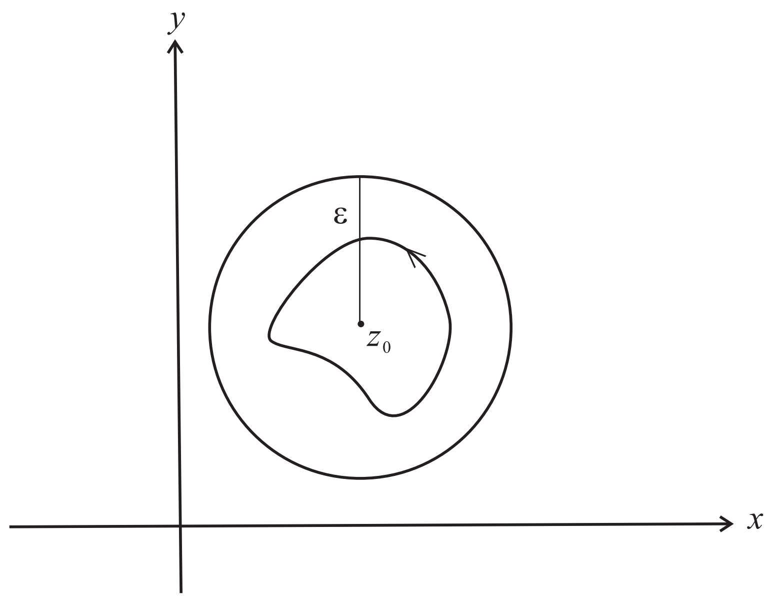
# 6. Residue calculus

Let $z_0$ be an isolated singularity of $f(z)$, then there exists a certain deleted neighborhood $N_\varepsilon = \{z : 0 < |z - z_0| < \varepsilon\}$ such that $f$ is analytic everywhere inside $N_\varepsilon$. We define

$$\text{Res}\,(f, z_0) = \frac{1}{2\pi i} \oint_C f(z)\,dz,$$

where $C$ is any simple closed contour around $z_0$ and inside $N_\varepsilon$.



Since $f(z)$ admits a Laurent expansion inside $N_\varepsilon$, where

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

then

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz = \text{Res}\,(f, z_0).$$

**Example**

$$\text{Res}\left(\frac{1}{(z-z_0)^k}, z_0\right) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{cases}$$

$$\text{Res}\,(e^{1/z}, 0) = 1 \text{ since } e^{1/z} = 1 + \frac{1}{1!}\frac{1}{z} + \frac{2}{2!}\frac{1}{z^2} + \cdots \quad , \ |z| > 0$$

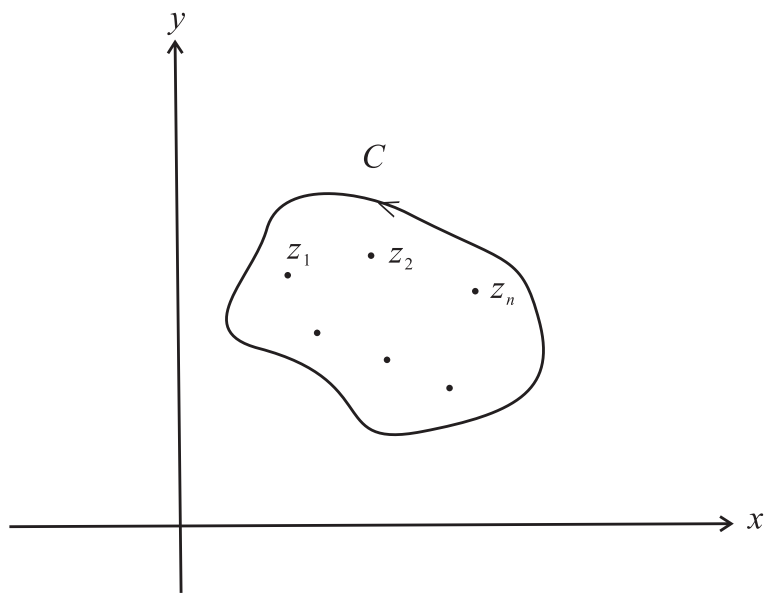
$$\text{Res}\left(\frac{1}{(z-1)(z-2)}, 1\right) = \frac{1}{1-2} \text{ by the Cauchy integral formula.}$$

## Cauchy residue theorem

Let $C$ be a simple closed contour inside which $f(z)$ is analytic everywhere except at the isolated singularities $z_1, z_2, \cdots, z_n$.

$$\oint_C f(z)\, dz = 2\pi i[\text{Res}\,(f, z_1) + \cdots + \text{Res}\,(f, z_n)].$$

This is a direct consequence of the Cauchy-Goursat Theorem.



**Example**

Evaluate the integral

$$\oint_{|z|=1} \frac{z+1}{z^2}\,dz$$

using

(i) direct contour integration,

(ii) the calculus of residues,

(iii) the primitive function $\log z - \dfrac{1}{z}$.

*Solution*

(i) On the unit circle, $z = e^{i\theta}$ and $dz = ie^{i\theta}\,d\theta$. We then have

$$\oint_{|z|=1} \frac{z+1}{z^2}\,dz = \int_0^{2\pi} (e^{-i\theta}+e^{-2i\theta})ie^{i\theta}\,d\theta = i\int_0^{2\pi} (1+e^{-i\theta})\,d\theta = 2\pi i.$$

(ii) The integrand $(z+1)/z^2$ has a double pole at $z = 0$. The Laurent expansion in a deleted neighborhood of $z = 0$ is simply $\dfrac{1}{z} + \dfrac{1}{z^2}$, where the coefficient of $1/z$ is seen to be 1. We have

$$\text{Res}\left(\frac{z+1}{z^2}, 0\right) = 1,$$

and so

$$\oint_{|z|=1} \frac{z+1}{z^2}\,dz = 2\pi i \text{Res}\left(\frac{z+1}{z^2}, 0\right) = 2\pi i.$$

(iii) When a closed contour moves around the origin (which is the branch point of the function $\log z$) in the anticlockwise direction, the increase in the value of $\arg z$ equals $2\pi$. Therefore,

$$\begin{aligned}\oint_{|z|=1} \frac{z+1}{z^2}\,dz &= \text{change in value of } \ln|z| + i\arg z - \tfrac{1}{z} \text{ in} \\ &\qquad \text{traversing one complete loop around the origin} \\ &= 2\pi i.\end{aligned}$$

## Computational formula

Let $z_0$ be a pole of order $k$. In a deleted neighborhood of $z_0$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots + \frac{b_k}{(z-z_0)^k}, \qquad b_k \neq 0.$$

Consider

$$g(z) = (z-z_0)^k f(z).$$

the principal part of $g(z)$ vanishes since

$$g(z) = b_k + b_{k-1}(z-z_0) + \cdots b_1 (z-z_0)^{k-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+k}.$$

By differenting $(k-1)$ times, we obtain

$$b_1 = \text{Res}\,(f, z_0) = \begin{cases} \frac{g^{(k-1)}(z_0)}{(k-1)!} & \text{if } g^{(k-1)}(z) \text{ is analytic at } z_0 \\ \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \left[ \frac{(z-z_0)^k f(z)}{(k-1)!} \right] & \text{if } z_0 \text{ is a removable} \\ & \text{singularity of } g^{(k-1)}(z) \end{cases}$$

**Simple pole**

$$k = 1: \qquad \text{Res}\,(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Suppose $f(z) = \dfrac{p(z)}{q(z)}$ where $p(z_0) \neq 0$ but $q(z_0) = 0, q'(z_0) \neq 0$.

$$\begin{aligned}
\text{Res}\,(f, z_0) &= \lim_{z \to z_0} (z - z_0) f(z) \\
&= \lim_{z \to z_0} (z - z_0) \frac{p(z_0) + p'(z_0)(z - z_0) + \cdots}{q'(z_0)(z - z_0) + \frac{q''(z_0)}{2!}(z - z_0)^2 + \cdots} \\
&= \frac{p(z_0)}{q'(z_0)}.
\end{aligned}$$

**Example**

Find the residue of

$$f(z) = \frac{e^{1/z}}{1-z}$$

at all isolated singularities.

*Solution*

(i) There is a simple pole at $z = 1$. Obviously

$$\text{Res}\,(f, 1) = \lim_{z \to 1} (z-1) f(z) = -e^{1/z}\Big|_{z=1} = -e.$$

(ii) Since

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots$$

has an essential singularity at $z = 0$, so does $f(z)$. Consider

$$\frac{e^{1/z}}{1-z} = (1 + z + z^2 + \cdots)\left(1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots\right), \quad \text{for } 0 < |z| < 1,$$

the coefficient of $1/z$ is seen to be

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e - 1 = \text{Res}\,(f, 0).$$

**Example**

Find the residue of

$$f(z) = \frac{z^{1/2}}{z(z-2)^2}$$

at all poles. Use the principal branch of the square root function $z^{1/2}$.

*Solution*

The point $z = 0$ is not a simple pole since $z^{1/2}$ has a branch point at this value of $z$ and this in turn causes $f(z)$ to have a branch point there. A branch point is not an isolated singularity.

However, $f(z)$ has a pole of order 2 at $z = 2$. Note that

$$\text{Res}\,(f, 2) = \lim_{z \to 2} \frac{d}{dz}\left(\frac{z^{1/2}}{z}\right) = \lim_{z \to 2}\left(-\frac{z^{1/2}}{2z^2}\right) = -\frac{1}{4\sqrt{2}},$$

where the principal branch of $2^{1/2}$ has been chosen (which is $\sqrt{2}$).

**Example**

Evaluate $\text{Res}\,(g(z)f'(z)/f(z), \alpha)$ if $\alpha$ is a pole of order $n$ of $f(z)$, $g(z)$ is analytic at $\alpha$ and $g(\alpha) \neq 0$.

*Solution*

Since $\alpha$ is a pole of order $n$ of $f(z)$, there exists a deleted neighborhood $\{z : 0 < |z - \alpha| < \varepsilon\}$ such that $f(z)$ admits the Laurent expansion:

$$f(z) = \frac{b_n}{(z-\alpha)^n} + \frac{b_{n-1}}{(z-\alpha)^{n-1}} + \cdots + \frac{b_1}{(z-\alpha)} + \sum_{n=0}^{\infty} a_n (z-\alpha)^n, \qquad b_n \neq 0.$$

Within the annulus of convergence, we can perform termwise differentiation of the above series

$$f'(z) = \frac{-nb_n}{(z-\alpha)^{n+1}} - \frac{(n-1)b_n}{(z-\alpha)^n} - \cdots - \frac{b_1}{(z-\alpha)^2} + \sum_{n=0}^{\infty} na_n(z-\alpha)^{n-1}.$$

Provided that $g(\alpha) \neq 0$, it is seen that

$$\begin{aligned} &= \lim_{z \to \alpha} g(z) \frac{(z-\alpha)\left[\frac{-nb_n}{(z-\alpha)^{n+1}} - \frac{(n-1)b_n}{(z-\alpha)^n} - \cdots - \frac{b_1}{(z-\alpha)^2} + \sum_{n=0}^{\infty} na_n(z-\alpha)^{n-1}\right]}{\frac{b_n}{(z-\alpha)^n} + \frac{b_{n-1}}{(z-\alpha)^{n-1}} + \cdots + \frac{b_1}{z-\alpha} + \sum_{n=0}^{\infty} a_n(z-\alpha)^n} \\ &= -ng(\alpha) \neq 0, \end{aligned}$$

so that $\alpha$ is a simple pole of $g(z)f'(z)/f(z)$. Furthermore,

$$\text{Res}\left(g\frac{f'}{f}, \alpha\right) = -ng(\alpha).$$

*Remark*

When $g(\alpha) = 0, \alpha$ becomes a removable singularity of $gf'/f$.

