2. Functions and limits: Analyticity and Harmonic Functions

Let S be a set of complex numbers in the complex plane. For every point $z = x + iy \in S$, we specific the rule to assign a corresponding complex number $w = u+iv$. This defines a function of the complex variable z, and the function is denoted by

$$w = f(z).$$

The set S is called the *domain of definition* of the function f and the collection of all values of w is called the *range* of f.

1. $f(z) = \text{Arg } z$ is defined everywhere except at $z = 0$, and Arg z can assume all possible real values in the interval $(-\pi, \pi]$.

2. The domain of definition of $f(z) = \dfrac{z+3}{z^2+1}$ is $\mathbb{C}\backslash\{i, -i\}$. What is the range of this function?

Real and imaginary parts of a complex function

Let $z = x + iy$. A complex function of the complex variable z may be visualized as a pair of real functions of the two real variables x and y. Let $u(x, y)$ and $v(x, y)$ be the real and imaginary parts of $f(z)$, respectively.

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy.$$

Consider the function

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy;$$

its real and imaginary parts are the real functions

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Example

Determine whether

$$w(z) = \frac{iz + 4}{2z + 3i}$$

is one-to-one.

Solution

For any two complex numbers z_1 and z_2, we have

$$\begin{aligned}
& w(z_1) = w(z_2) \\
\iff & \frac{iz_1 + 4}{2z_1 + 3i} = \frac{iz_2 + 4}{2z_2 + 3i} \\
\iff & 2iz_1z_2 + 8z_2 - 3z_1 + 12i = 2iz_1z_2 + 8z_1 - 3z_2 + 12i \\
\iff & z_1 = z_2.
\end{aligned}$$

Therefore, the function is one-to-one.

Complex velocity of a fluid source

Find a complex function $v(z)$ which gives the velocity of the flow at any point z due to a fluid source at the origin.

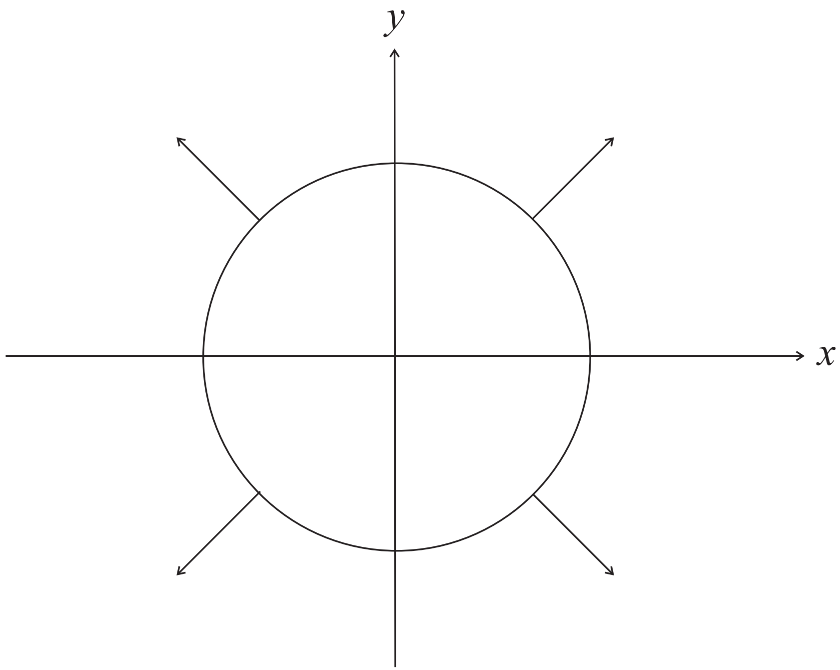
The direction of the velocity is radially outward from the fluid source and the magnitude of velocity is inversely proportional to the distance from the source. We observe

$$\text{Arg } v = \text{Arg } z \quad \text{and} \quad |v| = \frac{k}{|z|},$$

where k is some real positive constant. If we write $z = re^{i\theta}$, the velocity function is given by

$$v(z) = |v|e^{i\text{Arg } v} = \frac{k}{|z|}\, e^{i\theta} = \frac{k}{\overline{z}}, \quad z \neq 0.$$

The constant k is the strength of the source, which is related to the amount of fluid flowing out from the source per unit time.



Let $U(r)$ be the speed at any point on the circle $|z| = r$. From physics, the speed at any point in the flow field depends on the radial distance from the fluid source. We then have $U(r) = k/r$. Write m as the amount of fluid flowing out from the source per unit time so that

$$m = 2\pi r U(r) = 2\pi k$$

giving

$$k = \frac{m}{2\pi}.$$

Example

Consider the function $f(z) = z^2$ and write it as

$$f(z) = u(x, y) + iv(x, y) \qquad \text{where} \qquad z = x + iy.$$

1. Find the curves in the x-y plane such that $u(x, y) = \alpha$ and $v(x, y) = \beta$.

2. Find the curves in the u-v plane such that the preimage curves in the x-y plane are $x = a$ and $y = b$.

3. What is the image curve in the u-v plane of the closed curve: $r = 2(1 + \cos\theta)$ in the x-y plane?

Solution

For $z = x + iy$,

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy$$

so that

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

For all points on the hyperbola: $x^2 - y^2 = \alpha$ in the x-y plane, the image points in the w plane are on the coordinate curve $u = \alpha$.

Similarly, the points on the hyperbola: $2xy = \beta$ are mapped onto the coordinate curve $v = \beta$.

Recall the result:

$$(u+iv)^{1/2} = \pm\left[\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}} + i\frac{v}{|v|}\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}\right].$$

The image curve in the w-plane corresponding to $x = a$ is given by

$$\frac{u+\sqrt{u^2+v^2}}{2} = a^2 \quad \Longleftrightarrow \quad 4a^2(a^2-u) = v^2.$$

Similarly, the image curve in the w-plane corresponding to $y = b$ is given by

$$\frac{\sqrt{u^2+v^2}-u}{2} = b^2 \quad \Longleftrightarrow \quad 4b^2(b^2+u) = v^2.$$

Both image curves are parabolas in the w-plane.

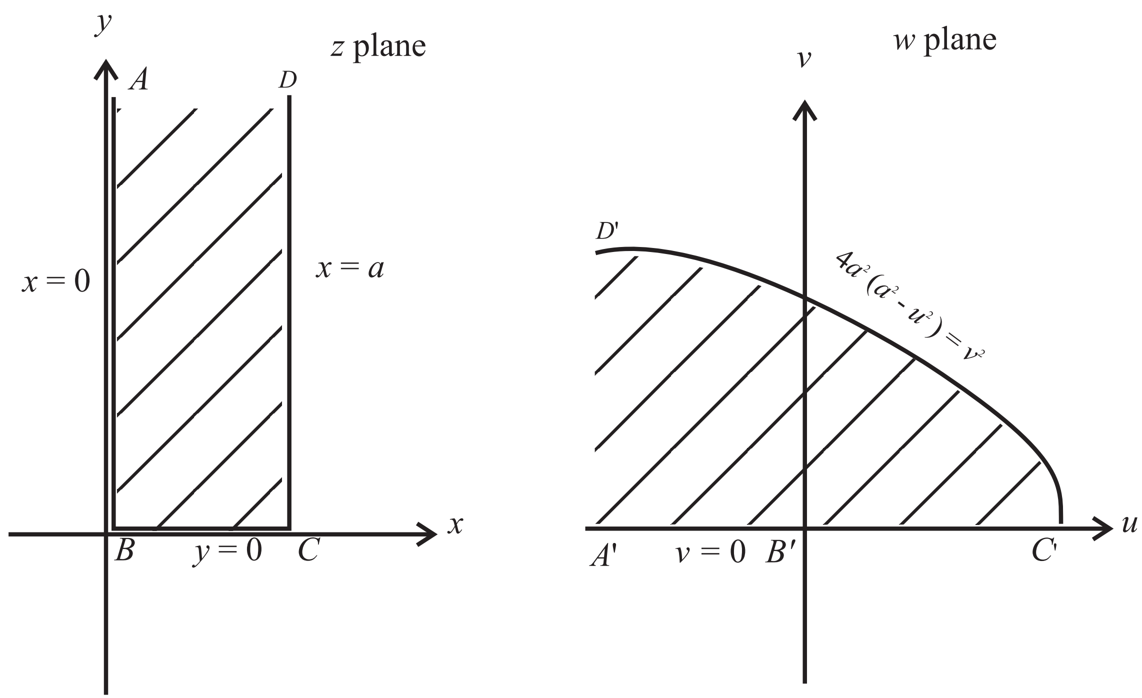
Mapping properties of the complex function $w = z^2$

(i) For $z = iy, y > 0$, we have $w = z^2 = -y^2$ so that the line: $x = 0, y > 0$ is mapped onto the line: $u < 0, v = 0$.

(ii) For $z = x$, we have $w = z^2 = x^2$ so that the line: $0 \leq x \leq a, y = 0$ is mapped onto the line: $0 \leq u \leq a^2, v = 0$.

(iii) For $z = a + iy, y > 0$, we have $u = a^2 - y^2$ and $v = 2aiy$ so that the line: $x = a, a > 0, y > 0$ is mapped onto the upper portion of parabola: $4a^2(a^2 - u) = v^2, v > 0$.

Hence, the semi-infinite strip: $\{(x, y) : 0 \leq x \leq a, y \geq 0\}$ in the z plane is mapped onto the semi-infinite parabolic wedge: $\{(u, v) : 4a^2(a^2 - u) \geq v^2, v \geq 0\}$ in the w-plane.



