MATH304

Homework 1

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1. If we write $z = re^{i\theta}$ and $w = Re^{i\phi}$, where $0 \le r < R$, show that

$$\mathrm{Re}\left(\frac{w+z}{w-z}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}.$$

This is called the Poisson Kernel (see Section 7.1.1).

2. Show that the distance of the point c from the line

$$\overline{a}z + a\overline{z} = b, \quad b \text{ is real},$$

is given by $\dfrac{(\overline{a}c + a\overline{c} - b)}{2|a|}$.

Remark

Distance is considered positive if the point c and the origin are on the opposite sides of the straight line and negative if otherwise.

3. When $|\alpha| < 1$ and $|\beta| < 1$, show that

$$\left|\frac{\alpha - \beta}{1 - \overline{\alpha}\beta}\right| < 1.$$

4. Let $u^{1/2}$ denote the root of $z^2 = u$ which has positive imaginary part, where u is a non-real complex number. For any two non-real complex numbers z_1 and z_2 having the same modulus value, find the value of a real number p such that

$$z_1 - z_2 = ipz_1^{1/2}z_2^{1/2}.$$

5. Suppose a complex number expressed in polar form is

$$z = r(\cos\theta + i\sin\theta).$$

Recalling the identities:

$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta) \quad \text{and} \quad \cos^2\frac{\theta}{2} = \frac{1}{2}(1 + \cos\theta),$$

show that

$$z^{1/2} = \pm\sqrt{r}\left(\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\frac{1-\cos\theta}{2}}\right) \quad \text{for} \quad 0 \le \theta \le \pi.$$

Explain why the preceding formula is invalid for $-\pi < \theta < 0$. Find the corresponding correct formula for this interval.

6. Show that the necessary and sufficient condition for the existence of z satisfying the following equation

$$|z - \alpha| + |z + \alpha| = 2|\beta|$$

is given by $|\alpha| \le |\beta|$. Find the maximum and minimum value of $|z|$.

7. Which of the following sets are connected set? Which of the following sets are domains?
   (a) The set $A \cup B$ and set $A \cap B$, where A consists of the points given by $|z - i| < 1$ while B is the set of points $|z - 1| < 1$.
   (b) The set $C \cup D$, where C consists of the points for which $|z| \le 1$ while B is given by $\operatorname{Re} z \ge 1$.

8. What are the boundary points of the sets defined below?
   (a) $|z| > 0$, (b) $\frac{1}{3} < \frac{1}{|z - i|} \le \frac{1}{2}$.

9. Consider the set of points (x, y) consisting of the solutions of

$$y = 0 \quad \text{and} \quad \sin \frac{\pi}{x} = 0$$

lying in the domain $0 < |z| < 1$, $z = x + iy$. What is the limit point for this set? Prove your result by showing mathematically that every neighborhood of this point contains at least one member of the given set.

10. Let the two points $P(\xi, \eta, \zeta)$ and $P'(\xi', \eta', \zeta')$ be two points on the Riemann sphere that correspond to z and z' in the complex plane. Show that the necessary and sufficient condition that the line joining P and P' passes through the center of the Riemann sphere is given by

$$z\overline{z}' + 1 = 0.$$