## Example

Suppose an even function $f(z)$ has a pole of order $n$ at $\alpha$. Within the deleted neighborhood $\{z : 0 < |z - \alpha| < \varepsilon\}, f(z)$ admits the Laurent expansion

$$f(z) = \frac{b_n}{(z-\alpha)^n} + \cdots + \frac{b_1}{(z-\alpha)} + \sum_{n=0}^{\infty} a_n (z-\alpha)^n, \quad b_n \neq 0.$$

Since $f(z)$ is even, $f(z) = f(-z)$ so that

$$f(z) = f(-z) = \frac{b_n}{(-z-\alpha)^n} + \cdots + \frac{b_1}{(-z-\alpha)} + \sum_{n=0}^{\infty} a_n (-z-\alpha)^n,$$

which is valid within the deleted neighborhood $\{z : 0 < |z + \alpha| < \varepsilon\}$. Hence, $-\alpha$ is a pole of order $n$ of $f(-z)$. Note that

$$\text{Res}\,(f(z), \alpha) = b_1 \quad \text{and} \quad \text{Res}\,(f(z), -\alpha) = -b_1$$

so that $\text{Res}\,(f(z), \alpha) = -\text{Res}\,(f(z), -\alpha)$. For an even function, if $z = 0$ happens to be a pole, then $\text{Res}\,(f, 0) = 0$.

**Example**

$$\oint_{|z|=2} \frac{\tan z}{z}\,dz = 2\pi i\left[\text{Res}\left(\frac{\tan z}{z}, \frac{\pi}{2}\right) + \text{Res}\left(\frac{\tan z}{z}, -\frac{\pi}{2}\right)\right]$$

since the singularity at $z = 0$ is removable. Observe that $\dfrac{\pi}{2}$ is a simple pole and $\cos z = -\sin\left(z - \dfrac{\pi}{2}\right)$, we have

$$\begin{aligned}
\text{Res}\left(\frac{\tan z}{z}, \frac{\pi}{2}\right) &= \lim_{z\to\frac{\pi}{2}} \frac{\left(z-\frac{\pi}{2}\right)\tan z}{z} \\
&= \lim_{z\to\frac{\pi}{2}} \frac{\left(z-\frac{\pi}{2}\right)\sin z}{z\left[-\left(z-\frac{\pi}{2}\right) + \frac{\left(z-\frac{\pi}{2}\right)^3}{6} + \cdots\right]} \\
&= \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}.
\end{aligned}$$

As $\tan z/z$ is even, we deduce that $\operatorname{Res}\left(\frac{\tan z}{z}, -\frac{\pi}{2}\right) = \frac{2}{\pi}$ using the result from the previous example. We then have

$$\oint_{|z|=2} \frac{\tan z}{z}\, dz = 0.$$

*Remark*

Let $p(z) = \sin z/z, q(z) = \cos z$, and observe that $p\left(\frac{\pi}{2}\right) = \frac{2}{\pi}, q\left(\frac{\pi}{2}\right) = 0$ and $q'\left(\frac{\pi}{2}\right) = -1 \neq 0$, then

$$\operatorname{Res}\left(\frac{\tan z}{z}, \frac{\pi}{2}\right) = p\left(\frac{\pi}{2}\right) \Big/ q'\left(\frac{\pi}{2}\right) = \frac{-2}{\pi}.$$

**Example**

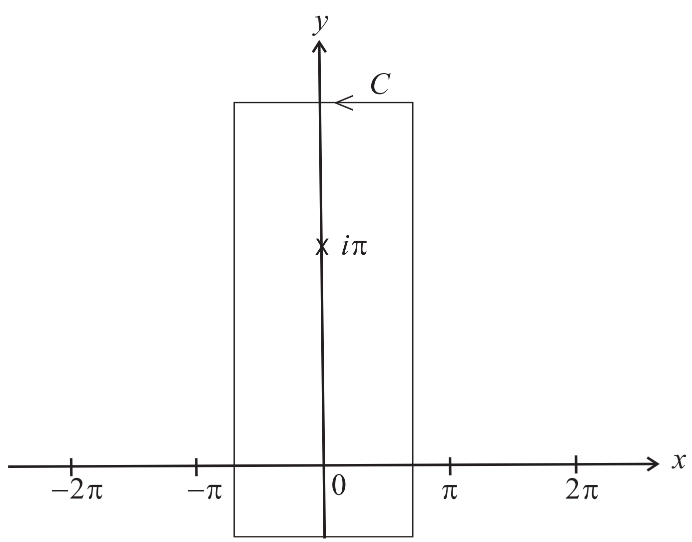
Evaluate

$$\oint_C \frac{z^2}{(z^2+\pi^2)^2 \sin z}\, dz.$$

*Solution*

$$\lim_{z\to 0} \frac{z}{\sin z}\frac{z}{(z^2+\pi^2)^2} = \left(\lim_{z\to 0}\frac{z}{\sin z}\right)\left(\lim_{z\to 0}\frac{z}{(z^2+\pi^2)^2}\right) = 0$$

so that $z = 0$ is a removable singularity.



It is easily seen that $z = i\pi$ is a pole of order 2.

$$
\begin{aligned}
\text{Res}\,(f, i\pi) &= \lim_{z \to i\pi} \frac{d}{dz}[(z - i\pi)^2 f(z)] \\
&= \lim_{z \to i\pi} \frac{d}{dz}\left[\frac{z^2}{(z + i\pi)^2 \sin z}\right] \\
&= \lim_{z \to i\pi} \frac{2z(z + i\pi)\sin z - z^2[(z + i\pi)\cos z + 2\sin z]}{(z + i\pi)^3 \sin^2 z} \\
&= \frac{2\sinh\pi + (-\pi\cosh\pi - \sinh\pi)}{-4\pi\sinh^2\pi} = -\frac{1}{4\pi\sinh\pi} + \frac{\cosh\pi}{4\pi\sinh^2\pi}.
\end{aligned}
$$

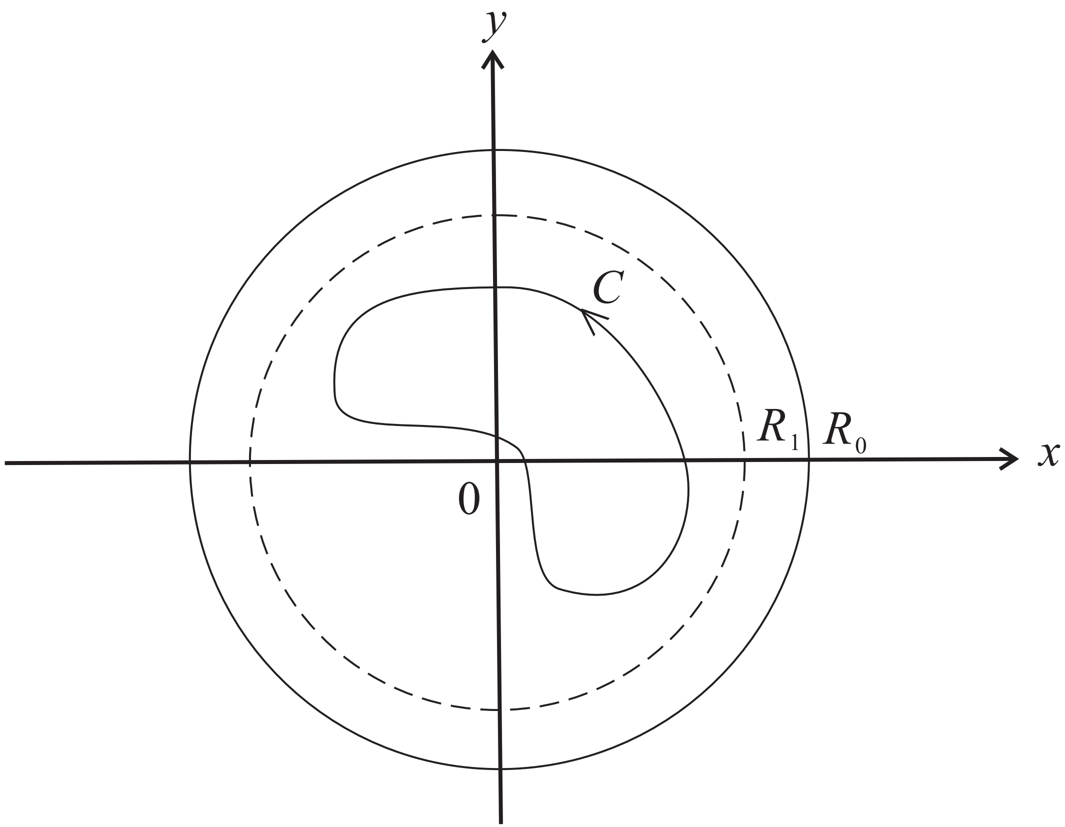
Recall that $\sin i\pi = i\sinh\pi$ and $\cos i\pi = \cosh\pi$. Hence,

$$
\begin{aligned}
\oint_C \frac{z^2}{(z^2 + \pi^2)^2 \sin z}\,dz &= 2\pi i \text{Res}\,(f, i\pi) \\
&= \frac{i}{2}\left(-\frac{1}{\sinh\pi} + \frac{\cosh\pi}{\sinh^2\pi}\right).
\end{aligned}
$$

**Theorem**

If a function $f$ is analytic everywhere in the finite plane except for a finite number of singularities interior to a positively oriented simple closed contour $C$, then

$$\oint_C f(z)\,dz = 2\pi i \text{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$



We construct a circle $|z| = R_1$ which is large enough so that $C$ is interior to it. If $C_0$ denotes a positively oriented circle $|z| = R_0$, where $R_0 > R_1$, then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \qquad R_1 < |z| < \infty, \tag{A}$$

where

$$c_n = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z)}{z^{n+1}}\, dz \quad n = 0, \pm 1, \pm 2, \cdots .$$

In particular,

$$2\pi i c_{-1} = \oint_{C_0} f(z)\, dz.$$

How to find $c_{-1}$? First, we replace $z$ by $1/z$ in Eq. (A) such that the domain of validity is a deleted neighborhood of $z = 0$.