The complex function $w = z^2$ maps a semi-infinite strip onto a semi-infinite parabolic wedge.

Let $Re^{i\phi}$ be the polar representation of w. Since $w = z^2$, and $z = re^{i\theta}$ in polar form, we have $Re^{i\phi} = r^2e^{2i\theta}$. Comparing, we deduce that

$$\phi = 2\theta \quad \text{and} \quad R = r^2.$$

Consider a point on the curve whose polar form is $r = 2(1+\cos\theta) = 4\cos^2\frac{\theta}{2}$, we see that

$$r^2 = 16\cos^4\frac{\theta}{2}.$$

Hence, we obtain

$$R = r^2 = 16\cos^4\frac{\theta}{2} = 16\cos^4\frac{\phi}{4}$$

so that the polar form of the image curve in the w-plane is

$$R = 16\cos^4\frac{\phi}{4}.$$

Limit of a complex function

Let $w = f(z)$ be defined in the point set S and z_0 be a limit point of S. The mathematical statement

$$\lim_{z \to z_0} f(z) = L, \quad z \in S,$$

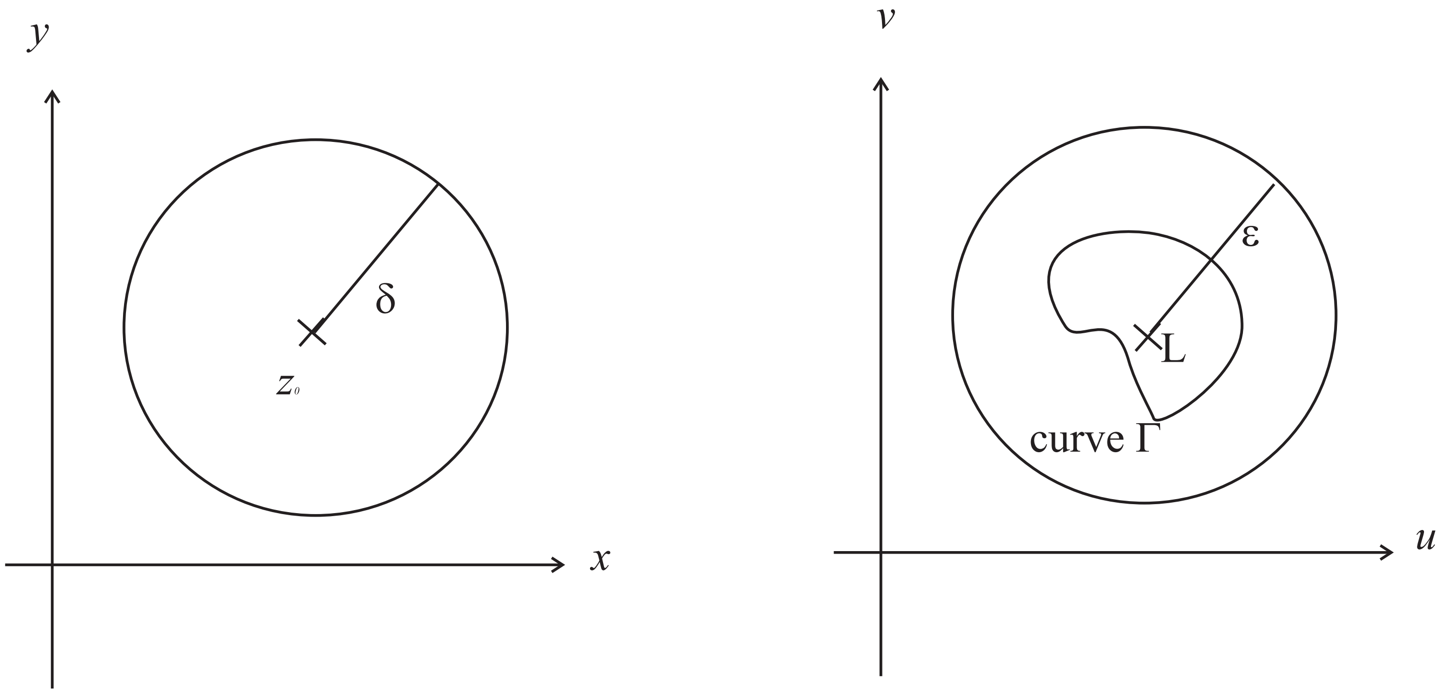
means that the value $w = f(z)$ can be made arbitrarily close to L if we choose z to be close enough, but not equal, to z_0. The formal definition of the limit of a function is stated as:

For any $\epsilon > 0$, there exists $\delta > 0$ (usually dependent on ϵ) such that

$$|f(z) - L| < \epsilon \quad \text{if} \quad 0 < |z - z_0| < \delta.$$

Remark

We require z_0 to be a limit point of S so that it would not occur that in some neighborhood of z_0 inside which $f(z)$ is not defined.



The circle $|z - z_0| = \delta$ in the z-plane is mapped onto the closed curve Γ in the w-plane. The annulus $0 < |z - z_0| < \delta$ in the z-plane is mapped onto the region enclosed by the curve Γ in the w-plane. The curve Γ lies completely inside the annulus $0 < |w - L| < \epsilon$.

- The function $f(z)$ needs not be defined at z_0 in order for the function to have a limit at z_0.

- The limit L, if it exists, must be unique.

- The value of L is independent of the direction along which z approaches z_0.

Example

Let S be the point set where $\frac{\sin z}{z}$ is defined. It can be shown that

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

though the function $\frac{\sin z}{z}$ is not defined at $z = 0$. Note that $z = 0$ is a limit point of S. Here, S is actually $\mathbb{C} \backslash \{0\}$.

Example

Prove that $\lim_{z \to i} z^2 = -1$.

We show that for given $\epsilon > 0$ there is a positive number δ such that

$$|z^2 - (-1)| < \epsilon \quad \text{whenever} \quad 0 < |z - i| < \delta.$$

Since

$$z^2 - (-1) = (z - i)(z + i) = (z - i)(z - i + 2i),$$

it follows from the the triangle inequality that

$$|z^2 - (-1)| = |z - i| \; |z - i + 2i| \leq |z - i|(|z - i| + 2). \qquad (1)$$

To ensure that the left-hand side of (1) is less than ϵ, we merely have to insist that z lies in a δ-neighborhood of i, where $\delta = \min(1, \epsilon/3)$. If so, then

$$|z - i|(|z - i| + 2) \leq \frac{\epsilon}{3}(1 + 2) = \epsilon.$$

Some properties of the limit of a function

If $L = \alpha + i\beta, f(z) = u(x, y) + iv(x, y), z = x + iy$ and $z_0 = x_0 + iy_0$, then

$$|u(x, y) - \alpha| \leq |f(z) - L| \leq |u(x, y) - \alpha| + |v(x, y) - \beta|,$$
$$|v(x, y) - \beta| \leq |f(z) - L| \leq |u(x, y) - \alpha| + |v(x, y) - \beta|.$$

It is obvious that $\lim_{z \to z_0} f(z) = L$ is equivalent to the following pair of limits

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = \alpha$$
$$\lim_{(x,y) \to (x_0,y_0)} v(x, y) = \beta.$$

Therefore, the study of the limiting behavior of $f(z)$ is equivalent to that of a pair of real functions $u(x, y)$ and $v(x, y)$.

Consequently, theorems concerning the limit and continuity of the sum, difference, product and quotient of complex functions can be inferred from those for real functions.

Suppose that

$$\lim_{z \to z_0} f_1(z) = L_1 \quad \text{and} \quad \lim_{z \to z_0} f_2(z) = L_2,$$

then

$$\lim_{z \to z_0} [f_1(z) \pm f_2(z)] = L_1 \pm L_2,$$
$$\lim_{z \to z_0} f_1(z) f_2(z) = L_1 L_2,$$
$$\lim_{z \to z_0} \frac{f_1(z)}{f_2(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

Limit at infinity

The definition of limit holds even when z_0 or L is the point at infinity. We can simply replace the corresponding neighborhood of z_0 or L by the neighborhood of ∞. The statement

$$\lim_{z \to \infty} f(z) = L$$

can be understood as

> For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $|f(z) - L| < \epsilon$ whenever $|z| > \frac{1}{\delta}$.

As for convention, z refers to a point in the finite complex plane. Hence, $|z| > \frac{1}{\delta}$ is a deleted neighborhood of ∞.

Theorem

If z_0 and w_0 are points in the z-plane and w-plane, respectively, then

(i) $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \dfrac{1}{f(z)} = 0$;

(ii) $\lim_{z \to \infty} f(z) = w_0$ if and only if $\lim_{z \to 0} f\left(\dfrac{1}{z}\right) = w_0$.

Proof

(i) $\lim_{z \to z_0} f(z) = \infty$ implies that for any $\epsilon > 0$, there is a positive number δ such that

$$|f(z)| > \frac{1}{\epsilon} \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

We may rewrite as

$$\left| \frac{1}{f(z)} - 0 \right| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Hence, $\lim_{z \to z_0} \dfrac{1}{f(z)} = 0$.

(ii) $\lim_{z\to\infty} f(z) = w_0$ implies that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad |z| > \frac{1}{\delta}.$$

Replacing z by $1/z$, we obtain

$$\left| f\left(\frac{1}{z}\right) - w_0 \right| < \epsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta,$$

hence $\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$.

