpha)\beta \\ \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2 \end{pmatrix}.$$

$$\begin{aligned} P[X_2 = 1, X_1 = 2, X_0 = 0] &= P[X_2 = 1 | X_1 = 2] P[X_1 = 2 | X_0 = 0] P[X_0 = 0] \\ &= [2\alpha(1-\alpha)\beta^2](0.3) \end{aligned}$$

(b) From $\boldsymbol{\pi}_{j+1} = \boldsymbol{\pi}_j P, j = 0, 1, \cdots, n$, we deduce that

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 P^n$$

Next, consider

$$\boldsymbol{\pi}_{n+1} = \boldsymbol{\pi}_n P,$$

by taking the limit $n \to \infty$ on both sides, we obtain

$$\boldsymbol{\pi}_\infty = \boldsymbol{\pi}_\infty P.$$

Suppose we take $\boldsymbol{\pi}_0 = (1 \quad 0 \quad 0)$, we have

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 P^n = \text{ first row in } P^n.$$

Similarly, $(0 \quad 1 \quad 0)P^n =$ second row of P^n and $(0 \quad 0 \quad 1)P^n =$ third row of P^n. By taking $n \to \infty$, the steady state pmf vector $\boldsymbol{\pi}_\infty$ should be independent of $\boldsymbol{\pi}_0$ so that we expect all rows of $\lim\limits_{n\to\infty} P^n$ to be identical.