Now

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n}, \qquad 0 < |z| < \frac{1}{R_1},$$

so that

$$c_{-1} = \text{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

*Remark*

By convention, we may define the residue at infinity by

$$\text{Res}\,(f, \infty) = -\frac{1}{2\pi i} \oint_C f(z)\, dz = -\text{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right),$$

where all singularities in the finite plane are included inside $C$. With the choice of the negative sign, we have

$$\sum_{all} \text{Res}\,(f, z_i) + \text{Res}\,(f, \infty) = 0.$$

**Example**
Evaluate

$$\oint_{|z|=2} \frac{5z-2}{z(z-1)}\, dz.$$

*Solution*

Write $f(z) = \dfrac{5z-2}{z(z-1)}$. For $0 < |z| < 1$,

$$\frac{5z-2}{z(z-1)} = \frac{5z-2}{z}\frac{-1}{1-z} = \left(5 - \frac{2}{z}\right)(-1 - z - z^2 - \cdots)$$

so that

$$\mathrm{Res}\,(f, 0) = 2.$$

For $0 < |z-1| < 1$,

$$\frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1}\frac{1}{1+(z-1)}$$
$$= \left(5 + \frac{3}{z-1}\right)\left[1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right]$$

so that

$$\text{Res}\,(f,1) = 3.$$

Hence,

$$\oint_{|z|=2} \frac{5z-2}{z(z-1)}\,dz = 2\pi i\,[\text{Res}\,(f,0) + \text{Res}\,(f,1)] = 10\pi i.$$

On the other hand, consider

$$\begin{aligned}
\frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{5-2z}{z(1-z)} = \frac{5-2z}{z}\frac{1}{1-z} \\
&= \left(\frac{5}{z} - 2\right)(1 + z + z^2 + \cdots) \\
&= \frac{5}{z} + 3 + 3z, \quad 0 < |z| < 1,
\end{aligned}$$

so that

$$\begin{aligned}
\oint_{|z|=2} \frac{5z-2}{z(z-1)}\, dz &= -2\pi i \mathrm{Res}\,(f, \infty) \\
&= 2\pi i \mathrm{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) = 10\pi i.
\end{aligned}$$

## Evaluation of integrals using residue methods

A wide variety of real definite integrals can be evaluated effectively by the calculus of residues.

### Integrals of trigonometric functions over $[0, 2\pi]$

We consider a real integral involving trigonometric functions of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\; d\theta,$$

where $R(x, y)$ is a rational function defined inside the unit circle $|z| = 1, z = x + iy$. The real integral can be converted into a contour integral around the unit circle by the following substitutions:

$$
\begin{aligned}
z &= e^{i\theta},\ dz = ie^{i\theta}\,d\theta = iz\,d\theta, \\
\cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right), \\
\sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right).
\end{aligned}
$$

The above integral can then be transformed into

$$
\begin{aligned}
& \int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta \\
= \ & \oint_{|z|=1} \frac{1}{iz}\,R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) dz \\
= \ & 2\pi i \left[\text{sum of residues of } \frac{1}{iz} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \text{inside } |z| = 1\right].
\end{aligned}
$$

**Example**

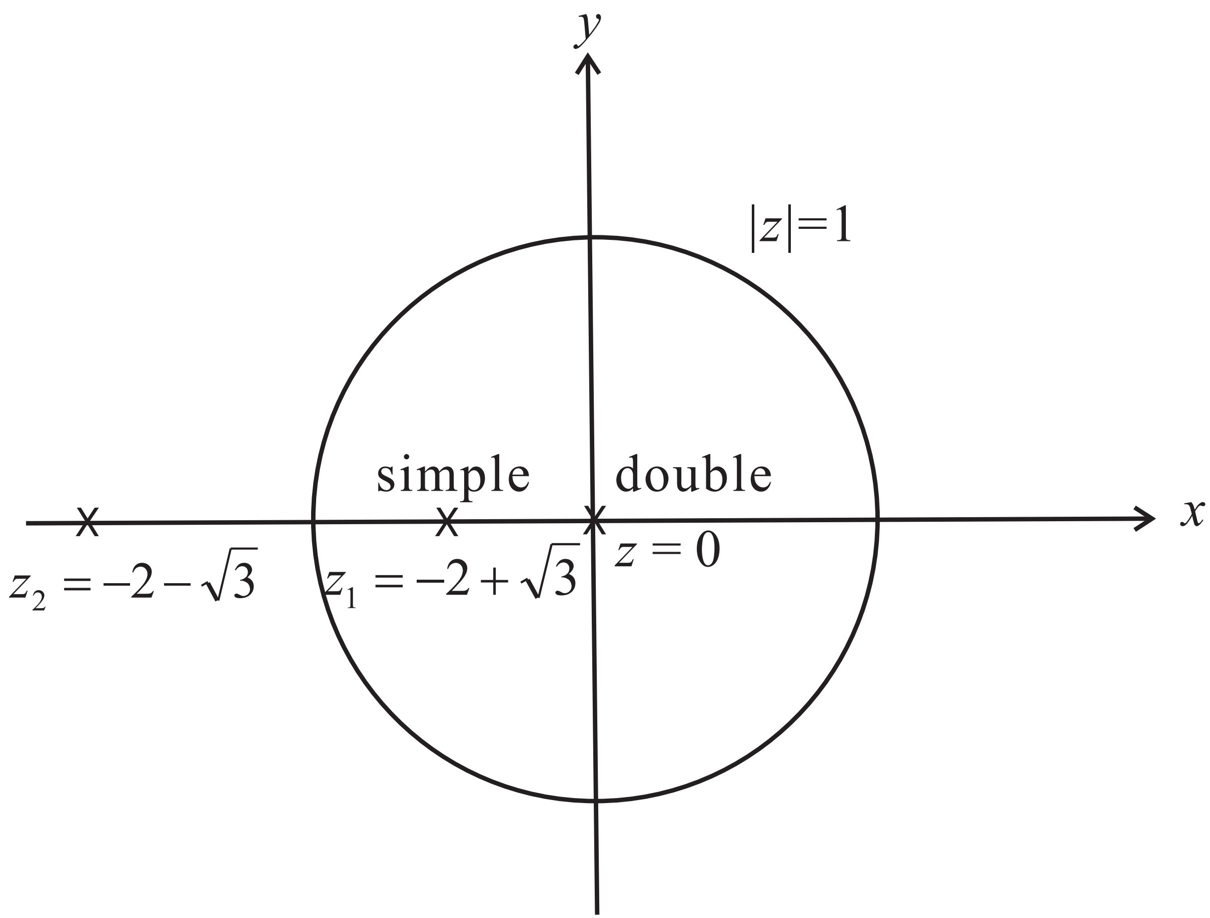
Compute $I = \int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}\, d\theta$.

*Solution*

$$\begin{aligned} \int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}\, d\theta &= -i \oint_{|z|=1} \frac{\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{z} \\ &= -i \oint_{|z|=1} \frac{z^4+1}{z^2(z^2+4z+1)}\, dz. \end{aligned}$$

The integrand has a pole of order two at $z = 0$. Also, the roots of $z^2+4z+1 = 0$, namely, $z_1 = -2-\sqrt{3}$ and $z_2 = -2+\sqrt{3}$, are simple poles of the integrand.

Write $f(z) = \dfrac{z^4+1}{z^2(z^2+4z+1)}$. Note that $z_1$ is inside but $z_2$ is outside $|z| = 1$.



$$
\begin{aligned}
\operatorname{Res}(f,0) &= \lim_{z\to 0}\frac{d}{dz}\frac{z^4+1}{z^2+4z+1} \\
&= \lim_{z\to 0}\frac{3z^3(z^2+4z+1)-(z^4+1)(2z+4)}{(z^2+4z+1)^2} = -4
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Res}(f,-2+\sqrt{3}) &= \left.\frac{z^4+1}{z^2}\right|_{z=-2+\sqrt{3}} \Big/ \left.\frac{d}{dz}(z^2+4z+1)\right|_{z=-2+\sqrt{3}} \\
&= \frac{(-2+\sqrt{3})^4+1}{(-2+\sqrt{3})^2}\cdot\frac{1}{2(-2+\sqrt{3})+4} = \frac{7}{\sqrt{3}}.
\end{aligned}
$$

$$
\begin{aligned}
I &= (-i)2\pi i\left[\operatorname{Res}(f,0)+\operatorname{Res}(f,-2+\sqrt{3})\right] \\
&= 2\pi\left(-4+\frac{7}{\sqrt{3}}\right).
\end{aligned}
$$

**Example**

Evaluate the integral

$$I = \int_0^{\pi} \frac{1}{a - b\cos\theta}\, d\theta, \qquad a > b > 0.$$

*Solution*

Since the integrand is symmetric about $\theta = \pi$, we have

$$I = \frac{1}{2}\int_0^{2\pi} \frac{1}{a - b\cos\theta}\, d\theta = \int_0^{2\pi} \frac{e^{i\theta}}{2ae^{i\theta} - b(e^{2i\theta} + 1)}\, d\theta.$$

The real integral can be transformed into the contour integral

$$I = i\oint_{|z|=1} \frac{1}{bz^2 - 2az + b}\, dz.$$

The integrand has two simple poles, which are given by the zeros of the denominator.

Let $\alpha$ denote the pole that is inside the unit circle, then the other pole will be $\frac{1}{\alpha}$. The two poles are found to be

$$\alpha = \frac{a - \sqrt{a^2 - b^2}}{b} \qquad \text{and} \qquad \frac{1}{\alpha} = \frac{a + \sqrt{a^2 - b^2}}{b}.$$

Since $a > b > 0$, the two roots are distinct, and $\alpha$ is inside but $\frac{1}{\alpha}$ is outside the closed contour of integration. We then have

$$\begin{aligned} I &= -\frac{1}{ib} \oint_{|z|=1} \frac{1}{(z-\alpha)\left(z - \frac{1}{\alpha}\right)}\, dz \\ &= -\frac{2\pi i}{ib} \text{ Res}\left(\frac{1}{(z-\alpha)\left(z-\frac{1}{\alpha}\right)}, \alpha\right) \\ &= -\frac{2\pi i}{ib\left(\alpha - \frac{1}{\alpha}\right)} = \frac{\pi}{\sqrt{a^2 - b^2}}. \end{aligned}$$

**Integral of rational functions**

$$\int_{-\infty}^{\infty} f(x)\, dx,$$

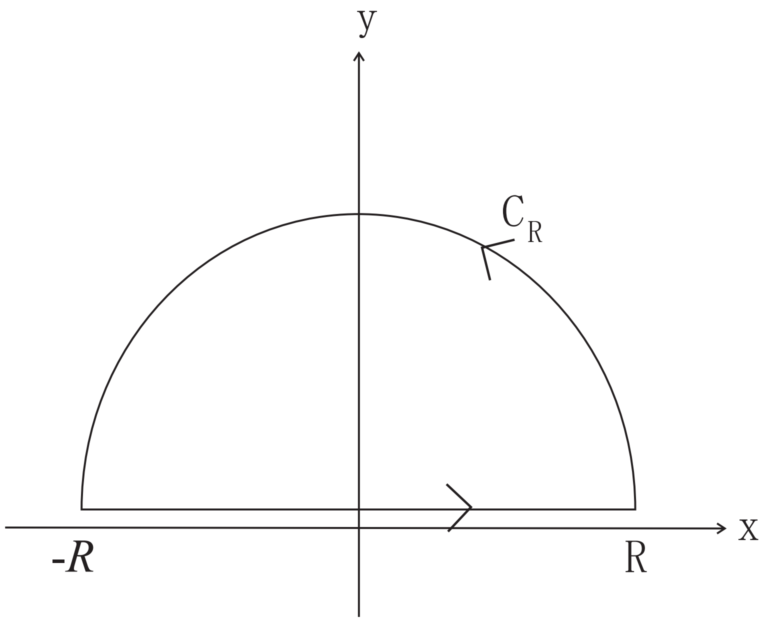
where

1. $f(z)$ is a rational function with no singularity on the real axis,
2. $\lim_{z \to \infty} z f(z) = 0$.

