MATH304, Spring 2007

Final Examination

Time allowed: 100 minutues **Course instructor:** *Prof. Y.K. Kwok*

[points]

1. (a) Find the radius of convergence of the following power series

$$3 + z + 3z^2 + z^3 + 3z^4 + z^5 + 3z^6 + \cdots$$

Hint: Use the root test. **[2]**

(b) Use Weierstrass' M-test to prove that

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2(n+1)}$$

is uniformly convergent for $|z| \leq 1$. **[2]**

(c) Show that

$$\sum_{n=1}^{\infty} \frac{z^4}{(1+z^4)^{n-1}} = 1 + z^4 \quad \text{for} \quad |1+z^4| > 1.$$

Furthermore, show that the series is uniformly convergent in the region: **[3]**

$$|1+z^4| \geq r, \text{ where } r > 1.$$

Hint: Define $S_n(z) = \sum_{k=1}^{n} \frac{z^4}{(1+z^4)^{k-1}}$, then

$$|S_n(z) - (1+z^4)| = \left| \frac{1}{(1+z^4)^{n-1}} \right|.$$

2. Consider the function

$$f(z) = \frac{z}{(z-\alpha)^2}, \quad \alpha \neq 0 \text{ and } |\alpha| \neq 1.$$

(a) Find the Laurent series of the function expanded at $z_0 = \alpha$ valid in $|z - \alpha| > 0$. **[1]**

(b) Find the Laurent series of the function expanded at $z_0 = 0$ valid in

(i) $|z| < |\alpha|$ **[2]**

(ii) $|\alpha| < |z| < \infty$. **[2]**

Explain why the Laurent series in $|z| < |\alpha|$ reduces to a Taylor series. **[1]**

(c) Is it meaningful to define (i) $\text{Res}(f, \alpha)$, (ii) $\text{Res}(f, 0)$? If yes, find the value of the residue. If not, explain why? **[2]**

(d) Find all possible values of the following integral

$$\oint_{|z|=1} \frac{z}{(z-\alpha)^2}\, dz, \quad |\alpha| \neq 1.$$

Distinguish the cases where $|\alpha| > 1$ and $|\alpha| < 1$. **[2]**

3. Let $p(z)$ and $q(z)$ be analytic at z_0, while z_0 is a simple zero of $p(z)$ and z_0 is a triple zero of $q(z)$. That is,

$$p(z_0) = 0, \quad p'(z_0) \neq 0$$
$$q(z_0) = q'(z_0) = q''(z_0) = 0, \quad q'''(z_0) \neq 0.$$

(a) Show that $f(z) = p(z)/q(z)$ has a double pole at z_0. **[1]**

(b) Compute **[3]**

$$\text{Res}\,(f, z_0).$$

Hint: In terms of Taylor series expansion of $p(z)$ and $q(z)$ at z_0, we have

$$f(z) = \frac{p(z)}{q(z)} = \frac{p'(z_0)(z-z_0) + p''(z_0)(z-z_0)^2/2! + \cdots}{q'''(z_0)(z-z_0)^3/3! + q''''(z_0)(z-z_0)^4/4! + \cdots}.$$

If f has a double pole at z_0, then

$$\text{Res}\,(f, z_0) = \lim_{z\to z_0} \frac{d}{dz}[(z-z_0)^2 f].$$

4. (a) Locate each of the isolated singularities of the function $f(z) = \pi \cot \dfrac{\pi}{z}$, and determine whether it is a removable singularity, a pole or an essential singularity. If the singularity is removable, then give the limit of the function at the point. If the singularity is a pole, then give the order of the pole, and compute the residue at the singularity. **[3]**

(b) Is $z = 0$ an isolated singularity of $f(z)$? Give detailed explanation to your answer. **[1]**

5. Find the Cauchy principal value of the following improper integral:

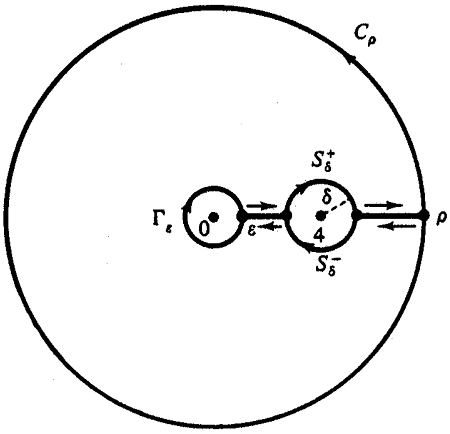
$$J = \int_0^\infty \frac{1}{x^\lambda (x-4)}\, dx, \quad 0 < \lambda < 1.$$

[7]

Hint: Choose the branch of the power function z^λ as

$$z^\lambda = e^{\lambda \log z} = e^{\lambda(\ln r + i\theta)}, \quad z = re^{i\theta}, 0 < \theta < 2\pi,$$

where the branch cut of $\log z$ is taken to be along the positive real axis. Note that $z = 4$ is a simple pole of the integrand function which lies along the positive real axis. Choose the closed contour C as shown:



Give your justification that the contour integrals along C_ρ and Γ_ϵ tend to zero as $\rho \to \infty$ and $\epsilon \to 0$, respectively.

Distribution of points in this problem

(i) Relate the contour integral along the closed contour C with the principal value of the improper integral J. **[2]**

(ii) Show

$$\lim_{\rho\to\infty} \int_{C_\rho} \frac{1}{z^\lambda (z-4)}\, dz = 0 \quad \text{and} \quad \lim_{\epsilon\to 0^+} \int_{\Gamma_\epsilon} \frac{1}{z^\lambda (z-4)}\, dz = 0.$$

[2]

(iii) Evaluate

$$\lim_{\delta\to 0^+} \int_{S_{\delta+}} \frac{1}{z^\lambda (z-4)}\, dz \quad \text{and} \quad \lim_{\delta\to 0^+} \int_{S_{\delta-}} \frac{1}{z^\lambda (z-4)}\, dz.$$

[2]

(iv) Finally, find the principal value of the integral. **[1]**

6. Consider the evaluation of the following integral

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\, dx, \quad 0 < p < 1.$$

The result of integrating

$$f(z) = \frac{e^{pz}}{1+e^z}$$

around a closed rectangular contour C, including the real axis from $x = -A$ to $x = A$ and the line $y = 2\pi$ from $x = A$ to $x = -A$ can be written in the form

$$\int_{-A}^{A} \frac{e^{px}}{1+e^x}\, dx - e^{2p\pi i} \int_{-A}^{A} \frac{e^{px}}{1+e^x}\, dx + \int_{s_1} f(z)\, dz + \int_{s_2} f(z)\, dz$$
$$= 2\pi i \text{Res}\left(\frac{e^{pz}}{1+e^z}, \pi i\right)$$

where s_1 and s_2 are the closing segments of the rectangle. The closing segments are the vertical lines $x = A$ and $x = -A, 0 \le y \le 2\pi$, oriented in the anti-clockwise direction.

(a) Show that the integrals along s_1 and s_2 tend to zero as $A \to \infty$, provided that $0 < p < 1$. Hence evaluate

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\,dx, \quad 0 < p < 1.$$

[3]

(b) Do the results in part (a) also hold when p is complex, where $0 < \operatorname{Re} p < 1$? Explain your result in details. **[2]**

(c) Hence, evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x}\,dx.$$

[3]

Hint: Take $p = \frac{1}{2}(1+i)$.

— *End* —