Example

$$\lim_{z\to -1} \frac{iz+3}{z+1} = \infty \quad \text{since} \quad \lim_{z\to -1} \frac{z+1}{iz+3} = 0$$

and

$$\lim_{z\to\infty} \frac{2z+i}{z+1} = 2 \quad \text{since} \quad \lim_{z\to 0} \frac{\frac{2}{z}+i}{\frac{1}{z}+1} = \lim_{z\to 0} \frac{2+iz}{1+z} = 2.$$

Continuity of a complex function

Continuity of a complex function is defined in the same manner as for a real function. The complex function $f(z)$ is said to be *continuous* at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The statement implicitly implies the existence of both $\lim_{z \to z_0} f(z)$ and $f(z_0)$. Alternatively, the statement can be understood as:

For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that
$|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Example

Discuss the continuity of the following functions at $z = 0$.

(a) $f(z) = \frac{\text{Im } z}{1+|z|}$;

(b) $f(z) = \begin{cases} 0 & z = 0 \\ \frac{\text{Re } z}{|z|} & z \neq 0 \end{cases}$.

Solution

(a) Let $z = x + iy$, then $\frac{\text{Im } z}{1+|z|} = \frac{y}{1+\sqrt{x^2+y^2}}$. Now,

$$\lim_{z \to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{y}{1+\sqrt{x^2+y^2}} = 0 = f(0).$$

Therefore, $f(z)$ is continuous at $z = 0$.

(b) Let $z = x + iy$, then

$$\frac{\text{Re } z}{|z|} = \frac{x}{\sqrt{x^2 + y^2}}.$$

Suppose z approaches 0 along the half straight line $y = mx$ $(x > 0)$, then

$$\lim_{\substack{z \to 0, \\ y = mx, x > 0}} \frac{\text{Re } z}{|z|}$$

$$= \lim_{x \to 0^+} \frac{x}{\sqrt{x^2 + m^2 x^2}} = \lim_{x \to 0^+} \frac{x}{x\sqrt{1 + m^2}} = \frac{1}{\sqrt{1 + m^2}}.$$

Since the limit depends on m, $\lim_{z \to 0} f(z)$ does not exist. Therefore, $f(z)$ cannot be continuous at $z = 0$.

Properties of continuous functions

A complex function is said to be continuous in a region R if it is continuous at every point in R.

Since the continuity of a complex function is defined using the concept of limits, it can be shown that $\lim_{z \to z_0} f(z) = f(z_0)$ is equivalent to

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} u(x,y) &= u(x_0,y_0), \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) &= v(x_0,y_0). \end{aligned}$$

For example, consider $f(z) = e^z$; its real and imaginary parts are, respectively, $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. Since both $u(x,y)$ and $v(x,y)$ are continuous at any point (x_0, y_0) in the finite x-y plane, we conclude that e^z is continuous at any point $z_0 = x_0 + iy_0$ in $\mathbb{C}$.

Examples of continuous functions in $\mathbb{C}$ are polynomials, exponential functions and trigonometric functions.

Sum, difference, product and quotient of continuous functions remain to be continuous.

Uniform continuity

Suppose $f(z)$ is continuous in a region R, then by hypothesis, at each point z_0 inside R and for any $\epsilon > 0$, we can find $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

Usually δ depends on ϵ and z_0 together. However, if we can find a single value of δ for each ϵ, independent of z_0 chosen in R, we say that $f(z)$ is *uniformly continuous* in the region R.

Example

Prove that $f(z) = z^2$ is uniformly continuous in the region $|z| \leq R$, R is finite.

Solution

We must show that given any $\epsilon > 0$, we can find $\delta > 0$ such that $|z^2 - z_0^2| < \epsilon$ when $|z - z_0| < \delta$, where δ depends *only* on ϵ and not on the particular point z_0 of the region.

If z and z_0 are any points in $|z| \leq R$, then

$$|z^2 - z_0^2| = |z + z_0|\ |z - z_0| \leq \{|z| + |z_0|\}|z - z_0| < 2R|z - z_0|.$$

Thus if $|z - z_0| < \delta$, it follows that $|z^2 - z_0^2| < 2R\delta$. Choosing $\delta = \frac{\epsilon}{2R}$, we see that $|z^2 - z_0^2| < \epsilon$ when $|z - z_0| < \delta$, where δ depends only on ϵ but not on z_0. Hence, $f(z) = z^2$ is uniformly continuous in the region.

Example

Prove that $f(z) = 1/z$ is not uniformly continuous in the region $0 < |z| < 1$.

Solution

Given $\delta > 0$, let z_0 and $z_0 + \xi$ be any two points of the region such that

$$|z_0 + \xi - z_0| = |\xi| = \delta.$$

Then

$$|f(z_0 + \xi) - f(z_0)| = \left| \frac{1}{z_0 + \xi} - \frac{1}{z_0} \right| = \frac{|\xi|}{|z_0||z_0 + \xi|} = \frac{\delta}{|z_0||z_0 + \xi|}$$

can be made larger than any positive number by choosing z_0 sufficiently close to 0. Hence, the function cannot be uniformly continuous in the region.

Differentiation of complex functions

Let the complex function $f(z)$ be *single-valued* in a neighborhood of a point z_0. The derivative of $f(z)$ at z_0 is defined by

$$\begin{aligned}\frac{df}{dz}(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}, \qquad \triangle z = z - z_0,\end{aligned}$$

provided that the above limit exists. The value of the limit must be independent of the path of z approaching z_0.

Many formulas for the computation of derivatives of complex functions are the same as those for the real counterparts.

The existence of the derivative of a complex function at a point implies the continuity of the function at the same point. Suppose $f'(z_0)$ exists, we can write

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0) = 0,$$

so that

$$\lim_{z \to z_0} f(z) = f(z_0).$$

This shows that $f(z)$ is continuous at z_0. However, continuity of $f(z)$ may not imply the differentiability of $f(z)$ at the same point.

It may occur that a complex function can be differentiable at a given point but not so in any neighborhood of that point.

Example

Show that the functions $\overline{z}$ and Re z are nowhere differentiable, while $|z|^2$ is differentiable only at $z = 0$.

Solution

The derivative of $\overline{z}$ is given by

$$\frac{d}{dz}\overline{z} = \lim_{\triangle z \to 0} \frac{\overline{z} + \triangle\overline{z} - \overline{z}}{\triangle z} = \lim_{\triangle z \to 0} \frac{\triangle\overline{z}}{\triangle z} = \lim_{\triangle z \to 0} [e^{-2i \text{ Arg } \triangle z}].$$

The value of the limit depends on the path approaching z. Therefore, $\overline{z}$ is nowhere differentiable.

Similarly,

$$\begin{aligned}\frac{d}{dz}\text{Re } z &= \frac{d}{dz}\,\frac{1}{2}\,(z+\overline{z}) \\ &= \frac{1}{2}\lim_{\triangle z\to 0}\frac{(z+\overline{z}+\triangle z+\triangle\overline{z})-(z+\overline{z})}{\triangle z} \\ &= \frac{1}{2}\lim_{\triangle z\to 0}\frac{\triangle z+\triangle\overline{z}}{\triangle z}=\frac{1}{2}+\frac{1}{2}\lim_{\triangle z\to 0}\frac{\triangle\overline{z}}{\triangle z}.\end{aligned}$$

Again, Re z is shown to be nowhere differentiable.

Lastly, the derivative of $|z|^2$ is given by

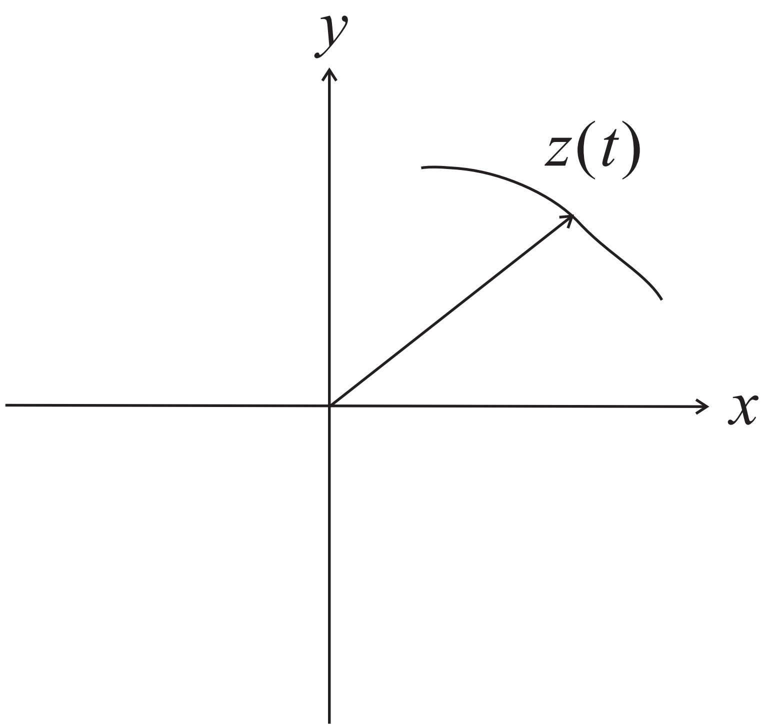
$$\frac{d}{dz}|z|^2 = \lim_{\triangle z\to 0}\frac{|z+\triangle z|^2-|z|^2}{\triangle z}=\lim_{\triangle z\to 0}\left[\overline{z}+z\,\frac{\triangle\overline{z}}{\triangle z}+\triangle\overline{z}\right].$$

The above limit exists only when $z=0$, that is, $|z|^2$ is differentiable only at $z=0$.