It can be shown that

$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i$ [sum of residues at the poles of $f$ in the upper half-plane].

Integrate $f(z)$ around a closed contour $C$ that consists of the upper semi-circle $C_R$ and the diameter from $-R$ to $R$.




By the Residue Theorem

$$\begin{aligned}\oint_C f(z)\,dz &= \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \\ &= 2\pi i\,[\text{sum of residues at the poles of } f \text{ inside } C].\end{aligned}$$

As $R \to \infty$, all the poles of $f$ in the upper half-plane will be enclosed inside $C$. To establish the claim, it suffices to show that as $R \to \infty$,

$$\lim_{R \to \infty} \oint_{C_R} f(z)\, dz = 0.$$

The modulus of the above integral is estimated by the modulus inequality as follows:

$$\begin{aligned}
\left| \int_{C_R} f(z)\, dz \right| &\le \int_0^\pi |f(Re^{i\theta})|\, R\, d\theta \\
&\le \max_{0 \le \theta \le \pi} |f(Re^{i\theta})|\, R \int_0^\pi d\theta \\
&= \max_{z \in C_R} |zf(z)| \pi,
\end{aligned}$$

which goes to zero as $R \to \infty$, since $\lim_{z \to \infty} zf(z) = 0$.

**Example**

Evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\, dx$$

by the residue method.

*Solution*

The complex function $f(z) = \dfrac{z^4}{1+z^6}$ has simple poles at $i$, $\dfrac{\sqrt{3}+i}{2}$ and $\dfrac{-\sqrt{3}+i}{2}$ in the upper half-plane, and it has no singularity on the real axis. The integrand observes the property $\lim_{z\to\infty} zf(z) = 0$. We obtain

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \left[\mathrm{Res}(f,i) + \mathrm{Res}\left(f, \frac{\sqrt{3}+i}{2}\right) + \mathrm{Res}\left(f, \frac{-\sqrt{3}+i}{2}\right)\right].$$

The residue value at the simple poles are found to be

$$\text{Res}(f, i) = \left.\frac{1}{6z}\right|_{z=i} = -\frac{i}{6},$$

$$\text{Res}\left(f, \frac{\sqrt{3}+i}{2}\right) = \left.\frac{1}{6z}\right|_{z=\frac{\sqrt{3}+i}{2}} = \frac{\sqrt{3}-i}{12},$$

and

$$\text{Res}\left(f, \frac{-\sqrt{3}+i}{2}\right) = \left.\frac{1}{6z}\right|_{z=\frac{-\sqrt{3}+i}{2}} = -\frac{\sqrt{3}+i}{12},$$

so that

$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\,dx = 2\pi i\left(-\frac{i}{6} + \frac{\sqrt{3}-i}{12} - \frac{\sqrt{3}+i}{12}\right) = \frac{2\pi}{3}.$$

**Integrals involving multi-valued functions**

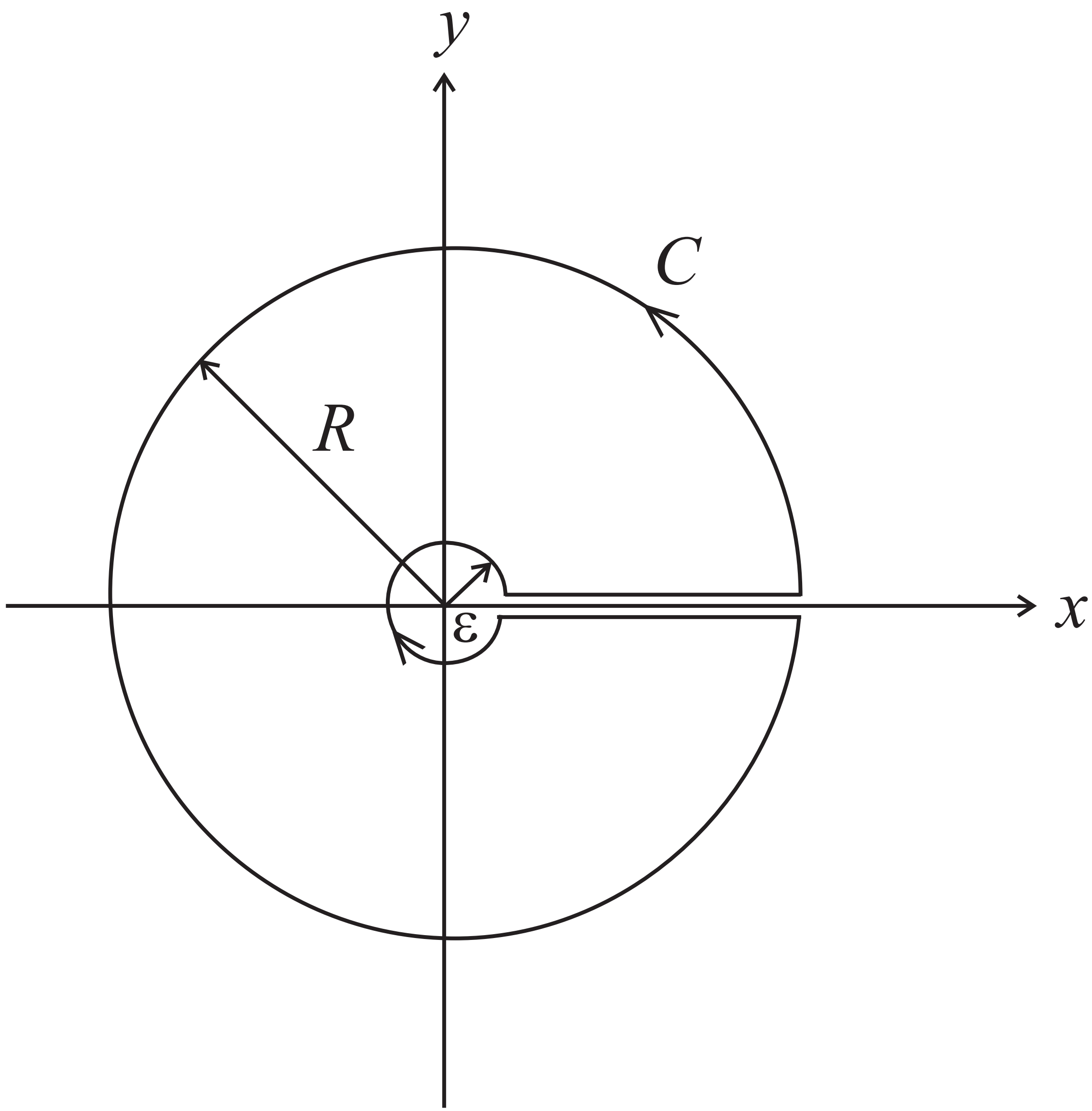
Consider a real integral involving a fractional power function

$$\int_0^\infty \frac{f(x)}{x^\alpha}\,dx, \quad 0 < \alpha < 1,$$

1. $f(z)$ is a rational function with no singularity on the positive real axis, including the origin.

2. $\lim_{z\to\infty} f(z) = 0$.

We integrate $\phi(z) = \dfrac{f(z)}{z^\alpha}$ along the closed contour as shown.




The closed contour $C$ consists of an infinitely large circle and an infinitesimal circle joined by line segments along the positive $x$-axis.

(i) line segment from $\varepsilon$ to $R$ along the upper side of the positive real axis: $z = x, \varepsilon \leq x \leq R$;

(ii) the outer large circle $C_R : z = Re^{i\theta}, \quad 0 < \theta < 2\pi$;

(iii) line segment from $R$ to $\varepsilon$ along the lower side of the positive real axis

$$z = xe^{2\pi i}, \quad \varepsilon \leq x \leq R;$$

(iv) the inner infinitesimal circle $C_\varepsilon$ in the clockwise direction

$$z = \varepsilon e^{i\theta}, \quad 0 < \theta < 2\pi.$$

Establish: $\lim_{R\to\infty}\int_{C_R}\phi(z) = 0$ and $\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\phi(z) = 0.$

$$\begin{aligned}\left|\int_{C_R}\phi(z)\,dz\right| &\leq \int_0^{2\pi}|\phi(Re^{i\theta})Re^{i\theta}|\,d\theta \leq 2\pi\max_{z\in C_R}|z\phi(z)| \\ \left|\int_{C_\varepsilon}\phi(z)\,dz\right| &\leq \int_0^{2\pi}|\phi(\varepsilon e^{i\theta})|\varepsilon\,d\theta \leq 2\pi\max_{z\in C_\epsilon}|z\phi(z)|.\end{aligned}$$

It suffices to show that $z\phi(z)\to 0$ as either $z\to\infty$ or $z\to 0$.

1. Since $\lim_{z\to\infty} f(z) = 0$ and $f(z)$ is a rational function,

   deg (denominator of $f(z)$) $\geq 1 +$ deg (numerator of $f(z)$).

   Further, $1-\alpha < 1, z\phi(z) = z^{1-\alpha}f(z)\to 0$ as $z\to\infty$.

2. Since $f(z)$ is continuous at $z=0$ and $f(z)$ has no singularity at the origin, $z\phi(z) = z^{1-\alpha}f(z)\sim 0\cdot f(0) = 0$ as $z\to 0$.

