1. Complex numbers

A *complex number* z is defined as an ordered pair

$$z = (x, y)\,,$$

where x and y are a pair of real numbers. In usual notation, we write

$$z = x + iy\,,$$

where i is a symbol. The operations of addition and multiplication of complex numbers are defined in a meaningful manner, which force $i^2 = -1$. The set of all complex numbers is denoted by $\mathbb{C}$. Write

$$\text{Re } z = x\,, \quad \text{Im } z = y\,.$$

Since complex numbers are defined as ordered pairs, two complex numbers (x_1, y_1) and (x_2, y_2) are equal if and only if both their real parts and imaginary parts are equal. Symbolically,

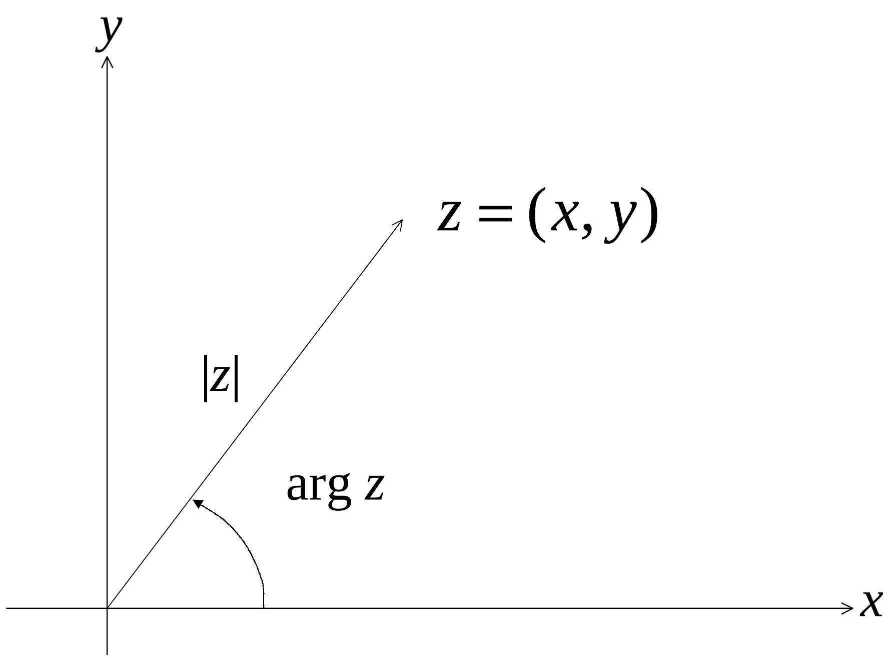
$$(x_1, y_1) = (x_2, y_2) \qquad \text{if and only if} \qquad x_1 = x_2 \quad \text{and} \quad y_1 = y_2\,.$$

A complex number $z = (x, y)$, or as $z = x + iy$, is defined by a pair of real numbers x and y; so does for a point (x, y) in the x-y plane. We associate a one-to-one correspondence between the complex number $z = x + iy$ and the point (x, y) in the x-y plane. We refer the plane as the *complex plane* or z-plane.

Polar coordinates

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Modulus of z : $\quad |z| = r = \sqrt{x^2 + y^2}.$



Vectorical representation of a complex number in the complex plane

Obviously, Re $z \le |z|$ and Im $z \le |z|$; and

$$z = x + iy = r(\cos\theta + i\sin\theta),$$

where θ is called the argument of z, denoted by arg z.

The principal value of arg z, denoted by Arg z, is the particular value of arg z chosen within the principal interval $(-\pi, \pi]$. We have

$$\text{arg } z = \text{Arg } z + 2k\pi \qquad k \text{ any integer}, \qquad \text{Arg } z \in (-\pi, \pi]\,.$$

Note that arg z is a multi-valued function.

Complex conjugate

The complex conjugate $\overline{z}$ of $z = x + iy$ is defined by

$$\overline{z} = x - iy\,.$$

In the complex plane, the conjugate $\overline{z} = (x, -y)$ is the reflection of the point $z = (x, y)$ with respect to the real axis.

Standard results on conjugates and moduli

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, (ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, (iii) $\overline{\dfrac{z_1}{z_2}} = \dfrac{\overline{z_1}}{\overline{z_2}}$,

(iv) $|z_1 z_2| = |z_1|\ |z_2|$ (v) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

Example

Find the square roots of $a + ib$, where a and b are real constants.

Solution

Let $u + iv$ be a square root of $a + ib$; and so $(u + iv)^2 = a + ib$. Equating the corresponding real and imaginary parts, we have

$$u^2 - v^2 = a \quad \text{and} \quad 2uv = b\,.$$

By eliminating v, we obtain a fourth degree equation for u:

$$4u^4 - 4au^2 - b^2 = 0\,.$$

The two real roots for u are found to be

$$u = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}\,.$$

From the relation $v^2 = u^2 - a$, we obtain

$$v = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}.$$

Apparently, there are four possible values for $u+iv$. However, there can be only two values of the square root of $a+ib$. By virtue of the relation $2uv = b$, one must choose u and v such that their product has the same sign as b. This leads to

$$u+iv = \pm\left(\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} + i\frac{b}{|b|}\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right),$$

provided that $b \neq 0$. The special case where $b = 0$ is trivial. As a numerical example, take $a = 3, b = -4$, then

$$\pm\left(\sqrt{\frac{3+5}{2}} - i\sqrt{\frac{5-3}{2}}\right) = \pm(2-i).$$

De' Moivre's theorem

Any complex number with unit modulus can be expressed as $\cos\theta + i\sin\theta$. By virtue of the complex exponential function*, we have

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The above formula is called the *Euler formula*. As motivated by the Euler formula, one may deduce that

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta,$$

where n can be any positive integer.

* The complex exponential function is defined by

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y), \quad z = x + iy.$$

To prove the theorem, we consider the following cases:

(i) The theorem is trivial when $n = 0$.

(ii) When n is a positive integer, the theorem can be proven easily by mathematical induction.