Complex velocity and acceleration

We treat $z(t)$ as the position vector of a particle with time parameter t. Then the velocity of the particle is

$$\frac{dz(t)}{dt} = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}.$$



Decomposition into various components

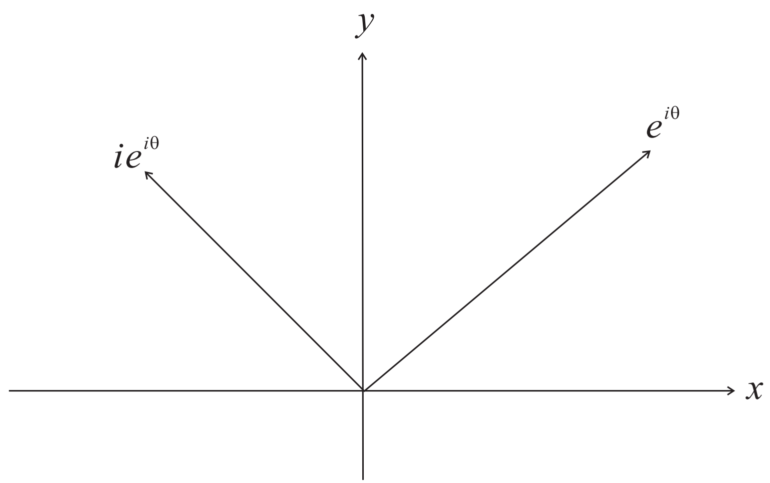
Write $z(t)$ in its polar form

$$z(t) = r(t)e^{i\theta(t)}$$

$u = \dot{z} = \dot{r}e^{i\theta} + r\dot{\theta}(ie^{i\theta})$

$u_r = \dot{r}$ = radial component of complex velocity

$u_\theta = r\dot{\theta}$ = tangential component of complex velocity



Note that $e^{i\theta}$ and $ie^{i\theta}$ represent the unit vector along the radial and tangential direction, respectively.

$$a = \frac{du}{dt} = \underbrace{(\ddot{r} - r\dot{\theta}^2)e^{i\theta}}_{\substack{\text{radial component} \\ \text{of acc} = a_r}} + \underbrace{(2\dot{r}\dot{\theta} + r\ddot{\theta})ie^{i\theta}}_{\substack{\text{tangential component} \\ \text{of acc} = a_\theta}}$$

$a_r = \ddot{r} - r\dot{\theta}^2, a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$

Each term has its individual terminology

$$\begin{aligned} \ddot{r} &= \text{radial acceleration} \\ -r\dot{\theta}^2 &= \text{centripetal acceleration} \\ r\ddot{\theta} &= \text{tangential acceleration} \\ 2\dot{r}\dot{\theta} &= \text{Coriolis acceleration} \end{aligned}$$

Example

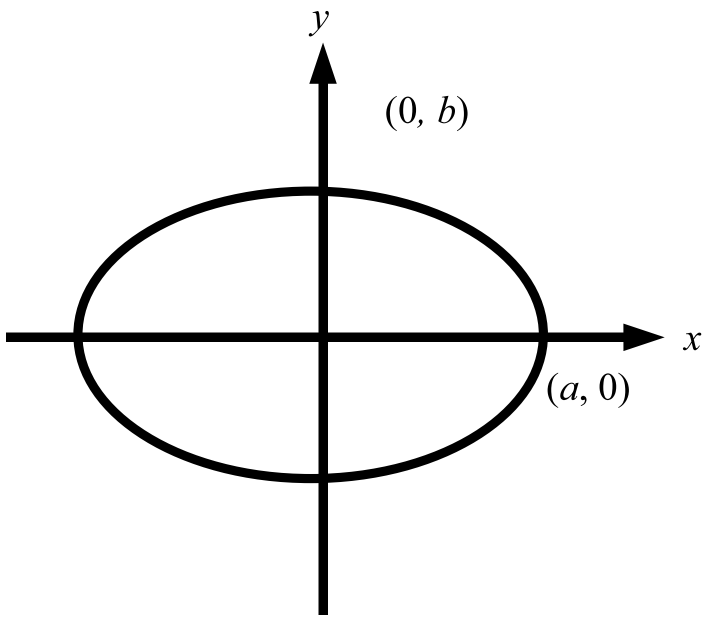
Consider the motion along the ellipse

$$z(t) = a\cos\omega t + ib\sin\omega t$$
$$\frac{d^2z(t)}{dt^2} = -a\omega^2\cos\omega t - ib\omega^2\sin\omega t = -\omega^2 z(t) = -\omega^2|z(t)|e^{i\theta}.$$

The acceleration at any point is always directed toward the origin. The radial component of acceleration $= -\omega^2|z|$.

Note that $x(t)$ and $y(t)$ are in simple harmonic motions, since

$$\frac{d^2x}{dt^2} = -\omega^2 x \text{ and } \frac{d^2y}{dt^2} = -\omega^2 y.$$



The periodic trajectory is an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In particular, when $a = b$, the trajectory becomes a circle of radius $a\,(= b)$. In this case, $r = a$ so that $\ddot{r} = 0$. Hence, the radial component of acceleration consists of only one term, namely, the centripetal term whose value is $-a\omega^2$. Here, $z(t) = ae^{i\omega t}$ so that $\theta(t) = \omega t$. This gives $\dot{\theta} = \omega$, which is the angular speed.

Cauchy-Riemann relations

What are the necessary and sufficient conditions for the existence of the derivative of a complex function?

- The necessary conditions are given by the Cauchy-Riemann equations.
- The sufficient conditions require, in addition to the Cauchy-Riemann relations, the continuity of all first order partial derivatives of u and v.

Let $f(z)$ be single-valued in a neighborhood of the point $z_0 = x_0 + iy_0$, and it is differentiable at z_0, that is, the limit

$$f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}$$

exists. This limit is independent of the direction along which $\triangle z$ approaches 0.

(i) First, we take $\triangle z \to 0$ in the direction parallel to the x-axis, that is, $\triangle z = \triangle x$. We then have

$$\begin{aligned} f(z_0 + \triangle z) - f(z_0) &= u(x_0 + \triangle x, y_0) + iv(x_0 + \triangle x, y_0) \\ &\quad - u(x_0, y_0) - iv(x_0, y_0), \end{aligned}$$

so that

$$\begin{aligned} f'(z_0) &= \lim_{\triangle x \to 0} \frac{u(x_0 + \triangle x, y_0) - u(x_0, y_0)}{\triangle x} \\ &\quad + i \lim_{\triangle x \to 0} \frac{v(x_0 + \triangle x, y_0) - v(x_0, y_0)}{\triangle x} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

(ii) Next, we let $\triangle z \to 0$ in the direction parallel to the y-axis, that is $\triangle z = i\triangle y$. Now, we have

$$
\begin{aligned}
f(z_0 + \triangle z) - f(z_0) &= u(x_0, y_0 + \triangle y) + iv(x_0, y_0 + \triangle y) \\
&\quad - u(x_0, y_0) - iv(x_0, y_0),
\end{aligned}
$$

so that

$$
\begin{aligned}
f'(z_0) &= \lim_{\triangle y \to 0} \frac{u(x_0, y_0 + \triangle y) - u(x_0, y_0)}{i\triangle y} \\
&\quad + i \lim_{\triangle y \to 0} \frac{v(x_0, y_0 + \triangle y) - v(x_0, y_0)}{i\triangle y} \\
&= \frac{1}{i} \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0).
\end{aligned}
$$

Combining the above two equations, we obtain

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Equating the respective real and imaginary parts gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The results are called the Cauchy-Riemann relations. They are the necessary conditions for the existence of the derivative of a complex function.

Theorem

Given $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$, *and assume that*

1. *Cauchy-Riemann relations hold at a point* $z_0 = x_0 + iy_0$,

2. u_x, u_y, v_x, v_y *are all continuous at* (x_0, y_0).

Then, $f'(z_0)$ *exists and it is given by*

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

Proof

Since $u(x,y)$ and $v(x,y)$ have continuous first order partials at (x_0,y_0) and satisfy the Cauchy-Riemann relations at the same point, we have

$$u(x,y)-u(x_0,y_0)=u_x(x_0,y_0)(x-x_0)-v_x(x_0,y_0)(y-y_0)+\epsilon_1(|\Delta z|) \quad (i)$$

$$v(x,y)-v(x_0,y_0)=v_x(x_0,y_0)(x-x_0)+u_x(x_0,y_0)(y-y_0)+\epsilon_2(|\Delta z|), \quad (ii)$$

where $\Delta z=(x+iy)-(x_0+iy_0), \epsilon_1$ and ϵ_2 represent higher order terms that satisfy

$$\lim_{|\Delta z|\to 0}\frac{\epsilon_1(|\Delta z|)}{|\Delta z|}=\lim_{|\Delta z|\to 0}\frac{\epsilon_2(|\Delta z|)}{|\Delta z|}=0, \quad |\Delta z|=\sqrt{(x-x_0)^2+(y-y_0)^2}. \quad (iii)$$

Adding (i) and i times (ii) together, we obtain

$$f(z)-f(z_0)=[u_x(x_0,y_0)+iv_x(x_0,y_0)](z-z_0)+\epsilon_1(|\Delta z|)+i\epsilon_2(|\Delta z|),$$

and subsequently,

$$\frac{f(z)-f(z_0)}{z-z_0}-[u_x(x_0,y_0)+iv_x(x_0,y_0)]=\frac{\epsilon_1(|\Delta z|)+i\epsilon_2(|\Delta z|)}{z-z_0}.$$

Note that

$$\left|\frac{\epsilon_1(|\Delta z|)+i\epsilon_2(|\Delta z|)}{z-z_0}\right|\leq\frac{\epsilon_1(|\Delta z|)}{|\Delta z|}+\frac{\epsilon_2(|\Delta z|)}{|\Delta z|};$$

and as $\Delta z\to 0$, also $|\Delta z|\to 0$.