The argument of the principal branch of $z^\alpha$ is chosen to be $0 \le \theta < 2\pi$, as dictated by the contour.

$$\begin{aligned}
\oint_C \phi(z)\,dz &= \int_{C_R} \phi(z)\,dz + \int_{C_\epsilon} \phi(z)\,dz \\
&\quad + \int_\epsilon^R \frac{f(x)}{x^\alpha}\,dx + \int_R^\epsilon \frac{f(xe^{2\pi i})}{x^\alpha e^{2\alpha\pi i}}\,dx \\
&= 2\pi i\ [\text{sum of residues at all the isolated singularities} \\
&\quad \text{of } f \text{ enclosed inside the closed contour } C].
\end{aligned}$$

By taking the limits $\epsilon \to 0$ and $R \to \infty$, the first two integrals vanish. The last integral can be expressed as

$$-\int_0^\infty \frac{f(x)}{x^\alpha e^{2\alpha\pi i}}\,dx = -e^{-2\alpha\pi i}\int_0^\infty \frac{f(x)}{x^\alpha}\,dx.$$

Combining the results,

$$\begin{aligned}
\int_0^\infty \frac{f(x)}{x^\alpha}\,dx &= \frac{2\pi i}{1 - e^{-2\alpha\pi i}}\ [\text{sum of residues at all the isolated} \\
&\quad \text{singularities of } f \text{ in the finite complex plane}].
\end{aligned}$$

**Example**

Evaluate $\displaystyle\int_0^\infty \frac{1}{(1+x)x^\alpha}\,dx, \quad 0<\alpha<1.$

*Solution*

$f(z) = \dfrac{1}{(1+z)z^\alpha}$ is multi-valued and has an isolated singularity at $z=-1$. By the Residue Theorem,

$$
\begin{aligned}
&\oint_C \frac{1}{(1+z)z^\alpha}\;dz \\
=\;& (1-e^{-2\alpha\pi i})\int_\epsilon^R \frac{dx}{(1+x)x^\alpha} + \int_{C_R}\frac{dz}{(1+z)z^\alpha} + \int_{C_\epsilon}\frac{dz}{(1+z)z^\alpha} \\
=\;& 2\pi i\ \mathrm{Res}\left(\frac{1}{(1+z)z^\alpha}, -1\right) = \frac{2\pi i}{e^{\alpha\pi i}}.
\end{aligned}
$$

The moduli of the third and fourth integrals are bounded by

$$\left|\int_{C_R} \frac{1}{(1+z)z^\alpha}\,dz\right| \le \frac{2\pi R}{(R-1)R^\alpha} \sim R^{-\alpha} \to 0 \qquad \text{as} \qquad R \to \infty,$$

$$\left|\int_{C_\epsilon} \frac{1}{(1+z)z^\alpha}\,dz\right| \le \frac{2\pi \epsilon}{(1-\epsilon)\epsilon^\alpha} \sim \epsilon^{1-\alpha} \to 0 \qquad \text{as} \qquad \epsilon \to 0.$$

On taking the limits $R \to \infty$ and $\epsilon \to 0$, we obtain

$$(1 - e^{-2\alpha\pi i})\int_0^\infty \frac{1}{(1+x)x^\alpha}\,dx = \frac{2\pi i}{e^{\alpha\pi i}};$$

so

$$\int_0^\infty \frac{1}{(1+x)x^\alpha}\,dx = \frac{2\pi i}{e^{\alpha\pi i}\,(1-e^{-2\alpha\pi i})} = \frac{\pi}{\sin \alpha\pi}.$$

**Example**

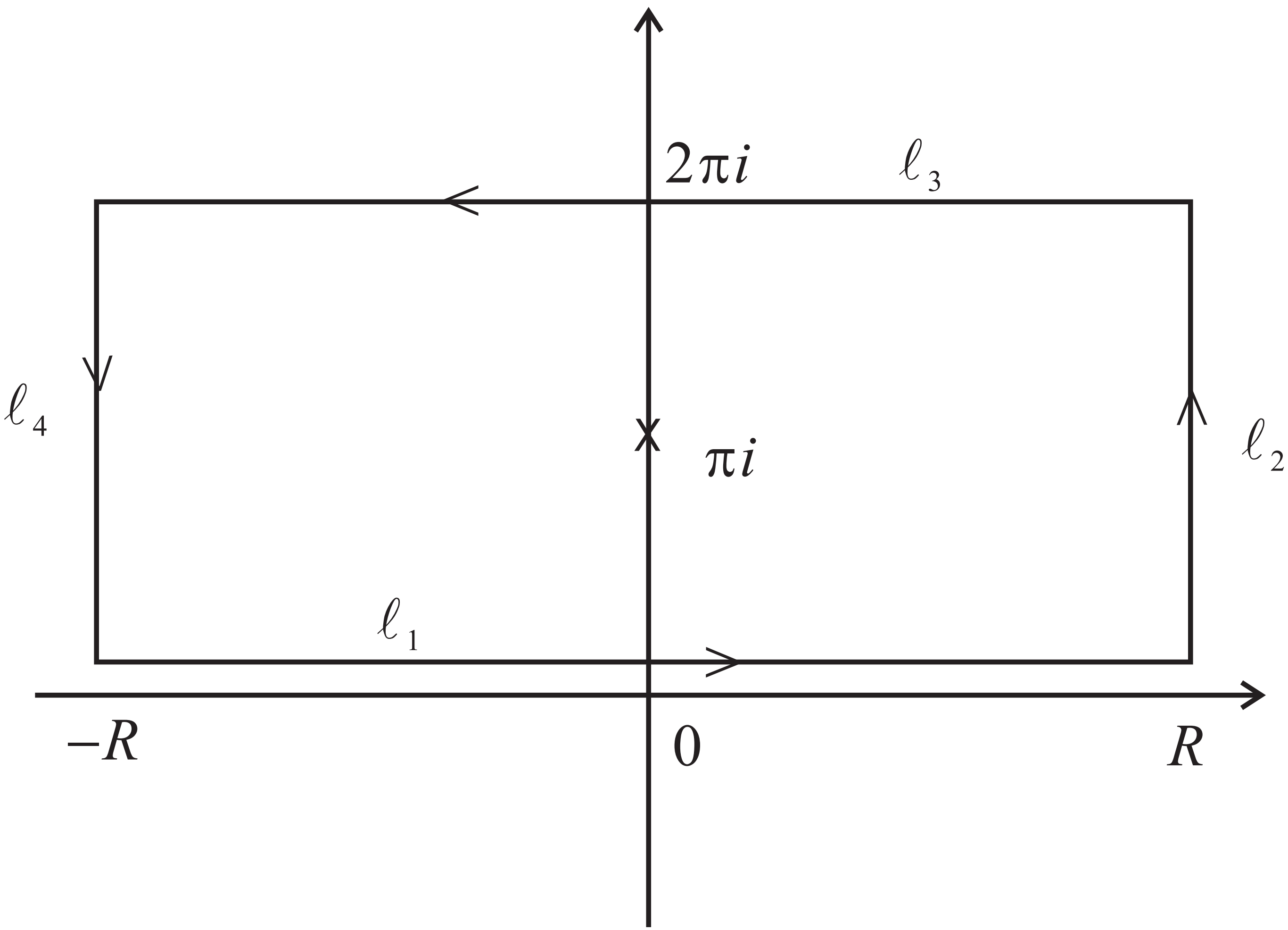
Evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x}\, dx, \qquad 0 < \alpha < 1.$$

*Solution*

The integrand function in its complex extension has infinitely many poles in the complex plane, namely, at $z = (2k+1)\pi i$, $k$ is any integer. We choose the rectangular contour as shown

$$\begin{aligned}
l_1 &: \ y = 0, && -R \le x \le R, \\
l_2 &: \ x = R, && 0 \le y \le 2\pi, \\
l_3 &: \ y = 2\pi, && -R \le x \le R, \\
l_4 &: \ x = -R, && 0 \le y \le 2\pi.
\end{aligned}$$




The chosen closed rectangular contour encloses only one simple pole at $z = \pi i$.

The only simple pole that is enclosed inside the closed contour $C$ is $z = \pi i$. By the Residue Theorem, we have

$$
\begin{aligned}
\oint_C \frac{e^{\alpha z}}{1+e^z}\,dz &= \int_{-R}^{R} \frac{e^{\alpha x}}{1+e^x}\,dx + \int_0^{2\pi} \frac{e^{\alpha(R+iy)}}{1+e^{R+iy}}\,idy \\
&\quad + \int_R^{-R} \frac{e^{\alpha(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx + \int_{2\pi}^{0} \frac{e^{\alpha(-R+iy)}}{1+e^{-R+iy}}\,idy \\
&= 2\pi i \text{ Res}\left(\frac{e^{\alpha z}}{1+e^z}, \pi i\right) \\
&= 2\pi i \left.\frac{e^{\alpha z}}{e^z}\right|_{z=\pi i} = -2\pi i e^{\alpha \pi i}.
\end{aligned}
$$

Consider the bounds on the moduli of the integrals as follows:

$$\left|\int_0^{2\pi} \frac{e^{\alpha(R+iy)}}{1+e^{R+iy}}\, idy\right| \leq \int_0^{2\pi} \frac{e^{\alpha R}}{e^R - 1}\, dy \sim O(e^{-(1-\alpha)R}),$$

$$\left|\int_{2\pi}^{0} \frac{e^{\alpha(-R+iy)}}{1+e^{-R+iy}}\, idy\right| \leq \int_0^{2\pi} \frac{e^{-\alpha R}}{1 - e^{-R}}\, dy \sim O(e^{-\alpha R}).$$

As $0 < \alpha < 1$, both $e^{-(1-\alpha)R}$ and $e^{-\alpha R}$ tend to zero as $R$ tends to infinity. Therefore, the second and the fourth integrals tend to zero as $R \to \infty$. On taking the limit $R \to \infty$, the sum of the first and third integrals becomes

$$(1 - e^{2\alpha\pi i}) \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x}\, dx = -2\pi i e^{\alpha\pi i};$$

so

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x}\, dx = \frac{2\pi i}{e^{\alpha\pi i} - e^{-\alpha\pi i}} = \frac{\pi}{\sin \alpha\pi}.$$

**Example**

Evaluate

$$\int_0^\infty \frac{1}{1+x^3}\,dx.$$

*Solution*

Since the integrand is not an even function, it serves no purpose to extend the interval of integration to $(-\infty, \infty)$. Instead, we consider the branch cut integral

$$\oint_C \frac{\text{Log } z}{1+z^3}\,dz,$$

where the branch cut is chosen to be along the positive real axis whereby $0 \leq \text{Arg } z < 2\pi$. Now

$$\begin{aligned} \oint_C \frac{\text{Log } z}{1+z^3}\,dz &= \int_\epsilon^R \frac{\ln x}{1+x^3}\,dx + \int_R^\epsilon \frac{\text{Log } (xe^{2\pi i})}{1+(xe^{2\pi i})^3}\,dx \\ &\quad + \oint_{C_R} \frac{\text{Log } z}{1+z^3}\,dz + \oint_{C_\epsilon} \frac{\text{Log } z}{1+z^3}\,dz \end{aligned}$$

$$= \; 2\pi i \sum_{j=1}^{3} \text{Res}\left(\frac{\text{Log } z}{1+z^3}, z_j\right),$$

where $z_j, j = 1, 2, 3$ are the zeros of $1/(1+z^3)$. Note that

$$\left|\oint_{C_\epsilon} \frac{\text{Log } z}{1+z^3}\, dz\right| \; = \; O\left(\frac{\epsilon \ln \epsilon}{1}\right) \longrightarrow 0 \text{ as } \epsilon \to 0;$$

$$\left|\oint_{C_R} \frac{\text{Log } z}{1+z^3}\, dz\right| \; = \; O\left(\frac{R \ln R}{R^3}\right) \longrightarrow 0 \text{ as } R \to \infty.$$