(iii) When n is a negative integer, let $n = -m$ where m is a positive integer. We then have

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^n &= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta} \\
&= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)} \\
&= \cos m\theta - i\sin m\theta = \cos n\theta + i\sin n\theta.
\end{aligned}
$$

How do we generalize the formula to $(\cos\theta + i\sin\theta)^s$, where s is a rational number?

Let $s = p/q$, where p and q are irreducible integers. Note that the modulus of $\cos\theta + i\sin\theta$ is one, so does $(\cos\theta + i\sin\theta)^s$. Hence, the polar representation of $(\cos\theta + i\sin\theta)^s$ takes the form $\cos\phi + i\sin\phi$ for some ϕ. Now, we write

$$\cos\phi + i\sin\phi = (\cos\theta + i\sin\theta)^s = (\cos\theta + i\sin\theta)^{p/q}.$$

Taking the power of q of both sides

$$\cos q\phi + i\sin q\phi = \cos p\theta + i\sin p\theta,$$

which implies

$$q\phi = p\theta + 2k\pi \quad \text{or} \quad \phi = \frac{p\theta + 2k\pi}{q}, \quad k = 0, 1, \cdots, q-1.$$

The value of ϕ corresponding to k that is beyond the above set of integers would be equal to one of those values defined in the equation plus some multiple of 2π.

There are q distinct roots of $(\cos\theta + i\sin\theta)^{p/q}$, namely,

$$\cos\left(\frac{p\theta + 2k\pi}{q}\right) + i\sin\left(\frac{p\theta + 2k\pi}{q}\right), \quad k = 0, 1, \cdots, q-1.$$

n^{th} root of unity

By definition, any n^{th} roots of unity satisfies the equation

$$z^n = 1\,.$$

By de' Moivre's theorem, the n distinct roots of unity are

$$z = e^{i2k\pi/n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, ..., n-1\,.$$

If we write $\omega_n = e^{i2\pi/n}$, then the n roots are $1, \omega_n, \omega_n^2, ..., \omega_n^{n-1}$.

Alternatively, if we pick any one of the n^{th} roots and call it α, then the other $n-1$ roots are given by $\alpha\omega_n, \alpha\omega_n^2, ..., \alpha\omega_n^{n-1}$. This is obvious since each of these roots satisfies

$$(\alpha\omega_n^k)^n = \alpha^n(\omega_n^n)^k = 1, \quad k = 0, 1, \cdots, n-1.$$

In the complex plane, the n roots of unity correspond to the n vertices of a regular n-sided polygon inscribed inside the unit circle, with one vertex at the point $z = 1$. The vertices are equally spaced on the circumference of the circle. The successive vertices are obtained by increasing the argument by an equal amount of $2\pi/n$ of the preceding vertex.

Suppose the complex number in the polar form is represented by $r(\cos\phi + i\sin\phi)$, its n^{th} roots are given by

$$r^{1/n}\left(\cos\frac{\phi + 2k\pi}{n} + i\sin\frac{\phi + 2k\pi}{n}\right), \qquad k = 0, 1, 2, ..., n-1,$$

where $r^{1/n}$ is the real positive n^{th} root of the real positive number r. The roots are equally spaced along the circumference with one vertex being at $r^{1/n}[\cos(\phi/n) + i\sin(\phi/n)]$.

Triangle inequalities

For any two complex numbers z_1 and z_2, we can establish

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + z_2\overline{z_1} \\ &= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\overline{z_2})\,. \end{aligned}$$

By observing that $\mathrm{Re}(z_1\overline{z_2}) \leq |z_1\overline{z_2}|$, we have

$$\begin{aligned} |z_1 + z_2|^2 &\leq |z_1|^2 + |z_2|^2 + 2|z_1\overline{z_2}| \\ &= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2\,. \end{aligned}$$

Since moduli are non-negative, we take the positive square root on both sides and obtain

$$|z_1 + z_2| \leq |z_1| + |z_2|\,.$$

To prove the other half of the triangle inequalities, we write

$$|z_1| = |(z_1 + z_2) + (-z_2)| \leq |z_1 + z_2| + |-z_2|$$

giving

$$|z_1| - |z_2| \leq |z_1 + z_2|\,.$$

By interchanging z_1 and z_2 in the above inequality, we have

$$|z_2| - |z_1| \leq |z_1 + z_2|\,.$$

Combining all results together

$$\Big||z_1| - |z_2|\Big| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

Example

Find an upper bound for $|z^5 - 4|$ if $|z| \leq 1$.

Solution

Applying the triangle inequality, we get

$$|z^5 - 4| \leq |z^5| + 4 = |z|^5 + 4 \leq 1 + 4 = 5,$$

since $|z| \leq 1$. Hence, if $|z| \leq 1$, an upper bound for $|z^5 - 4|$ is 5.

In general, the triangle inequality is considered as a crude inequality, which means that it will not yield least upper bound estimate. Can we find a number smaller than 5 that is also an upper bound, or is 5 the least upper bound?

Yes, we may use the Maximum Modulus Theorem (to be discussed later). The upper bound of the modulus will correspond to a complex number that lies on the boundary, where $|z| = 1$.