It then follows from the results in (iii) that

$$\lim_{\Delta z \to 0} \frac{\epsilon_1(|\Delta z|) + i\epsilon_2(|\Delta z|)}{z - z_0} = 0.$$

Hence, we obtain

$$\begin{aligned} & f'(z_0) \\ = & \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ = & u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0). \end{aligned}$$

Example

Discuss the differentiability of the function

$$f(z) = f(x + iy) = \sqrt{|xy|}$$

at $z = 0$.

Solution

Write $f(x + iy) = u(x, y) + iv(x, y)$ so that

$$u(x, y) = \sqrt{|xy|} \quad \text{and} \quad v(x, y) = 0.$$

Since $u(x, 0)$ and $u(0, y)$ are identically equal to zero, and so

$$u_x(0, 0) = u_y(0, 0) = 0.$$

Also, obviously

$$v_x(0, 0) = v_y(0, 0) = 0,$$

the Cauchy-Riemann relations are satisfied at the point $(0, 0)$.

However, suppose z approaches the origin along the ray: $x = \alpha t, y = \beta t$, assuming that α and β cannot be zero simultaneously.

For $z = \alpha t + i\beta t$, we then have

$$\frac{f(z) - f(0)}{z - 0} = \frac{f(z)}{z} = \frac{\sqrt{|\alpha\beta|}}{\alpha + i\beta}.$$

The limit of the above quantity as $z \to 0$ depends on the choices of α and β, and so the value is non-unique.

Therefore, $f(z)$ is not differentiable at $z = 0$ though the Cauchy-Riemann relations are satisfied at $z = 0$. Note that the first order partial derivatives of $u(x, y)$ is NOT continuous at $(0, 0)$.

Example

Show that the function $f(z) = z\text{Re }(z)$ is nowhere differentiable except at the origin; hence find $f'(0)$. Is $f(z)$ continuous at $z = 0$? Explain why or why not?

Solution

When $z_0 = 0$, we have

$$f'(0) = \lim_{z \to 0} \frac{\Delta f}{\Delta z} = \lim_{z \to 0} \frac{z\text{Re } z}{z} = 0.$$

When $z_0 \neq 0$, we write $z_0 = x_0 + iy_0$. Let $z = x + iy$, we then have

$$\begin{aligned}
\lim_{z \to z_0} \frac{\Delta f}{\Delta z} &= \lim_{z \to z_0} \frac{z \text{Re } z - z_0 \text{Re } z_0}{z - z_0} \\
&= \lim_{z \to z_0} \left[\frac{(z - z_0)\text{Re } z}{z - z_0} + \frac{z_0(\text{Re } z - \text{Re } z_0)}{z - z_0} \right] \\
&= \lim_{z \to z_0} \left(x + z_0 \frac{x - x_0}{z - z_0} \right).
\end{aligned}$$

Suppose we approach z_0 along the direction parallel to the y-axis

$$\lim_{\substack{z \to z_0 \\ x = x_0}} \frac{\Delta f}{\Delta z} = x_0. \qquad (i)$$

On the other hand, if we approach z_0 along the direction parallel to the x-axis, we obtain

$$\lim_{\substack{z \to z_0 \\ z = x + iy_0}} \frac{\Delta f}{\Delta z} = \lim_{x \to x_0} \frac{(x + iy_0)x - (x_0 + iy_0)x_0}{x - x_0} \qquad (ii)$$

$$= \lim_{x \to x_0} \frac{x^2 - x_0^2 + iy_0(x - x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} x + x_0 + iy_0 = 2x_0 + iy_0.$$

Since the two limits in Eqs. (i) and (ii) are not equal, the limit $\lim_{z \to z_0} \frac{\Delta f}{\Delta z}$ does not exist. Therefore, the function is not differentiable at any $z_0 \neq 0$.

Recall that differentiability of a complex function at a point implies continuity of the function at the same point. Since $f'(0)$ exists, so $f(z)$ is continuous at $z = 0$.

Analyticity

A function $f(z)$ is said to be analytic at some point z_0 if it is differentiable at every point of a certain neighborhood of z_0. In other words, $f(z)$ is analytic if and only if there exists a neighborhood $N(z_0;\epsilon)$, $\epsilon > 0$, such that $f'(z)$ exists for all $z \in N(z_0;\epsilon)$.

Since $z_0 \in N(z_0;\epsilon)$, analyticity at z_0 implies differentiability at z_0. The converse statement is not true, that is, differentiability of $f(z)$ at z_0 does not guarantee the analyticity of $f(z)$ at z_0.

For example, the function $f(z) = |z|^2$ is nowhere differentiable except at the origin, hence $f(z) = |z|^2$ is not analytic at $z = 0$.

Entire functions

If a function is analytic in the entire complex plane, then the function is called an *entire function*.

To show that $f(z)$ is analytic in an open region or domain $\mathcal{D}$, we may either show

(i) $f'(z)$ exists for all z in $\mathcal{D}$, or

(ii) the real and imaginary parts of $f(z)$ have continuous first order partials and their derivatives satisfy the Cauchy-Riemann relations at every point inside $\mathcal{D}$.

Remark

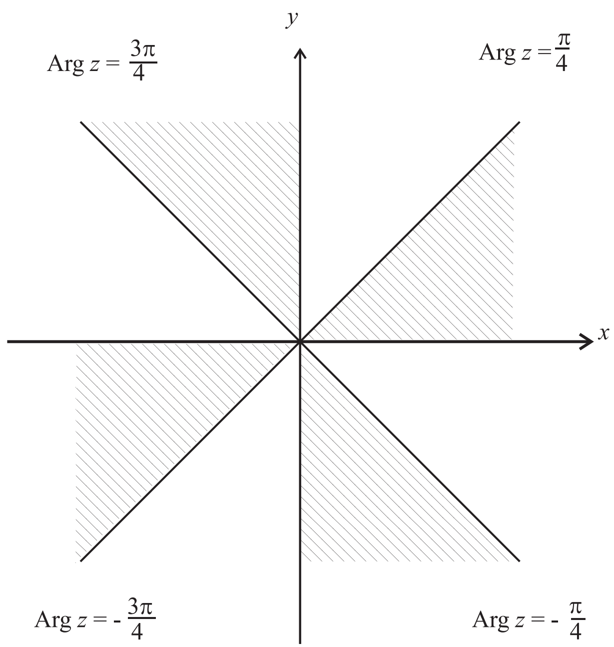
Since every point in an open region is an interior point, so if $f'(z)$ exists for all z in $\mathcal{D}$, then f is analytic everywhere inside $\mathcal{D}$.

Example

Find the domains in which the function

$$f(z) = |x^2 - y^2| + 2i|xy|, \qquad z = x + iy,$$

is analytic.



Domain of analyticity (shown in shaded areas) of $f(z) = |x^2 - y^2| + 2i|xy|$.

The functional values of $f(z)$ depend on the signs of $x^2 - y^2$ and xy:

(i) x^2-y^2 changes sign when (x,y) crosses the lines $x = y$ or $x = -y$;

(ii) xy changes sign when (x,y) crosses the x-axis or y-axis.

- When $x^2 - y^2 > 0$ and $xy > 0, f(z) = z^2$. When $x^2 - y^2 < 0$ and $xy < 0, f(z) = -z^2$. Both functions are known to be analytic.
- When $x^2 - y^2 > 0$ and $xy < 0, f(z) = x^2 - y^2 - 2ixy$. Inside these domains, the Cauchy-Riemann relations are not satisfied, and so $f(z)$ fails to be analytic. Inside the domains defined by $x^2-y^2 < 0$ and $xy > 0$, the function becomes $f(z) = -(x^2-y^2)+2ixy$, which is non-analytic.

The function is analytic within the following domains:

$$0 < \text{ Arg } z < \frac{\pi}{4}, \qquad \frac{\pi}{2} < \text{ Arg } z < \frac{3\pi}{4}$$

$$-\pi < \text{ Arg } z < -\frac{3\pi}{4} \qquad \text{and} \qquad -\frac{\pi}{2} < \text{ Arg } z < -\frac{\pi}{4}.$$

Example

Show that there is no entire function f such that $f'(z) = xy^2$ for all $z \in \mathbb{C}$.

Solution

Suppose that f is entire with $f'(z) = xy^2 = u_x + iv_x$ at all points. This gives $u_x = xy^2$ and $v_x = 0$. Then $u(x, y) = \frac{1}{2}x^2y^2 + F(y)$ and $v(x, y) = G(y)$, where F and G are arbitrary functions of y. On the other hand, given that f is entire, the Cauchy-Riemann equations are satisfied at all points, so that $v_y = xy^2$ and $u_y = 0$. This leads to a contradiction.