Hence

$$\lim_{\substack{R \to \infty \\ \epsilon \to 0}} \oint_C \frac{\text{Log } z}{1+z^3}\, dz \; = \; \int_0^\infty \frac{\ln x}{1+x^3}\, dx + \int_\infty^0 \frac{\text{Log } (xe^{2i\pi})}{1+(xe^{2i\pi})^3}\, dx$$

$$= \; -2\pi i \int_0^\infty \frac{1}{1+x^3}\, dx,$$

thus giving

$$\int_0^\infty \frac{1}{1+x^3}\, dx = -\sum_{j=1}^{3} \text{Res}\left(\frac{\text{Log } z}{1+z^3}, z_j\right).$$

The zeros of $1+z^3$ are $\alpha = e^{i\pi/3}, \beta = e^{i\pi}$ and $\gamma = e^{5\pi i/3}$. Sum of residues is given by

$$
\begin{aligned}
& \text{Res}\left(\frac{\text{Log } z}{1+z^3}, \alpha\right) + \text{Res}\left(\frac{\text{Log } z}{1+z^3}, \beta\right) + \text{Res}\left(\frac{\text{Log } z}{1+z^3}, \gamma\right) \\
= \; & \frac{\text{Log } \alpha}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\text{Log } \beta}{(\beta-\alpha)(\beta-\gamma)} + \frac{\text{Log } \gamma}{(\gamma-\alpha)(\gamma-\beta)} \\
= \; & -i\frac{\left[\frac{\pi}{3}(\beta-\gamma) + \pi(\gamma-\alpha) + \frac{5\pi}{3}(\alpha-\beta)\right]}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)} = -\frac{2\pi}{3\sqrt{3}}.
\end{aligned}
$$

Hence,

$$\int_0^\infty \frac{1}{1+x^3}\, dx = \frac{2\pi}{3\sqrt{3}}.$$

## Evaluation of Fourier integrals

A Fourier integral is of the form

$$\int_{-\infty}^{\infty} e^{imx} f(x)\, dx, \qquad m > 0,$$

1. $\lim_{z \to \infty} f(z) = 0$,
2. $f(z)$ has no singularity along the real axis.

*Remarks*

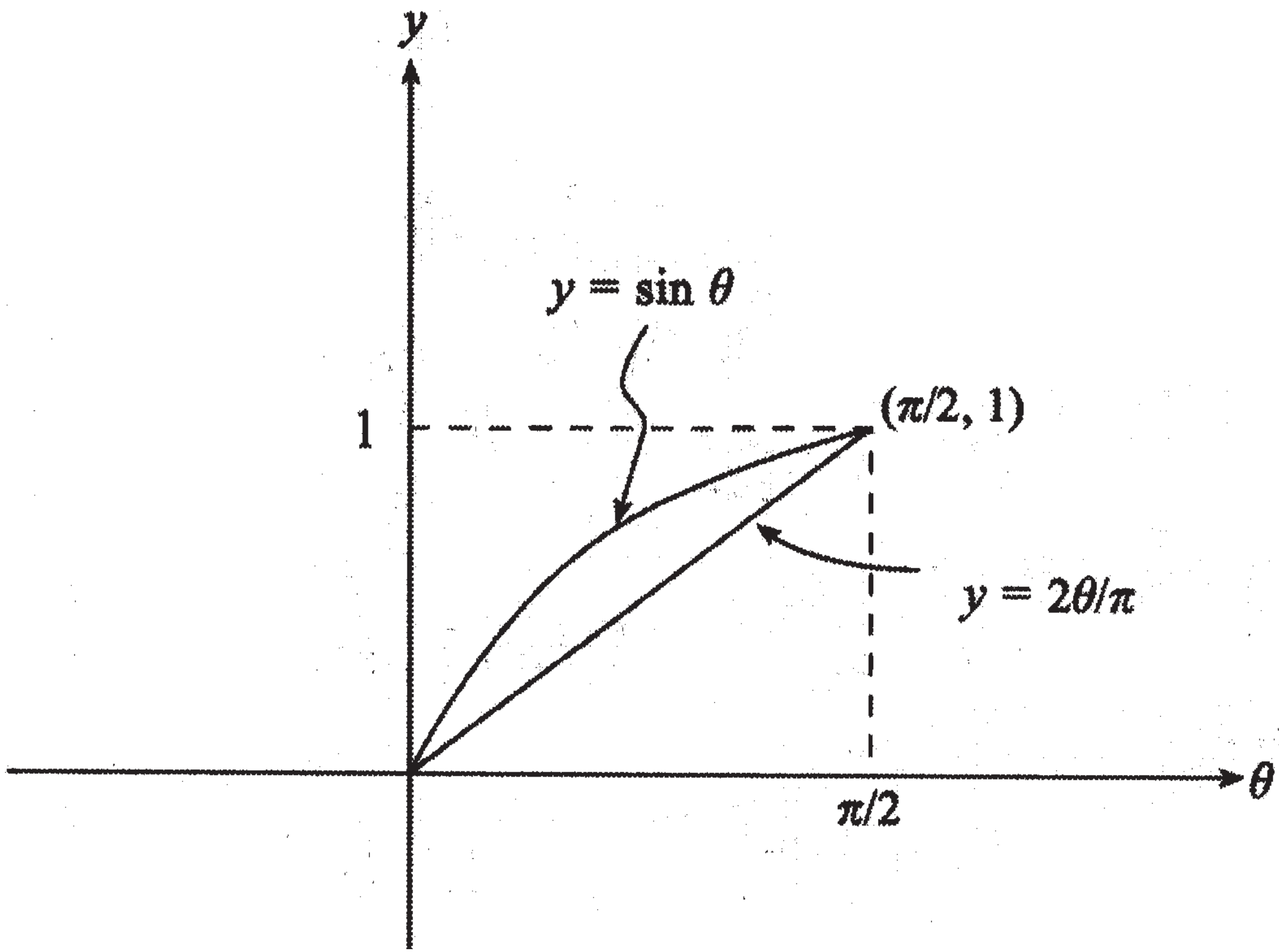
1. The assumption $m > 0$ is not strictly essential. The evaluation method works even when $m$ is negative or pure imaginary.
2. When $f(z)$ has singularities on the real axis, the Cauchy principal value of the integral is considered.

*Jordan Lemma*

We consider the modulus of the integral for $\lambda > 0$

$$\begin{aligned}
\left|\int_{C_R} f(z)e^{i\lambda z}\,dz\right| &\leq \int_0^{\pi} |f(Re^{i\theta})|\,|e^{i\lambda Re^{i\theta}}|\,R\,d\theta \\
&\leq \max_{z\in C_R}|f(z)|\,R\int_0^{\pi} e^{-\lambda R\sin\theta}\,d\theta \\
&= 2R\,\max_{z\in C_R}|f(z)|\int_0^{\frac{\pi}{2}} e^{-\lambda R\sin\theta}\,d\theta \\
&\leq 2R\,\max_{z\in C_R}|f(z)|\int_0^{\frac{\pi}{2}} e^{-\lambda R\frac{2\theta}{\pi}}\,d\theta \\
&= 2R\,\max_{z\in C_R}|f(z)|\,\frac{\pi}{2R\lambda}\,(1-e^{-\lambda R}),
\end{aligned}$$

which tends to 0 as $R\to\infty$, given that $f(z)\to 0$ as $R\to\infty$.


y
y = sin θ
1
(π/2, 1)
y = 2θ/π
θ
π/2

To evaluate the Fourier integral, we integrate $e^{imz}f(z)$ along the closed contour $C$ that consists of the upper half-circle $C_R$ and the diameter from $-R$ to $R$ along the real axis. We then have

$$\oint_C e^{imz} f(z)\, dz = \int_{-R}^{R} e^{imx} f(x)\, dx + \int_{C_R} e^{imz} f(z)\, dz.$$

Taking the limit $R \to \infty$, the integral over $C_R$ vanishes by virtue of the Jordan Lemma.

Lastly, we apply the Residue Theorem to obtain

$$\int_{-\infty}^{\infty} e^{imx} f(x)\, dx = 2\pi i \text{ [sum of residues at all the isolated singularities of } f \text{ in the upper half-plane]}$$

since $C$ encloses all the singularities of $f$ in the upper half-plane as $R \to \infty$.

**Example**

Evaluate the Fourier integral

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2 + x + 1}\, dx.$$

*Solution*

It is easy to check that $f(z) = \frac{1}{z^2+z+1}$ has no singularity along the real axis and $\lim_{z\to\infty} \frac{1}{z^2+z+1} = 0$. The integrand has two simple poles, namely, $z = e^{\frac{2\pi i}{3}}$ in the upper half-plane and $e^{-\frac{2\pi i}{3}}$ in the lower half-plane. By virtue of the Jordan Lemma, we have

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+x+1}\,dx \;=\; \text{Im}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+x+1}\,dx \;=\; \text{Im}\oint_C \frac{e^{2iz}}{z^2+z+1}\,dz,$$

where $C$ is the union of the infinitely large upper semi-circle and its diameter along the real axis.