Example

For a non-zero z and $-\pi < \text{Arg } z \leq \pi$, show that

$$|z-1| \leq \Big||z|-1\Big| + |z|\,|\text{Arg } z|.$$

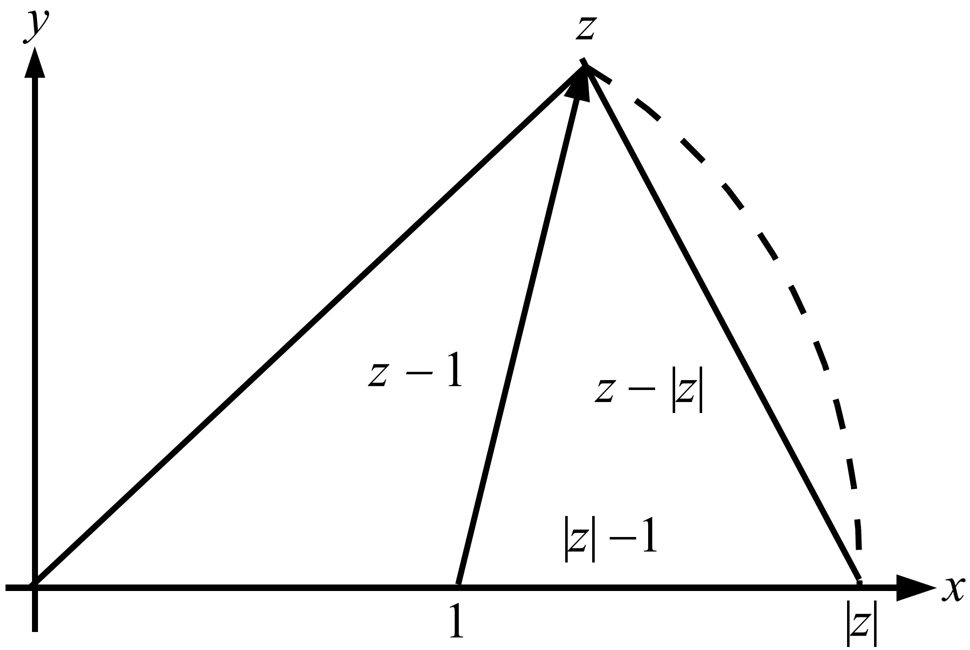
Solution

$$\begin{aligned}
|z-1| &\leq \Big|z-|z|+(|z|-1)\Big| \leq \Big|z-|z|\Big| + \Big||z|-1\Big| \\
&= |z|\,|\cos\theta + i\sin\theta - 1| + \Big||z|-1\Big|, \qquad \theta = \text{Arg } z \\
&= |z|\sqrt{(\cos\theta-1)^2+\sin^2\theta} + \Big||z|-1\Big| \\
&= |z|\left|2\sin\frac{\theta}{2}\right| + \Big||z|-1\Big| \leq |z||\text{Arg } z| + \Big||z|-1\Big|,
\end{aligned}$$

since $(\cos\theta-1)^2+\sin^2\theta = \left(1-2\sin^2\frac{\theta}{2}-1\right)^2 + 4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} = 4\sin^2\frac{\theta}{2}$

and $\left|\sin\frac{\theta}{2}\right| \leq \left|\frac{\theta}{2}\right|$.

Take $|z| > 1$ as an illustration



Geometric interpretation: Consider the triangle whose vertices are $1, z$ and $|z|$, by the Triangle Inequality, we have

$$|z-1| \leq \Big| |z| - 1 \Big| + \Big| z - |z| \Big|.$$

The chord joining $|z|$ and z is always shorter than the circular arc joining the same two points. Note that the arc length is given by $|z|\,|\text{Arg } z|$.

Geometric applications

How to find the equation of the perpendicular bisector of the line segment joining the two points z_1 and z_2? Since any point z on the bisector will be equidistant from z_1 and z_2, the equation of the bisector can be represented by

$$|z - z_1| = |z - z_2| \,.$$

For a given equation $f(x, y) = 0$ of a geometric curve, if we set $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$, the equation can be expressed in terms of the pair of conjugate complex variables z and $\overline{z}$ as

$$f(x, y) = f\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right) = F(z, \overline{z}) = 0 \,.$$

For example, the unit circle centered at the origin as represented by the equation $x^2 + y^2 = 1$ can be expressed as $z\overline{z} = 1$.

Further examples

(i) The set $\{z : |z-a| < r\}, a \in \mathbb{C}, r \in \mathbb{R}$, represents the set of points inside the circle centered at a with radius r but excluding the boundary.

(ii) The set $\{z : r_1 \leq |z-a| \leq r_2\}, a \in \mathbb{C}, r_1 \in \mathbb{R}, r_2 \in \mathbb{R}$, represents the annular region centered at a and bounded by circles of radii r_1 and r_2. Here, the boundary circles are included.

(iii) The set of points z such that $|z-\alpha| + |z-\beta| \leq 2d, \alpha$ and $\beta \in \mathbb{C}$ and $d \in \mathbb{R}$, is the set of all points on or inside the ellipse with foci α and β and with length of semi-major axis equals d. What is the length of the semi-minor axis?

Example

Find the region in the complex plane that is represented by

$$0 < \operatorname{Arg} \frac{z-1}{z+1} < \frac{\pi}{4}.$$

Solution

Let $z = x + iy$, and consider $\operatorname{Arg} \dfrac{z-1}{z+1} = \operatorname{Arg} \dfrac{x^2+y^2-1+2iy}{(x+1)^2+y^2}$, whose value lies between 0 and $\pi/4$ if and only if the following 3 conditions are satisfied

(i) $x^2 + y^2 - 1 > 0$, (ii) $y > 0$ and (iii) $\dfrac{2y}{x^2+y^2-1} < 1$.

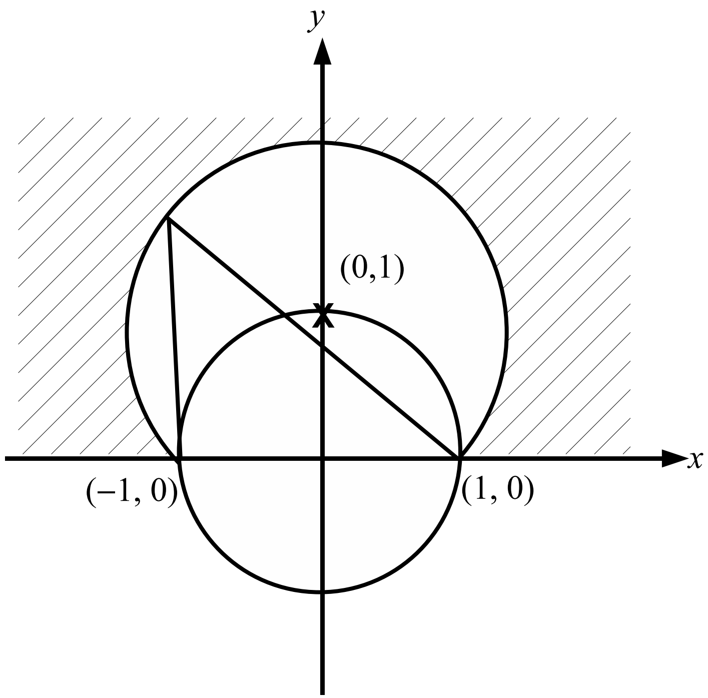
These 3 conditions correspond to

$$\operatorname{Re} \frac{z-1}{z+1} > 0, \quad \operatorname{Im} \frac{z-1}{z+1} > 0 \quad \text{and} \quad \operatorname{Arg} \frac{z-1}{z+1} < \frac{\pi}{4}.$$