Example

Suppose $f(z)$ and $g(z)$ are analytic inside the domain $\mathcal{D}$. Show that $f(z)$ is constant inside $\mathcal{D}$ if $|f(z)|$ is constant inside $\mathcal{D}$. Is $g(z)$ constant inside $\mathcal{D}$ if Re $(g(z))$ is constant inside $\mathcal{D}$? Explain why or why not.

Solution

Write $f(z) = u(x,y) + iv(x,y), z = x + iy$. Consider

$$|f(z)|^2 = u^2 + v^2 = \text{ constant}$$

so that

$$uu_x + vv_x = 0 \quad \text{and} \quad uu_y + vv_y = 0$$
$$\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note that $|f(z)| = 0 \Leftrightarrow f(z) = 0$. Now, consider the case where $f(z) = u(x,y) + iv(x,y) \neq 0$.

Since $\begin{pmatrix} u \\ v \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so we must have

$$\det \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = u_x v_y - u_y v_x = 0.$$

From the Cauchy-Riemann relations, $u_x = v_y, v_x = -u_y$; we obtain

$$u_x^2 + u_y^2 = 0 \text{ so } u_x = u_y = 0. \text{ that is, } u = \text{constant.}$$

Similarly, $v_x = v_y = 0$, so $v =$ constant. Hence, $f(z)$ is constant in $\mathcal{D}$.

Write $g(z) = \alpha(x, y) + i\beta(x, y), z = x + iy$. When $\alpha =$ constant, $\alpha_x = \alpha_y = 0$. From the Cauchy-Riemann relations, $\beta_x = \beta_y = 0$ so that $\beta =$ constant. Hence, $g(z)$ in constant in $\mathcal{D}$.

Example

Suppose $f(z)$ and $\overline{f(z)}$ are analytic in a domain $\mathcal{D}$. Show that $f(z)$ is constant in $\mathcal{D}$.

Solution

Write $f = u + iv$ and $\overline{f} = u - iv$.

Since f is analytic, we have $u_x = v_y$ and $u_y = -v_x$. Also, since $\overline{f}$ is analytic, we have $u_x = -v_y$ and $u_y = v_x$. Combining the results, we obtain

$$v_x = v_y = 0 \quad \text{and} \quad u_x = u_y = 0.$$

Hence, f is a constant.

Harmonic functions

A real-valued function $\phi(x, y)$ of two real variables x and y is said to be *harmonic* in a given domain $\mathcal{D}$ in the x-y plane if ϕ has continuous partial derivatives up to the second order in $\mathcal{D}$ and satisfies the Laplace equation

$$\phi_{xx}(x, y) + \phi_{yy}(x, y) = 0.$$

Analytic functions are closely related to harmonic functions. Suppose $f(z) = u(x, y) + iv(x, y)$ is an analytic function in $\mathcal{D}$, we will show that both the component functions $u(x, y)$ and $v(x, y)$ are harmonic in $\mathcal{D}$.

We state without proof the following result:

If a complex function is analytic at a point, then its real and imaginary parts have continuous partial derivatives of all orders at that point.

Suppose $f(z)$ is analytic in $\mathcal{D}$, then

$$u_x = v_y \quad \text{and} \quad v_x = -u_y.$$

Differentiating both sides of the equations with respect to x, we obtain

$$u_{xx} = v_{yx} \quad \text{and} \quad v_{xx} = -u_{yx} \quad \text{in } \mathcal{D}.$$

Similarly,

$$u_{xy} = v_{yy} \quad \text{and} \quad v_{xy} = -u_{yy}.$$

Since the above partial derivatives are all continuous, it is guaranteed that

$$u_{xy} = u_{yx} \quad \text{and} \quad v_{xy} = v_{yx}.$$

Combining the results,

$$v_{yy} = u_{xy} = u_{yx} = -v_{xx} \quad \text{and so} \quad v_{xx} + v_{yy} = 0,$$

and

$$-u_{yy} = v_{xy} = v_{yx} = u_{xx} \quad \text{and so} \quad u_{xx} + u_{yy} = 0.$$

Therefore, both $u(y, y)$ and $v(x, y)$ are harmonic functions.

Take any two harmonic functions, they normally do not form a complex analytic function.

For example, $\phi = e^x \cos y, \psi = 2xy$

$$f = \phi + i\psi = e^x \cos y + 2ixy \text{ is } \textit{NOT} \text{ analytic.}$$

Suppose ψ is changed to $e^x \sin y$, or ϕ is changed to $x^2 - y^2$, then

$$f_1 = e^x \cos y + ie^x \sin y = e^z$$

or

$$f_2 = x^2 - y^2 + 2ixy = z^2.$$

Trick: As necessary conditions, we require $\phi_x = \psi_y$ and $\phi_y = -\psi_x$.

Harmonic conjugate

Given two harmonic functions $\phi(x, y)$ and $\psi(x, y)$ and if they satisfy the Cauchy-Riemann relations throughout a domain $\mathcal{D}$, with

$$\phi_x = \psi_y \quad \text{and} \quad \phi_y = -\psi_x.$$

We call ψ a *harmonic conjugate* of ϕ in $\mathcal{D}$.

Note that harmonic conjugacy is not a symmetric relation because of the minus sign in the second Cauchy-Riemann relation. While ψ is a harmonic conjugate of $\phi, -\phi$ is a harmonic conjugate of ψ.

For example, $e^x \sin y$ is a harmonic conjugate of $e^x \cos y$ while $-e^x \cos y$ is a harmonic conjugate of $e^x \sin y$.

Theorem

A complex function $f(z) = u(x,y) + iv(x,y), z = x + iy$, is analytic in a domain $\mathcal{D}$ if and only if v is a harmonic conjugate to u in $\mathcal{D}$.

Proof

$\Rightarrow$ Given that $f = u + iv$ is analytic, then u and v are harmonic and Cauchy-Riemann relations are satisfied. Hence, v is a harmonic conjugate of u.

$\Leftarrow$ Given that v is a harmonic conjugate of u in $\mathcal{D}$, we have the satisfaction of the Cauchy-Riemann relations and the continuity of the first order partials of u and v in $\mathcal{D}$. Hence, $f = u + iv$ is differentiable for all points in $\mathcal{D}$. Since $\mathcal{D}$ is an open set, every point in $\mathcal{D}$ is an interior point, so f is analytic in $\mathcal{D}$..

Exact differentials

The differential $M(x,y)\,dx + N(x,y)\,dy$ is an exact differential if and only if M and N observe

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Under such scenario, then

$$M(x,y)\,dx + N(x,y)\,dy = dF$$

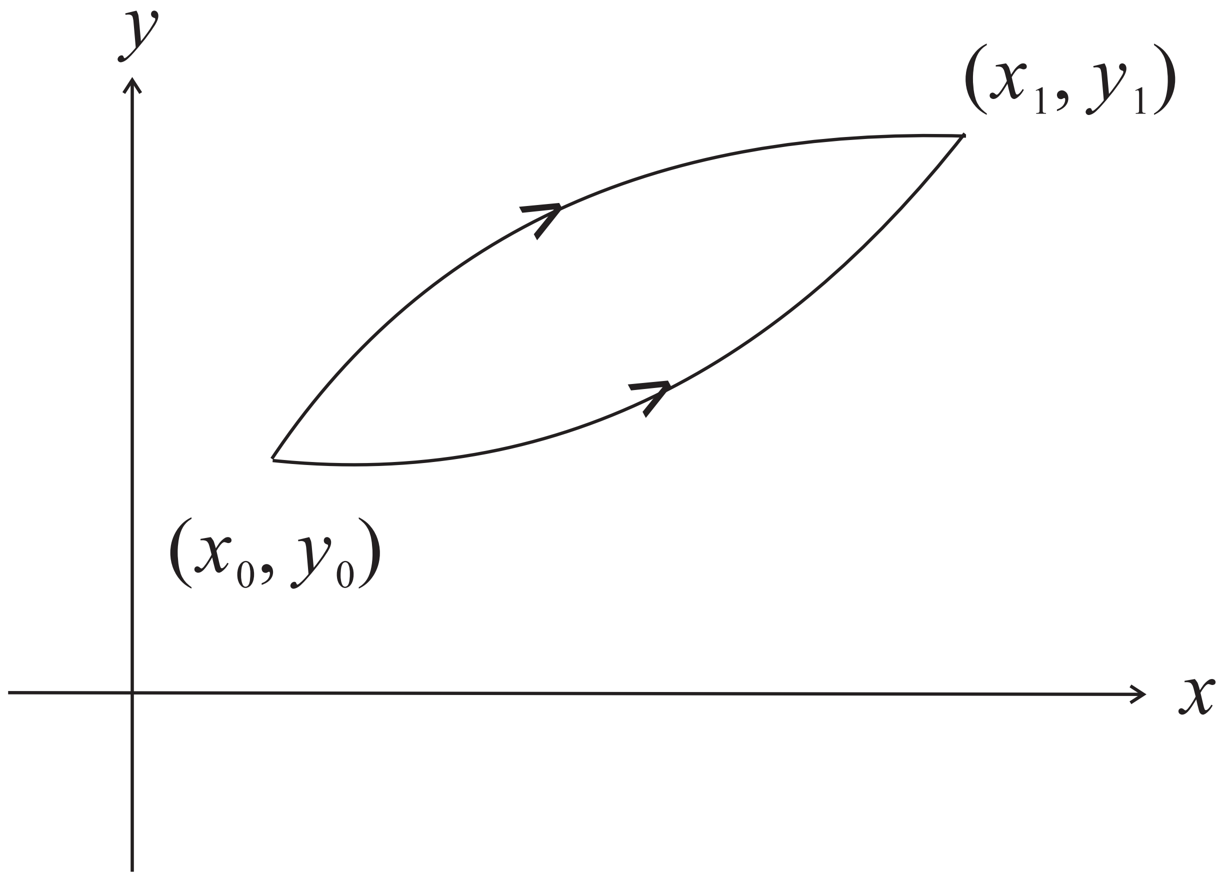
for some F, that is,

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N.$$

The line integral of the differential along any path joining (x_0, y_0) and (x_1, y_1) is given by

$$\int_{(x_0,y_0)}^{(x_1,y_1)} M\,dx + N\,dy = F(x_1, y_1) - F(x_0, y_0).$$

The integral value is path independent provided that there is no singular points enclosed inside the closed curve represented by the two paths of integration.