Note that

$$\begin{aligned}\oint_C \frac{e^{2iz}}{z^2+z+1}\,dz \;&=\; 2\pi i\,\text{Res}\left(\frac{e^{2iz}}{z^2+z+1}, e^{\frac{2\pi i}{3}}\right)\\ &=\; 2\pi i\,\frac{e^{2iz}}{2z+1}\bigg|_{z=e^{\frac{2\pi i}{3}}} \;=\; 2\pi i\,\frac{e^{2ie^{\frac{2\pi i}{3}}}}{2e^{\frac{2\pi i}{3}}+1}.\end{aligned}$$

Hence,

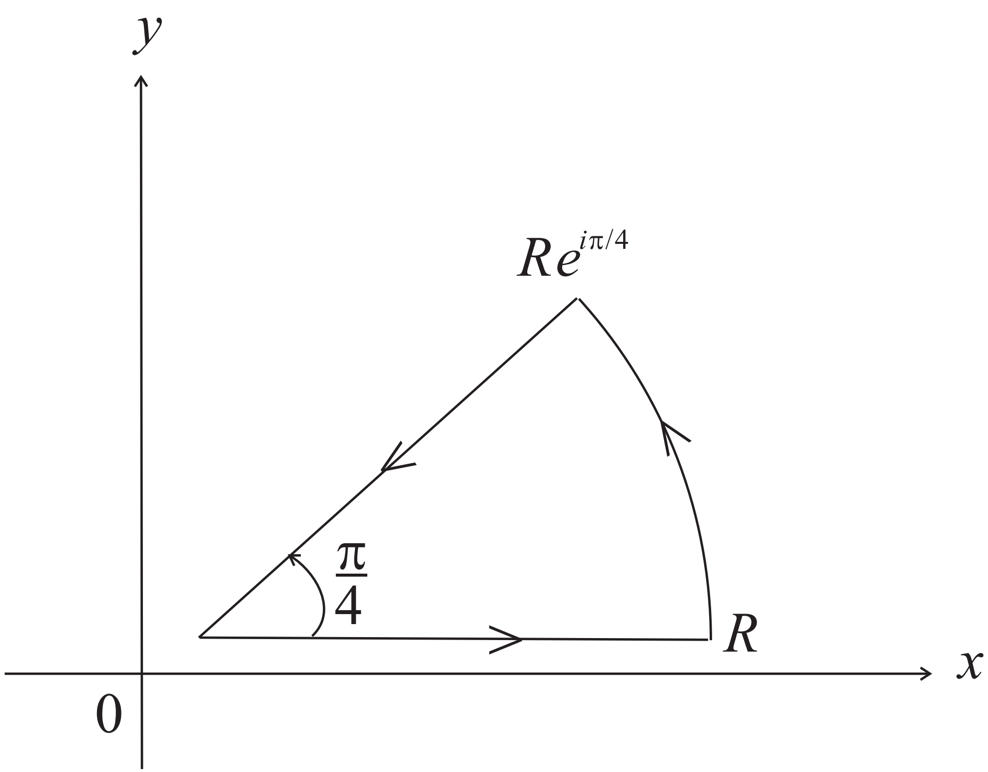
$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+x+1}\,dx \;=\; \text{Im}\left(2\pi i\,\frac{e^{2ie^{\frac{2\pi i}{3}}}}{2e^{\frac{2\pi i}{3}}+1}\right) = -\frac{2}{\sqrt{3}}\,\pi e^{-\sqrt{3}}\,\sin 1.$$

**Example**

Show that

$$\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

*Solution*



$$0 = \oint_C e^{iz^2}\,dz \;=\; \int_0^R e^{ix^2}\,dx + \int_0^{\frac{\pi}{4}} e^{iR^2 e^{2i\theta}} iRe^{i\theta}\,d\theta$$
$$+ \int_R^0 e^{ir^2 e^{i\pi/2}} e^{i\pi/4}\,dr.$$

Rearranging

$$\int_0^R (\cos x^2 + i\sin x^2)\,dx = e^{i\pi/4}\int_0^R e^{-r^2}\,dr - \int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2\sin 2\theta} iRe^{i\theta}\,d\theta.$$

Next, we take the limit $R \to \infty$. We recall the well-known result

$$e^{i\frac{\pi}{4}}\int_0^\infty e^{-r^2}\,dr = \frac{\sqrt{\pi}}{2}e^{i\pi/4} = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}}.$$

Also, we use the transformation $2\theta = \phi$ and observe $\sin\phi \geq \frac{2\phi}{\pi}, 0 \leq \phi \leq \frac{\pi}{2}$, to obtain

$$
\begin{aligned}
\left| \int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2 \sin^2\theta} iRe^{i\theta}\, d\theta \right| &\le \int_0^{\pi/4} e^{-R^2 \sin 2\theta} R\, d\theta \\
&= \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin\phi}\, d\phi \\
&\le \frac{R}{2} \int_0^{\pi/2} e^{-2R^2\phi/\pi}\, d\phi \\
&= \frac{\pi}{4R}(1 - e^{-R^2}) \longrightarrow 0 \text{ as } R \to \infty.
\end{aligned}
$$

We then obtain

$$
\int_0^\infty (\cos x^2 + i \sin x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} + i\frac{1}{2}\sqrt{\frac{\pi}{2}}
$$

so that

$$
\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.
$$

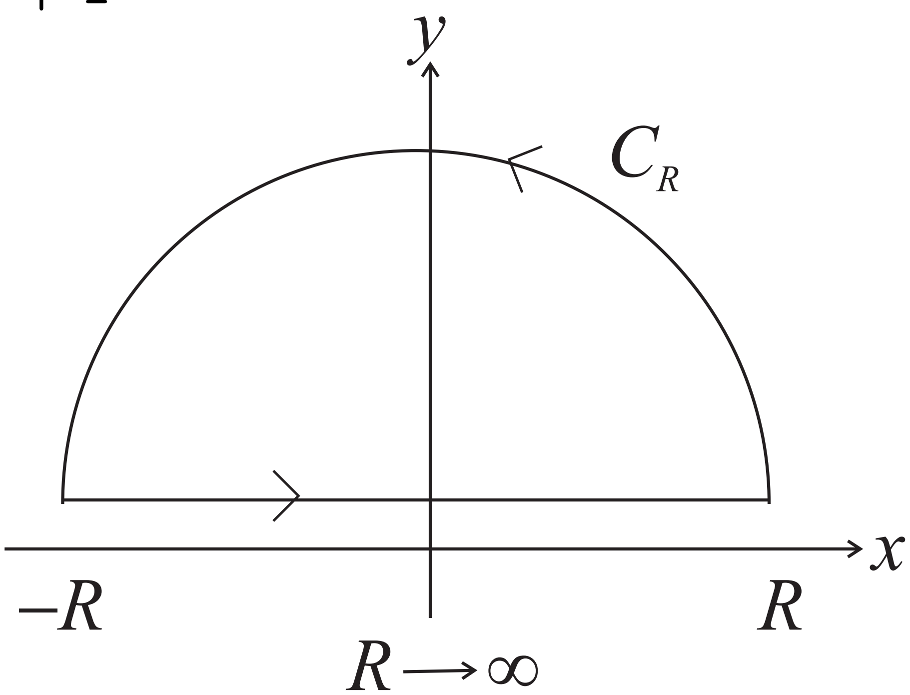
**Example**

Evaluate $\displaystyle\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx$.

*Hint:* Use $\text{Log}(i-x)+\text{Log}(i+x)=\text{Log}(i^2-x^2)=\ln(x^2+1)+\pi i$.

*Solution*

Consider $\displaystyle\oint_C \frac{\text{Log}(z+i)}{z^2+1}\,dz$ around $C$ as shown.



The only pole of $\dfrac{\text{Log}(z+i)}{z^2+1}$ in the upper half plane is the simple pole $z = i$. Consider

$$\begin{aligned} & 2\pi i \text{Res}\left(\frac{\text{Log}(z+i)}{z^2+1}, i\right) \\ = \quad & 2\pi i \lim_{z \to i} \frac{(z-i)\text{Log}(z+i)}{(z-i)(z+i)} = \pi \text{Log } 2i = \pi \ln 2 + \frac{\pi^2}{2} i. \end{aligned}$$

$$\int_{C_R} \frac{\text{Log}(z+i)}{z^2+1}\, dz = O\left(\frac{(\ln R)R}{R^2}\right) \to 0 \quad \text{as} \quad R \to \infty$$

$$\int_0^R \frac{\text{Log}(i-x)}{x^2+1}\, dx + \int_0^R \frac{\text{Log}(x+i)}{x^2+1}\, dx + \int_{C_R} \frac{\text{Log}(z+i)}{z^2+1}\, dz = \pi \ln 2 + \frac{\pi^2}{2} i$$

$$\text{and} \quad \text{Log}(i-x) + \text{Log}(i+x) = \ln(x^2+1) + \pi i.$$

From $\displaystyle\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\, dx + \int_0^\infty \frac{\pi i}{x^2+1}\, dx = \pi \ln 2 + \frac{\pi^2}{2} i$ and $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$ so that

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\, dx = \pi \ln 2.$$

## Cauchy principal value of an improper integral

Suppose a real function $f(x)$ is continuous everywhere in the interval $[a, b]$ except at a point $x_0$ inside the interval. The integral of $f(x)$ over the interval $[a, b]$ is an improper integral, which may be defined as

$$\int_a^b f(x)\,dx = \lim_{\epsilon_1, \epsilon_2 \to 0} \left[ \int_a^{x_0 - \epsilon_1} f(x)\,dx + \int_{x_0 + \epsilon_2}^b f(x)\,dx \right], \quad \epsilon_1,\ \epsilon_2 > 0.$$

In many cases, the above limit exists only when $\epsilon_1 = \epsilon_2$, and does not exist otherwise.

**Example**

Consider the following improper integral

$$\int_{-1}^{2} \frac{1}{x-1}\ dx,$$

show that the Cauchy principal value of the integral exists, then find the principal value.

*Solution*

Principal value of $\displaystyle\int_{1}^{2} \frac{1}{x-1}\ dx$ exists if the following limit exists.

$$\begin{aligned} & \lim_{\epsilon \to 0^+} \left[ \int_{-1}^{1-\epsilon} \frac{1}{x-1}\ dx + \int_{1+\epsilon}^{2} \frac{1}{x-1}\ dx \right] \\ = & \lim_{\epsilon \to 0^+} \left[ \ln|x-1| \Big|_{-1}^{1-\epsilon} + \ln|x-1| \Big|_{1+\epsilon}^{2} \right] \\ = & \lim_{\epsilon \to 0^+} \left[ (\ln \epsilon - \ln 2) + (\ln 1 - \ln \epsilon) \right] = -\ln 2. \end{aligned}$$

Hence, the principal value of $\displaystyle\int_{-1}^{2} \frac{1}{x-1}\ dx$ exists and its value is $-\ln 2$.