The last inequality can be expressed as $x^2 + (y-1)^2 > 2$. For $y > 0$, the region outside the circle: $x^2 + (y-1)^2 = 2$ is contained completely inside the region outside the circle: $x^2 + y^2 = 1$. Hence, the region represented by the above 3 inequalities is

$$R = \{x + iy : x^2 + (y-1)^2 > 2 \quad \text{and} \quad y > 0\}.$$

This is the region which is outside the circle $x^2 + (y-1)^2 = 2$ and lying in the upper half-plane.



Example

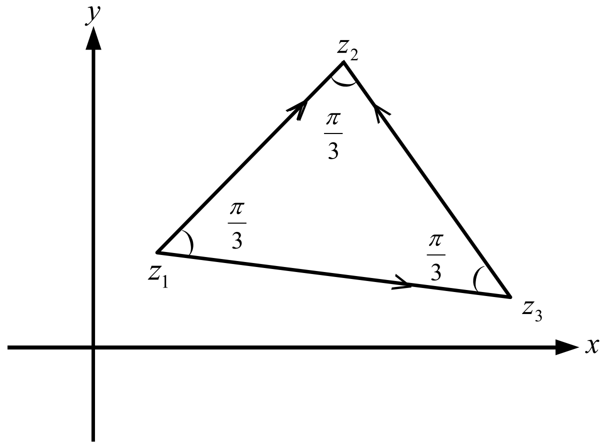
If z_1, z_2 and z_3 represent the vertices of an equilateral triangle, show that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Solution

From the figure, we observe that

$$\begin{aligned} z_2 - z_1 &= e^{i\pi/3}(z_3 - z_1) \\ z_1 - z_3 &= e^{i\pi/3}(z_2 - z_3). \end{aligned}$$



Taking the division

$$\frac{z_2 - z_1}{z_1 - z_3} = \frac{z_3 - z_1}{z_2 - z_3}$$

so that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Example

Suppose $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, show that

$$z_2 = \omega z_1 \quad \text{and} \quad z_3 = \omega^2 z_1,$$

where ω is a root of the quadratic equation: $z^2 + z + 1 = \mathbf{0}$. Note that ω is a cube root of unity. Also, the other root is given by ω^2.

Hence, explain why z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed inside the unit circle: $|z| = 1$.

Solution

Since $|z_1| = |z_2| = |z_3| = 1$, so $z_2 = e^{i\alpha} z_1$ and $z_3 = e^{i\beta} z_1$, where α and β are chosen within $(-\pi, \pi]$. Substituting into $z_1 + z_2 + z_3 = 0$, we obtain

$$e^{i\alpha} + e^{i\beta} + 1 = 0.$$

Equating the respective real and imaginary parts

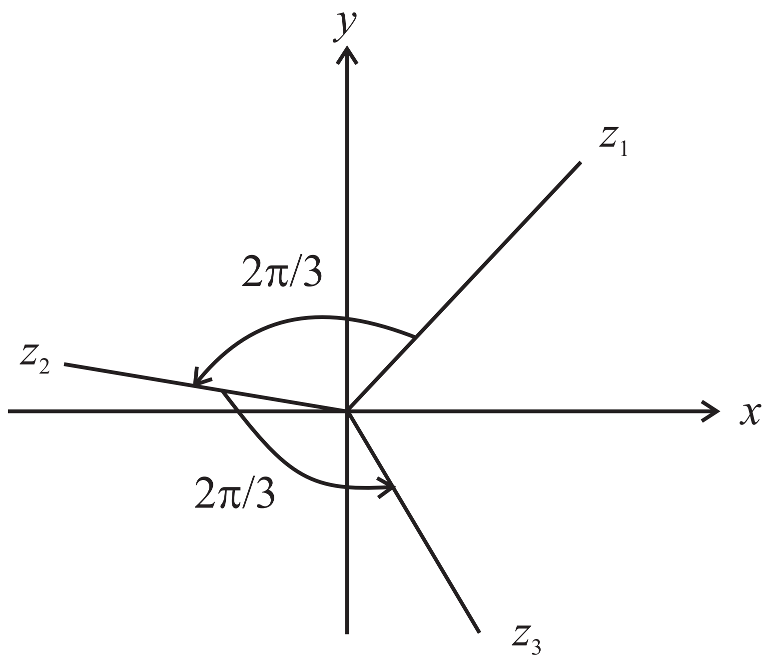
$$\cos\alpha + \cos\beta + 1 = 0 \quad \text{and} \quad \sin\alpha + \sin\beta = 0,$$

we obtain $\alpha = -\beta$ and $\cos\alpha = -1/2$.

Without loss of generality, we take α to be positive and obtain $\alpha = 2\pi/3$ and $\beta = -2\pi/3$. Both $e^{2\pi i/3}$ and $e^{-2\pi i/3}$ are the roots of $\omega^2 + \omega + 1 = 0$. Indeed, one is the square of the other, and vice versa.