Given that $\phi(x, y)$ is harmonic in a simply connected domain $\mathcal{D}$, it can be shown that it is always possible to obtain its harmonic conjugate $\psi(x, y)$ by integration. Starting with the differential form:

$$d\psi = \psi_x \, dx + \psi_y \, dy,$$

and using the Cauchy-Riemann relations, we have

$$d\psi = -\phi_y \, dx + \phi_x \, dy.$$

To obtain ψ, we integrate along some path Γ joining a fixed point (x_0, y_0) to (x, y), that is,

$$\psi(x, y) = \int_\Gamma - \phi_y \, dx + \phi_x \, dy.$$

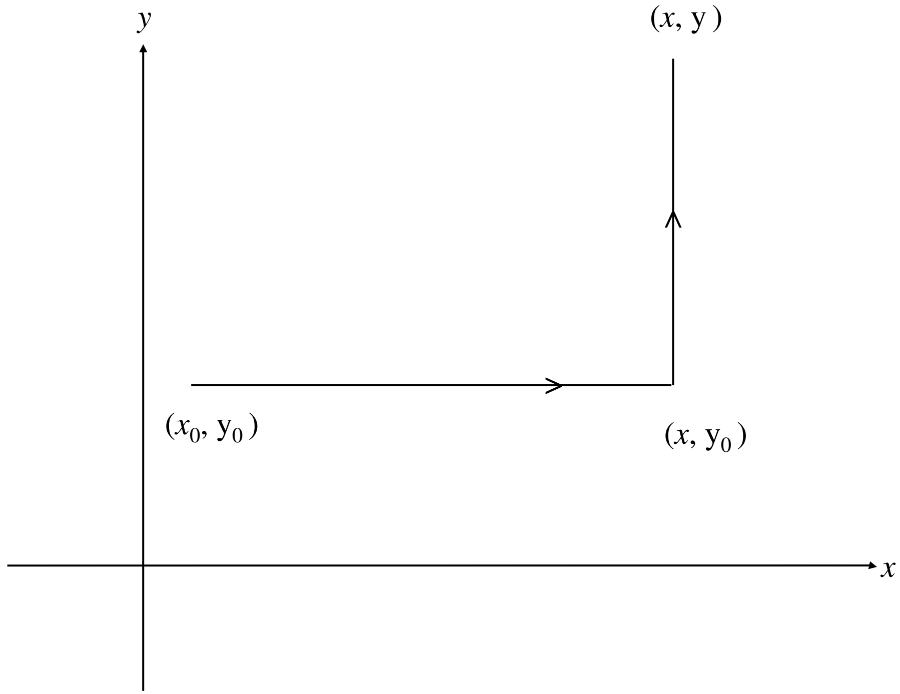
The above integral is an exact differential provided that

$$-(-\phi_y)_y + (\phi_x)_x = 0,$$

that is, ϕ is harmonic.

To ease the computation, we choose the path which consists of horizontal and vertical line segments as shown in the Figure.

$$\psi(x,y) = \int_{x_0}^{x} -\phi_y(x,y_0)\,dx + \int_{y_0}^{y} \phi_x(x,y)\,dy.$$



The choice of a different starting point (x_0, y_0) of the integration path simply leads to a different additive constant in $\psi(x,y)$. Recall that $\psi(x,y)$ is unique up to an additive constant.

Example

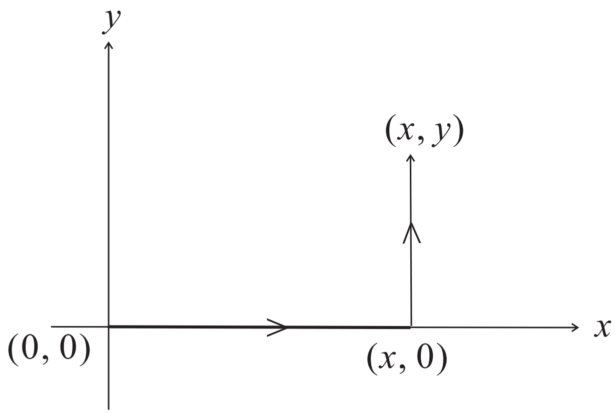
Find a harmonic conjugate of the harmonic function

$$u(x,y) = e^{-x}\cos y + xy.$$

Solution

1. Take $(x_0, y_0) = (0,0)$, $u_y(x,0) = x$ and $u_x(x,y) = -e^{x}\cos y + y$.

$$\begin{aligned} v(x,y) &= \int_0^x -x\,dx + \int_0^y (-e^{-x}\cos y + y)\,dy \\ &= -\frac{x^2}{2} - e^{-x}\sin y + \frac{y^2}{2}. \end{aligned}$$



2. From the first Cauchy-Riemann relation, we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -e^{-x} \cos y + y.$$

Integrating with respect to y, we obtain

$$v(x, y) = -e^{-x} \sin y + \frac{y^2}{2} + \eta(x),$$

where $\eta(x)$ is an arbitrary function arising from integration.

Using the second Cauchy-Riemann relation, we have

$$\frac{\partial v}{\partial x} = e^{-x} \sin y + \eta'(x) = -\frac{\partial u}{\partial y} = e^{-x} \sin y - x.$$

Comparing like terms, we obtain

$$\eta'(x) = -x,$$

and subsequently,

$$\eta(x) = -\frac{x^2}{2} + C, \text{ where } C \text{ is an arbitrary constant.}$$

Hence, a harmonic conjugate is found to be (taking C to be zero for convenience)

$$v(x, y) = -e^{-x} \sin y + \frac{y^2 - x^2}{2}.$$

The corresponding analytic function, $f = u + iv$, is seen to be

$$f(z) = e^{-z} - \frac{iz^2}{2}, \quad z = x + iy,$$

which is an entire function.

3. It is readily seen that

$$e^{-x}\cos y = \text{Re } e^{-z} \quad \text{and} \quad xy = \frac{1}{2}\text{Im } z^2.$$

A harmonic conjugate of Re e^{-z} is Im e^{-z}, while that of $\frac{1}{2}$Im z^2 is $-\frac{1}{2}$Re z^2. Therefore, a harmonic conjugate of $u(x,y)$ can be taken to be

$$v(x,y) = \text{Im } e^{-z} - \frac{1}{2}\text{Re } z^2 = -e^{-x}\sin y + \frac{y^2 - x^2}{2}.$$

Example

Show that $f'(z) = \dfrac{\partial u}{\partial x}(z,0) - i\dfrac{\partial u}{\partial y}(z,0)$. Use the result to find a harmonic conjugate of

$$u(x,y) = e^{-x}(x\sin y - y\cos y).$$

Solution

Observe that $f'(z) = \dfrac{\partial u}{\partial x}(x,y) - i\dfrac{\partial u}{\partial y}(x,y)$. Putting $y = 0$, we obtain

$$f'(x) = \frac{\partial u}{\partial x}(x,0) - i\frac{\partial u}{\partial y}(x,0).$$

Replacing x by z, we obtain $f'(z) = \dfrac{\partial u}{\partial x}(z,0) - i\dfrac{\partial u}{\partial y}(z,0)$.

Now, for $u(x,y) = e^{-x}(x \sin y - y \cos y)$, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^{-x} \sin y - x e^{-x} \sin y + y e^{-x} \cos y \\ \frac{\partial u}{\partial y} &= x e^{-x} \cos y + y e^{-x} \sin y - e^{-x} \cos y. \end{aligned}$$

We then have

$$f'(z) = \frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0) = -i(ze^{-z} - e^{-z}).$$

Integrating with respect to z, we obtain $f(z) = ize^{-z}$.

The imaginary part of $f(z) = v(x,y) = e^{-x}(y \sin y + x \cos y)$.

Theorem

If ψ is a harmonic conjugate of ϕ, then the two families of curves

$$\phi(x,y)=\alpha \quad \text{and} \quad \psi(x,y)=\beta$$

are mutually orthogonal to each other.

Proof

Consider a particular member from the first family

$$\phi(x,y)=\alpha_1,$$

the slope of the tangent to the curve at (x,y) is given by $\frac{dy}{dx}$ where

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0,$$

giving

$$\frac{dy}{dx}=-\frac{\partial\phi}{\partial x}\Big/\frac{\partial\phi}{\partial y}.$$

Similarly, the slope of the tangent to a member from the second family at (x, y) is given by

$$\frac{dy}{dx} = -\frac{\partial \psi}{\partial x} \Big/ \frac{\partial \psi}{\partial y}.$$

The product of the slopes of the two tangents to the two curves at the same point is found to be

$$\left(-\frac{\partial \phi}{\partial x} \Big/ \frac{\partial \phi}{\partial y}\right) \left(-\frac{\partial \psi}{\partial x} \Big/ \frac{\partial \psi}{\partial y}\right) = -1,$$

by virtue of the Cauchy-Riemann relations: $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$ and $\dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x}$. Hence, the two families of curves are mutually orthogonal to each other.