**Lemma**

If $f$ has a simple pole at $z = c$ and $T_r$ is the circular arc defined by

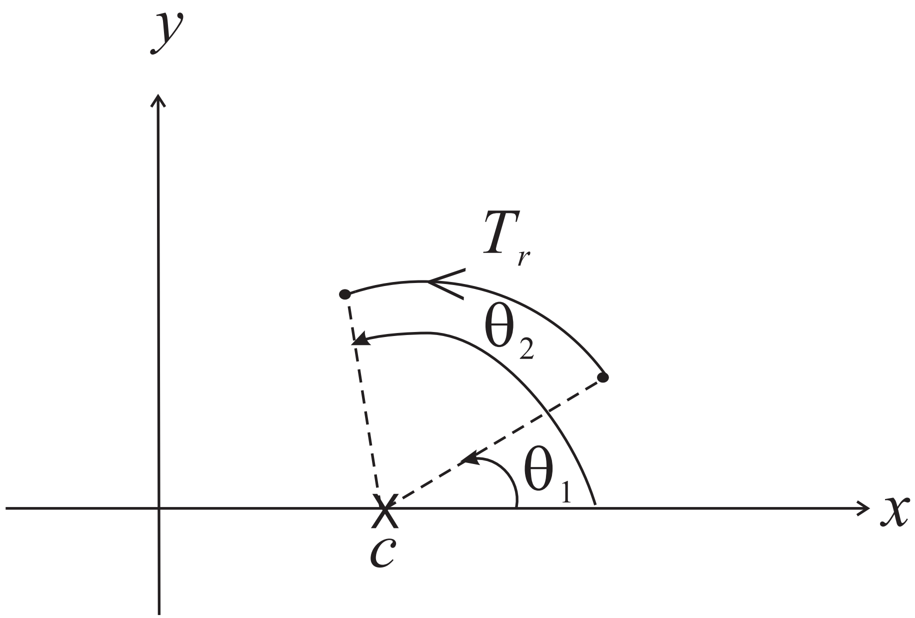
$$T_r : z = c + re^{i\theta} \quad (\theta_1 \le \theta \le \theta_2),$$

then

$$\lim_{r \to 0^+} \int_{T_r} f(z)\, dz = i(\theta_2 - \theta_1)\text{Res}\,(f, c).$$

In particular, for the semi-circular arc $S_r$

$$\lim_{r \to 0^+} \int_{S_r} f(z)\, dz = i\pi\text{Res}\,(f, c).$$




*Proof*

Since $f$ has a simple pole at $c$,

$$f(z) = \frac{a_{-1}}{z-c} + \underbrace{\sum_{k=0}^{\infty} a_k (z-c)^k}_{g(z)}, \quad 0 < |z-c| < R \quad \text{for some } R.$$

For $0 < r < R$, $\int_{T_r} f(z)\,dz = a_{-1} \int_{T_r} \frac{1}{z-c}\,dz + \int_{T_r} g(z)\,dz.$

Since $g(z)$ is analytic at $c$, it is bounded in some neighborhood of $z = c$. That is,

$$|g(z)| \leq M \quad \text{for} \quad |z - c| < r.$$

For $0 < r < R$,

$$\left| \int_{T_r} g(z)\, dz \right| \leq M \cdot \text{ arc length of } T_r = Mr(\theta_2 - \theta_1)$$

and so

$$\lim_{r \to 0+} \int_{T_r} g(z)\, dz = 0.$$

Finally,

$$\int_{T_r} \frac{1}{z - c}\, dz = \int_{\theta_1}^{\theta_2} \frac{1}{re^{i\theta}} ire^{i\theta}\, d\theta = i \int_{\theta_1}^{\theta_2} d\theta = i(\theta_2 - \theta_1)$$

so that

$$\lim_{r \to 0+} \int_{T_r} f(z)\, dz = a_{-1} i(\theta_2 - \theta_1) = \text{Res}\,(f, c) i(\theta_2 - \theta_1).$$

**Example**

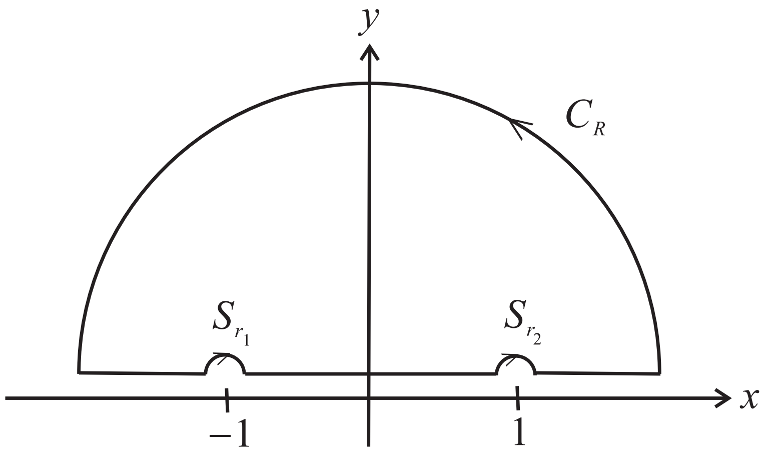
Compute the principal value of

$$\int_{-\infty}^{\infty} \frac{xe^{2ix}}{x^2-1}\, dx.$$

*Solution*

The improper integral has singularities at $x = \pm 1$. The principal value of the integral is defined to be

$$\lim_{\substack{R\to\infty \\ r_1, r_2 \to 0^+}} \left( \int_{-R}^{-1-r_1} + \int_{-1+r_1}^{1-r_2} + \int_{1+r_2}^{R} \right) \frac{xe^{2ix}}{x^2-1}\, dx.$$



Let

$$
\begin{aligned}
I_1 &= \int_{S_{r_1}} \frac{ze^{2iz}}{z^2-1}\,dz \\
I_2 &= \int_{S_{r_2}} \frac{ze^{2iz}}{z^2-1}\,dz \\
I_R &= \int_{C_R} \frac{ze^{2iz}}{z^2-1}\,dz.
\end{aligned}
$$

Now, $f(z) = \dfrac{ze^{2iz}}{z^2-1}$ is analytic inside the above closed contour.

By the Cauchy Integral Theorem

$$
\left(\int_{-R}^{-1-r_1} + \int_{-1+r_1}^{1-r_2} + \int_{1+r_2}^{R}\right) \frac{xe^{2ix}}{x^2-1}\,dx + I_1 + I_2 + I_R = 0.
$$

By the Jordan Lemma, and since $\frac{z}{z^2-1} \to 0$ as $z \to \infty$, so

$$\lim_{R\to\infty} I_R = 0.$$

Since $z = \pm 1$ are simple poles of $f$,

$$\begin{aligned}\lim_{r_1\to 0^+} I_1 &= -i\pi \text{Res}\,(f,-1) = -i\pi \lim_{z\to -1}(z+1)f(z) \\ &= (-i\pi)e^{-2i}/2.\end{aligned}$$

Similarly, $\lim_{r_2\to 0^+} I_2 = -i\pi \text{Res}\,(f,1) = \frac{-i\pi e^{2i}}{2}$.

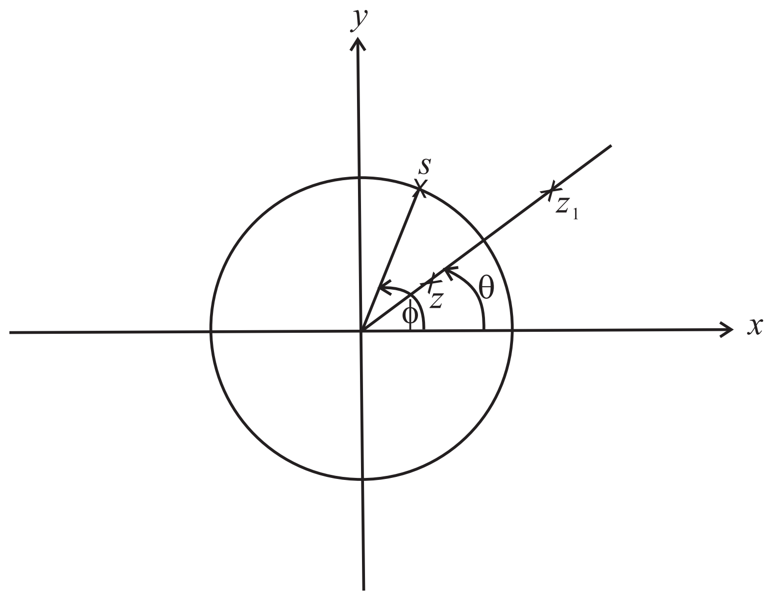
$$PV\int_{-\infty}^{\infty} \frac{xe^{2ix}}{x^2-1}\,dx = \frac{i\pi e^{-2i}}{2} + \frac{i\pi e^{2i}}{2} = i\pi\cos 2.$$

## Poisson integral formula

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s - z}\, ds.$$

Here, $C$ is the circle with radius $r_0$ centered at the origin. Write $s = r_0 e^{i\phi}$ and $z = re^{i\theta}$, $r > r_0$. We choose $z_1$ such that $|z_1|\ |z| = r_0^2$ and both $z_1$ and $z$ lie on the same ray so that

$$z_1 = \frac{r_0^2}{r} e^{i\theta} = \frac{r_0^2}{\overline{z}} = \frac{s\overline{s}}{\overline{z}}.$$



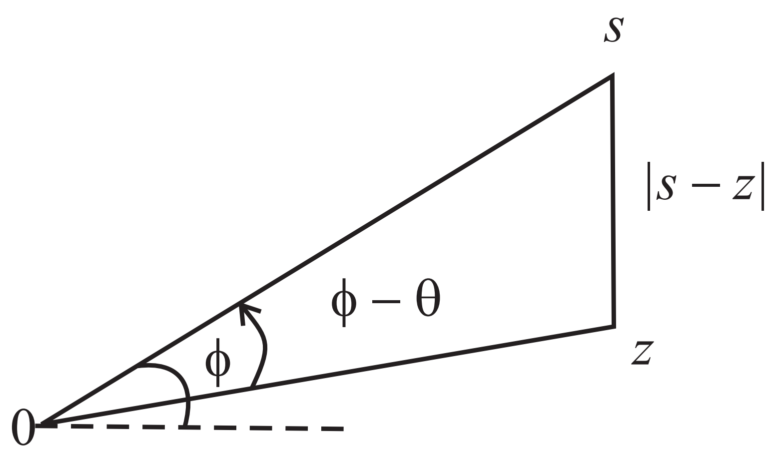
Since $z_1$ lies outside $C$, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C f(s)\left(\frac{1}{s-z}-\frac{1}{s-z_1}\right) ds \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{s}{s-z}-\frac{s}{s-z_1}\right) f(s)\, d\phi. \end{aligned}$$

The integrand can be expressed as

$$\frac{s}{s-z}-\frac{1}{1-\overline{s}/\overline{z}}=\frac{s}{s-z}+\frac{\overline{z}}{\overline{s}-\overline{z}}=\frac{r_0^2-r^2}{|s-z|^2}$$

and so $f(re^{i\theta})=\dfrac{r_0^2-r^2}{2\pi}\displaystyle\int_0^{2\pi}\frac{f(r_0e^{i\theta})}{|s-z|^2}\,d\phi$.



Now $|s-z|^2 = r_0^2 - 2r_0 r\cos(\phi-\theta) + r^2 > 0$ (from the cosine rule). Taking the real part of $f$, where $f = u + iv$, we obtain

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{r_0^2 - r^2}{r_0^2 - 2r_0 r\cos(\phi-\theta) + r^2}}_{P(r_0,r,\phi-\theta)} u(r_0,\phi)\,d\phi, \qquad r < r_0.$$

Knowing $u(r_0,\phi)$ on the boundary, $u(r,\theta)$ is uniquely determined.

The kernel function $P(r_0,r,\phi-\theta)$ is called the *Poisson kernel*.

$$\begin{aligned} P(r_0,r,\phi-\theta) &= \frac{r_0^2 - r^2}{|s-z|^2} = \mathrm{Re}\left(\frac{s}{s-z} + \frac{\overline{z}}{\overline{s}-\overline{z}}\right) \\ &= \mathrm{Re}\left(\frac{s}{s-z} + \frac{z}{s-z}\right) \\ &= \mathrm{Re}\left(\frac{s+z}{s-z}\right) \text{ which is harmonic for } |z| < r_0. \end{aligned}$$