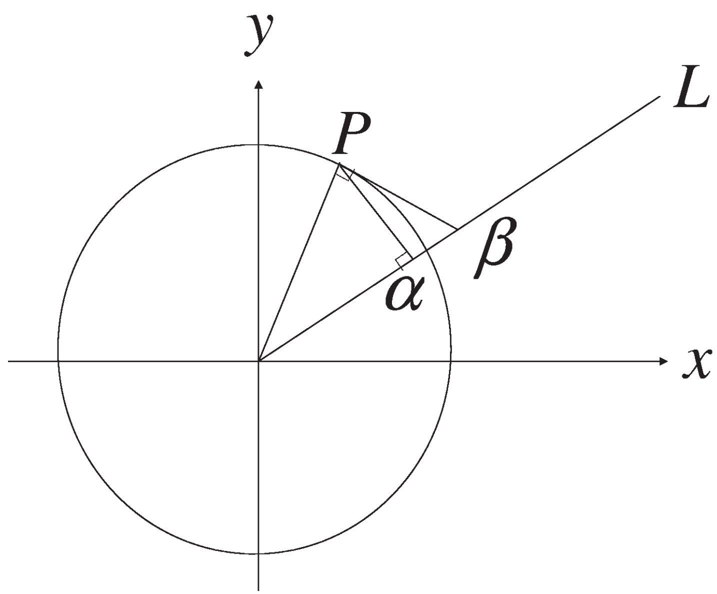
Suppose we write $\omega = e^{2\pi i/3}$, then $\omega^2 = e^{-2\pi i/3}$. Hence, $z_2 = \omega z_1$ and $z_3 = \omega^2 z_1$. The 3 points z_1, z_2 and z_3 all lie on the unit circle $|z| = 1$ since $|z_1| = |z_2| = |z_3| = 1$.

Consider the triangle formed by these 3 points: z_1, z_3 and z_3. If z_1 is one vertex, then z_2 is obtained by rotating the position vector of z_1 by $2\pi/3$ in anti-clockwise sense. Likewise, z_3 is obtained by rotating the position vector of z_2 by the same amount. Hence, z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed on the unit circle: $|z| = 1$.



Symmetry with respect to a circle

Given a point α in the complex plane, we would like to construct the symmetry point of α with respect to the circle $C_R : |z| = R$. The symmetry point of α is defined to be $\beta = R^2/\overline{\alpha}$. Conversely, since we may write $\alpha = R^2/\overline{\beta}$, we can as well consider α to be the symmetry point of β. The two points α and β are said to be symmetric with respect to the circle C_R.



Construction of a pair of symmetric points with respect to the circle C_R.

(i) Assume that $|\alpha| < R$ and so the inversion point β will be outside the circle C_R. Observe that

$$\text{Arg}\,\beta = \text{Arg}\,\frac{R^2}{\overline{\alpha}} = \text{Arg}\,\frac{1}{\overline{\alpha}} = \text{-Arg}\,\overline{\alpha} = \text{Arg}\,\alpha,$$

one concludes that α and β both lie on the same ray emanating from the origin.

The inversion point β can be constructed as follows: draw the circle C_R and a ray L from the origin through α. We then draw a perpendicular to L through α which intersects the circle C_R at P. The point of intersection of the tangent line to the circle C_R at P and the ray L then gives β. By the property of similar triangles, we have

$$\frac{|\beta|}{R} = \frac{R}{|\alpha|}.$$

(ii) When $|\alpha| = R$, the inversion point is just itself. This is because when $|\alpha| = R$, we have $|\beta| = \dfrac{R^2}{|\overline{\alpha}|} = R$. Together with Arg $\beta =$ Arg α, we obtain $\beta = \alpha$.

(iii) When $|\alpha| > R$, the inversion point β will be inside the circle C_R. In fact, α may be considered as the inversion point of β.

To reverse the method of construction, we find a tangent to the circle which passes through α and call the point of tangency P. A ray L is the drawn from the origin through α and a perpendicular is dropped from P to L. The point of intersection of the perpendicular with the ray L then gives β.

Limit points

Neighborhood of z_0, denoted by $N(z_0;\epsilon)$, is defined as

$$N(z_0;\epsilon) = \{z : |z - z_0| < \epsilon\}.$$

Sometimes, $N(z_0;\epsilon)$ may be called an open disc centered at z_0 with radius ϵ. A deleted neighborhood of z_0 is $N(z_0;\epsilon)\backslash\{z_0\}$. Write it as $\widehat{N}(z_0;\epsilon)$.

A point z_0 is a *limit point* or an *accumulation point* of a point set S if every neighborhood of z_0 contains a point of S other than z_0. That is, $\widehat{N}(z_0;\epsilon) \cap S \neq \phi$, for any ϵ.

- Since this is true for any neighborhood of z_0, S must contain infinitely many points. In other words, if S consists of discrete number of points, then there is no limit point of S.
- A limit point z_0 may or may not belong to the set S.

Example

Show that $z = 1$ is a limit point of the following point set:

$$A = \left\{ z : z = (-1)^n \frac{n}{n+1}, n \text{ is an integer} \right\}.$$

Solution

Given any $\epsilon > 0$, we want to show that there exists one point in A other than $z = 1$. In fact, there exist an infinite number of points such that

$$\left| (-1)^n \frac{n}{n+1} - 1 \right| < \epsilon \,.$$

If we choose n to be even and positive and $n + 1 > \frac{1}{\epsilon}$, then $\left| \frac{n}{n+1} - 1 \right| < \epsilon$. Therefore, for any ϵ, $N(1; \epsilon)$ contains at least one point in A other than $z = 1$.

Some topological definitions

- A point z_0 is called an *interior point* of a set S if there exists a neighborhood of z_0 with all of these points belong to S.

- If every neighborhood of z_0 contains points of S and also points not belonging to S then z_0 is called a *boundary point*.

- The set of all boundary points of a set S is called the *boundary* of S. If a point is neither an interior nor a boundary point, then it is called an *exterior point* of S.

Remark

If z_0 is not a boundary point, then there exists a neighborhood of z_0 such that it is completely inside S or completely outside S. In the former case, it is an interior point. In the latter case, it is an exterior point.

Example

Show that the boundary of

$$B_r(z_0) = \{z : |z - z_0| < r\}$$

is the circle: $|z - z_0| = r$.

Solution

Pick a point z_1 on the circle $|z-z_0| = r$. Every disk that is centered at z_1 will contain (infinitely many) points in $B_r(z_0)$ and (infinitely many) points not in $B_r(z_0)$. Hence, every point on the circle $|z - z_0| = r$ is a boundary point of $B_r(z_0)$.