Steady state temperature distribution

In a two-dimensional steady state temperature field, the temperature function $T(x,y)$ is harmonic:

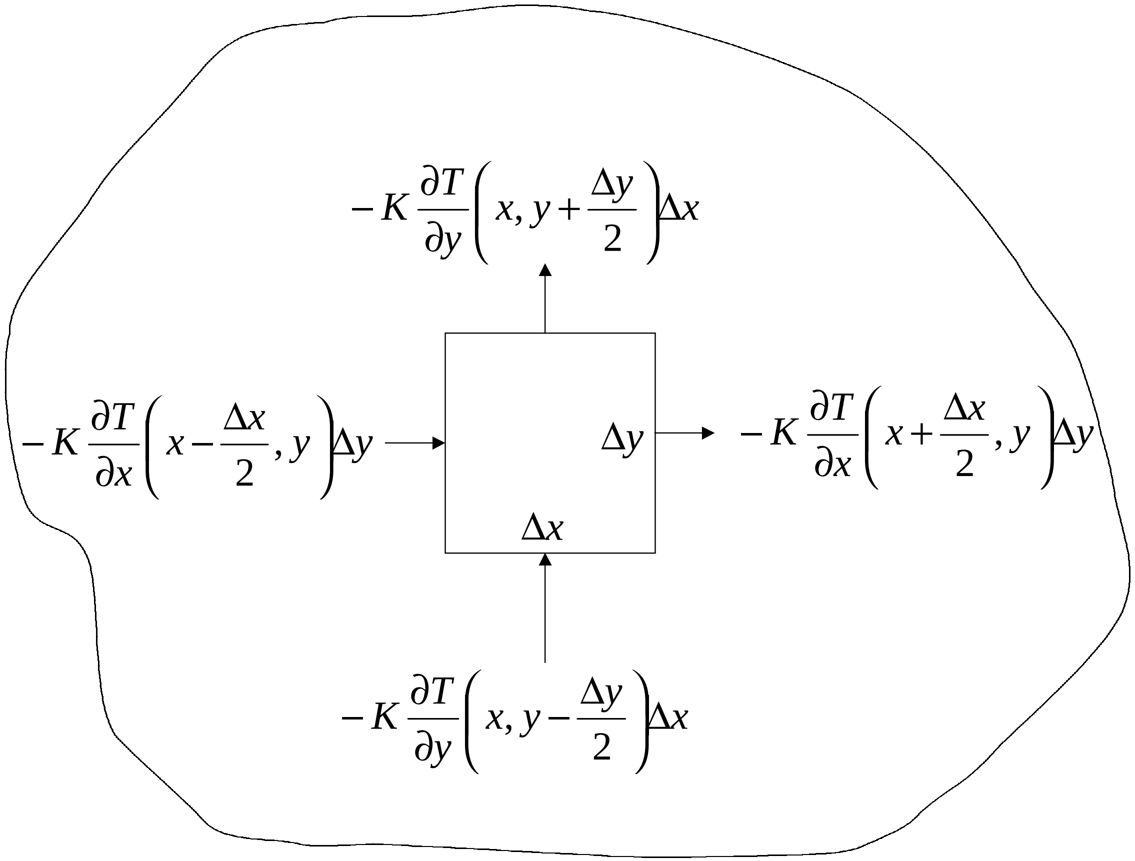
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

From the empirical law of heat conduction

$$Q = \text{ heat flux across a surface } = -K\frac{\partial T}{\partial n}, \quad K > 0,$$

where K is called the thermal conductivity of the material and $\frac{\partial T}{\partial n}$ denotes the normal derivative of T with respect to the surface.

Steady state temperature distribution prevails if there is no heat source or sink inside the body and there is no net heat flux across the bounding surface.



An infinitesimal control volume of widths Δx and Δy is contained inside a two-dimensional body. The heat fluxes across the four sides of the rectangular control volume are shown.

- Within a unit time interval, the amount of heat flowing across the left vertical side *into* the rectangular control volume is

$$-K\frac{\partial T}{\partial x}\Big(x-\frac{\Delta x}{2},y\Big)\Delta y.$$

Likewise, the amount of heat flowing across the right vertical side *out of* the control volume is

$$-K\frac{\partial T}{\partial x}\left(x+\frac{\Delta x}{2},y\right)\Delta y.$$

- The net accumulation of heat per unit time per unit volume inside the control volume is

$$K\left[\frac{\frac{\partial T}{\partial x}\left(x+\frac{\Delta x}{2},y\right)-\frac{\partial T}{\partial x}\left(x-\frac{\Delta x}{2},y\right)}{\Delta x}\right.$$
$$\left.+\frac{\frac{\partial T}{\partial y}\left(x,y+\frac{\Delta y}{2}\right)-\frac{\partial T}{\partial y}\left(x,y-\frac{\Delta y}{2}\right)}{\Delta y}\right].$$

- Taking the limits $\Delta x \to 0$ and $\Delta y \to 0$, we then obtain

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0.$$

  Hence, the steady state temperature function is a harmonic function.

Example

Supposing the isothermal curves of a steady state temperature field are given by the family of parabolas

$$y^2 = \alpha^2 + 2\alpha x, \quad \alpha \text{ is real positive,}$$

in the complex plane, find the general solution of the temperature function $T(x, y)$. Also, find the family of flux lines of the temperature field.

Solution

First, we solve for the parameter α in the equation of the isothermal curves. This gives

$$\alpha = -x + \sqrt{x^2 + y^2},$$

where the positive sign is chosen since $\alpha > 0$.

A naive guess may suggest that the temperature function $T(x, y)$ is given by

$$T(x, y) = -x + \sqrt{x^2 + y^2}.$$

However, since $T(x, y)$ has to be harmonic, the above function cannot be a feasible solution. We set

$$T(x, y) = f(t)$$

where $t = \sqrt{x^2 + y^2} - x$, and f is some function to be determined such that $T(x, y)$ is harmonic. To solve for $f(t)$, we first compute

$$\begin{aligned} \frac{\partial^2 T}{\partial x^2} &= f''(t)\left(\frac{\partial t}{\partial x}\right)^2 + f'(t)\frac{\partial^2 t}{\partial x^2} \\ &= f''(t)\left(\frac{x}{\sqrt{x^2+y^2}} - 1\right)^2 + f'(t)\frac{y^2}{(x^2+y^2)^{3/2}}, \\ \frac{\partial^2 T}{\partial y^2} &= f''(t)\frac{y^2}{x^2+y^2} + f'(t)\frac{x^2}{(x^2+y^2)^{3/2}}. \end{aligned}$$

Since $T(x,y)$ satisfies the Laplace equation

$$2\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)f''(t)+\frac{1}{\sqrt{x^2+y^2}}f'(t)=0 \quad \text{or} \quad \frac{f''(t)}{f'(t)}=-\frac{1}{2t}.$$

Integrating once gives

$$\ln f'(t)=-\frac{1}{2}\ln t+C \quad \text{or} \quad f'(t)=\frac{C'}{\sqrt{t}}.$$

Integrating twice gives

$$f(t)=C_1\sqrt{t}+C_2,$$

where C_1 and C_2 are arbitrary constants. The temperature function is

$$T(x,y)=f(t)=C_1\sqrt{\sqrt{x^2+y^2}-x}+C_2.$$

When expressed in polar coordinates

$$T(r,\theta)=C_1\sqrt{r(1-\cos\theta)}+C_2=C_1\sqrt{2r}\sin\frac{\theta}{2}+C_2.$$

Since $T(r, \theta)$ can be expressed as $\sqrt{2} C_1 \text{Im } z^{1/2} + C_2$, the harmonic conjugate of $T(r, \theta)$ is easily seen to be

$$F(r, \theta) = -\sqrt{2} C_1 \text{Re}\, z^{1/2} + C_3 = -C_1 \sqrt{2r} \cos \frac{\theta}{2} + C_3,$$

where C_3 is another arbitrary constant. Note that

$$\sqrt{2r} \cos \frac{\theta}{2} = \sqrt{r + r \cos \theta} = \sqrt{\sqrt{x^2 + y^2} + x}$$

so that

$$F(x, y) = -C_1 \sqrt{\sqrt{x^2 + y^2} + x} + C_3.$$

The family of curves defined by

$$x + \sqrt{x^2 + y^2} = \beta \quad \text{or} \quad y^2 = \beta^2 - 2\beta x, \qquad \beta > 0,$$

are orthogonal to the isothermal curves $y^2 = \alpha^2 + 2\alpha x, \alpha > 0$.

Physically, the direction of heat flux is normal to the isothermal lines. Therefore, the family of curves orthogonal to the isothermal lines are called the *flux lines*. These flux lines indicate the flow directions of heat in the steady state temperature field.

The flux function $F(r, \theta)$ is a harmonic conjugate of the temperature function. The families of curves: $T(r, \theta) = \alpha$ and $F(r, \theta) = \beta, \alpha$ and β being constant, are mutually orthogonal to each other.