No other points are boundary points

- Since points inside the circle are interior points, they cannot be boundary points.
- Given a point outside the circle, we can enclose it in a disk that does not intersect the disk $B_r(z_0)$. Hence, such a point is not boundary point.

In this example, none of the boundary points of $B_r(z_0)$ belong to $B_r(z_0)$.

Open and closed sets

- A set which consists only of interior points is called an *open set*.

- Every point of an open set has a neighborhood contained completely in the set.

- A set is closed if it contains all its boundary points.

- The closure $\overline{S}$ of a set S is the closed set consisting of all points in S together with the boundary of S.

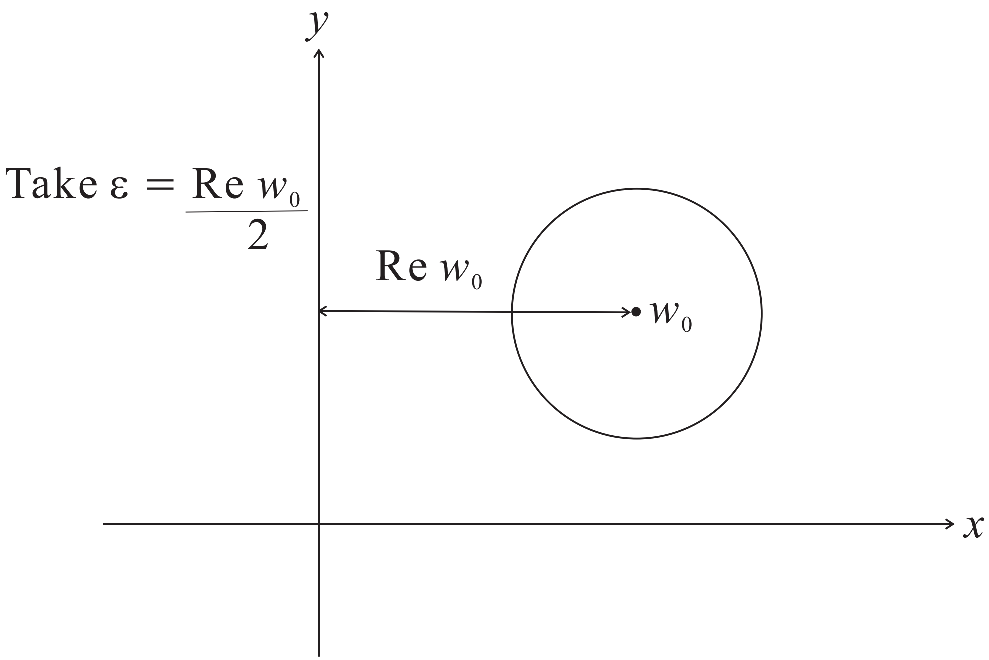
We may think of any two-dimensional set without a boundary as an open set. For example, the set $S = \{z : |z| < 1\}$ is an open set. The closure $\overline{S}$ is the set $\{z : |z| \leq 1\}$.

Example

The set $R = \{z : \text{Re } z > 0\}$ is an open set. To see this, for any point $w_0 \in R$; then $\sigma_0 = \text{Re } w_0 > 0$. Let $\varepsilon = \sigma_0/2$ and suppose that $|z - w_0| < \varepsilon$. Then $-\varepsilon < \text{Re } (z - w_0) < \varepsilon$ and so

$$\text{Re } z = \text{Re } (z - w_0) + \text{Re } w_0 > -\varepsilon + \sigma_0 = \sigma_0/2 > 0.$$

Consequently, z also lies in R. Hence, each point w_0 of R is an interior point and so R is open.



Theorem

A set D is open if and only if it contains no point of its boundary.

Proof

- Suppose that D is an open set, and let P be a boundary point of D. If P is in D, then there is an open disc centered at P that lies within D (since D is open). Hence, P is not in the boundary of D.

- Suppose D is a set that contains none of its boundary points, for any $z_0 \in D$, z_0 cannot be a boundary point of D. Hence, there is some disc centered at z_0 that is either a subset of D or a subset of the complement of D. The latter is impossible since z_0 itself is in D. Hence, each point of D is an interior point so that D is open.

Corollary

A set C is closed if and only if its complement $D = \{z : z \notin C\}$ is open. To see the claim, we observe that the boundary of a set coincides exactly with the boundary of the complement of that set (as a direct consequence of the definition of boundary point). Recall that a closed set contains all its boundary points. Its complement shares the same boundary, but these boundary points are not contained in the complement, so the complement is open.

Remark

There are sets that are neither open nor closed since they contain part, but not all, of their boundary. For example,

$$S = \{z : 1 < |z| \leq 2\}$$

is neither open nor closed.

Theorem

A set S is closed if and only if S contains all its limit points.

Proof

Write $\widehat{N}(z;\epsilon)$ as the deleted ϵ-neighborhood of z, and S' as the complement of S. Note that for $z \notin S, N(z;\epsilon) \cap S = \widehat{N}(z;\epsilon) \cap S$.

$$\begin{aligned}
S \text{ is closed} \quad &\Leftrightarrow \quad S' \text{ is open} \\
&\Leftrightarrow \quad \text{given } z \notin S \text{, there exists } \epsilon > 0 \text{ such that } N(z;\epsilon) \subset S' \\
&\Leftrightarrow \quad \text{given } z \notin S \text{, there exists } \epsilon > 0 \text{ such that } \widehat{N}(z;\epsilon) \cap S = \phi \\
&\Leftrightarrow \quad \text{no point of } S' \text{ is a limit point of } S
\end{aligned}$$

Compact set

- A *bounded set* S is one that can be contained in a large enough circle centered at the origin, that is, $|z| < M$ for some large enough constant M for all points z in S where M is some sufficiently large constant.

- An *unbounded* set is one which is not bounded.

- A set which is both closed and bounded is called a *compact set*.

Motivation for defining connectedness

From basic calculus, if $f'(x) = 0$ for all x in (a, b), then f is a constant.

The above result is *not* true if the domain of definition of the function is not connected. For example

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } 2 < x < 3 \end{cases},$$

whose domain of definition is $(0, 1) \cup (2, 3)$. We have $f'(x) = 0$ for all x in $(0, 1) \cup (2, 3)$, but f is not constant.

This example illustrates the importance of connectedness in calculus.

Domain and region

- An open set S is said to be *connected* if any two points of S can be joined by a continuous curve lying entirely inside S. For example, a neighborhood $N(z_0; \epsilon)$ is connected.

- An *open connected* set is called an *open region* or *domain*.

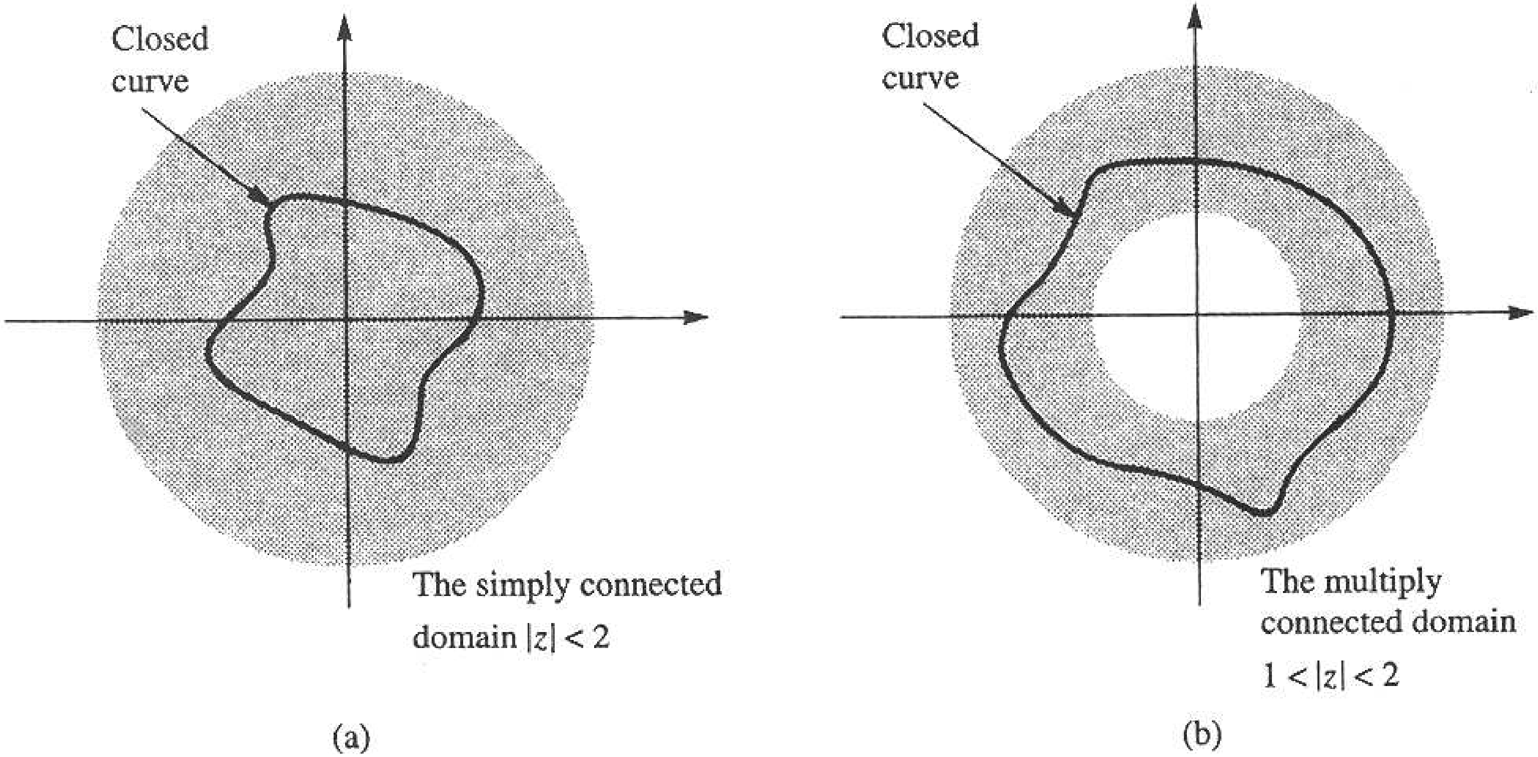
- To the point set S, we add all of its limit points, then the new set $\overline{S}$ is the closure of S.

- The closure of an open region is called a *closed region*.

- To an open region we may add none, some or all its limit points, and call the new set a *region*.

Simply and multiply connected domains

Loosely speaking, a simply connected domain contains no holes, but a multiply connected domain has one or more holes.



- When any closed curve is constructed in a simply connected domain, every point inside the curve lies in the domain.

- It is always possible to construct some closed curve inside a multiply connected domain in such a way that one or more points inside the curve do not belong to the domain.

Complex infinity

- Unlike the real number system, which has $+\infty$ and $-\infty$, there is only one infinity in the complex number system. This is because the complex number field $\mathbb{C}$ is not an ordered field*. Recall that in the real number system, $+\infty$ ($-\infty$) is an upper (lower) bound of every subset of the extended real number system (set of all real numbers augmented with $+\infty$ and $-\infty$).

*$\mathbb{C}$ is not an ordered field can be argued as follows. Suppose $\mathbb{C}$ is an ordered field, then for any non-zero $x \in \mathbb{C}$, we must have either x or $-x$ being positive, and $x^2 = (-x)^2$ being positive. Consider i and $-i$, we have $i^2 = (-i)^2 = -1$, which is negative. A contradiction is encountered.

Extended complex plane

It is convenient to augment the complex plane with the point at infinity, denoted by ∞. The set $\{\mathbb{C} \cup \infty\}$ is called the *extended complex plane*. The algebra involving ∞ are defined as follows:

$$\begin{aligned}
z + \infty = \infty\,, &\qquad \text{for all } z \in \mathbb{C} \\
z \cdot \infty = \infty\,, &\qquad \text{for all } z \in \mathbb{C}/\{0\} \\
z/\infty = 0\,, &\qquad \text{for all } z \in \mathbb{C} \\
z/0 = \infty\,, &\qquad \text{for all } z \in \mathbb{C}/\{0\}.
\end{aligned}$$

In particular, we have $-1\cdot\infty = \infty$ and expressions like $0\cdot\infty, \infty/\infty, \infty \pm \infty$ and $0/0$ are not defined. Topologically, any set of the form $\{z : |z| > R\}$ where $R \geq 0$ is called a *neighborhood of* ∞.

- A set D contains the point at infinity if there is a large number M such that D contains all the points z with $|z| > M$.

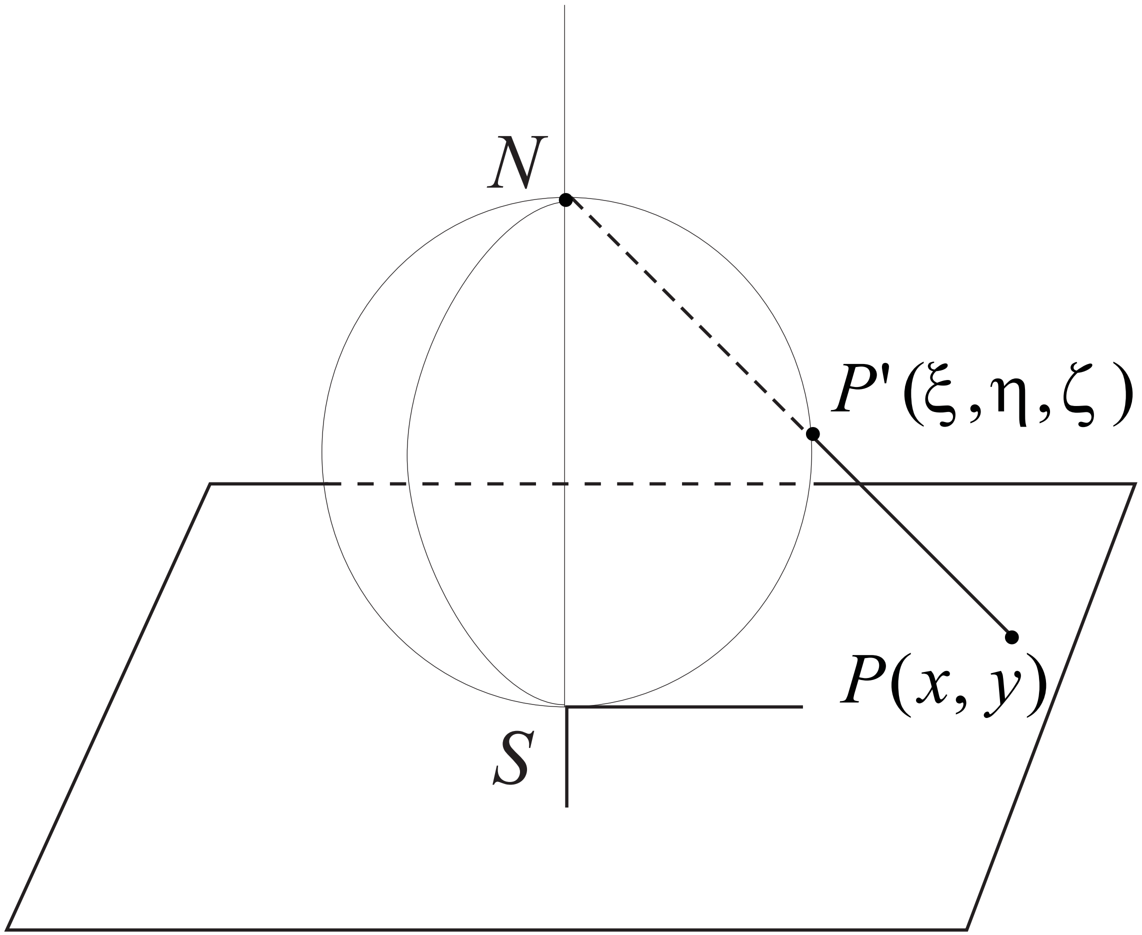
Examples

- The open half-plane $\operatorname{Re} z > 0$ does not contain the point at infinity since it does not contain any neighborhood of ∞.

- The open set $D = \{z : |z+1| + |z-1| > 1\}$ does.

One "reaches" the point at infinity by letting $|z|$ increase without bound, with no restriction at all on $\arg z$. One way to visualize all this is to let $w = 1/z$ and think about $|w|$ being very small: an open set containing the point at infinity will become an open set containing $w = 0$.

The statement "z approaches infinity" is identical with "w converges to zero".



Riemann sphere is sitting on the complex plane.

Riemann sphere and stereographic projection

- In order to visualize the point at infinity, we consider the *Riemann sphere* that has radius 1/2 and is tangent to the complex plane at the origin (see Figure). We call the point of contact the south pole (denoted by S) and the point diametrically opposite S the north pole (denoted by N).

- Let z be an arbitrary complex number in the complex plane, represented by the point P. We draw the straight line PN which intersects the Riemann sphere at a unique point P' distinct from N.

- There exists a one-to-one correspondence between points on the Riemann sphere, except N, and all the finite points in the complex plane. We assign the north pole N as the point at infinity. This correspondence is known as the *stereographic projection*.

The coordinates are related by

$$x = \frac{\xi}{1-\zeta} \quad \text{and} \quad y = \frac{\eta}{1-\zeta}. \tag{A}$$

The equation of the Riemann sphere is given by

$$\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \quad \text{or} \quad \xi^2 + \eta^2 + \zeta^2 = \zeta. \tag{B}$$

We substitute $\xi = x(1-\zeta)$ and $\eta = y(1-\zeta)$ into eq. (B) to obtain

$$\zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1} = \frac{|z|^2}{|z|^2 + 1}.$$

Once ζ is available, we use eq. (A) to obtain

$$\begin{aligned}
\xi &= \frac{x}{x^2 + y^2 + 1} = \frac{1}{2}\frac{z + \overline{z}}{|z|^2 + 1}, \\
\eta &= \frac{y}{x^2 + y^2 + 1} = \frac{1}{2i}\frac{z - \overline{z}}{|z|^2 + 1}.
\end{aligned}$$