MATH304 — Complex Analysis

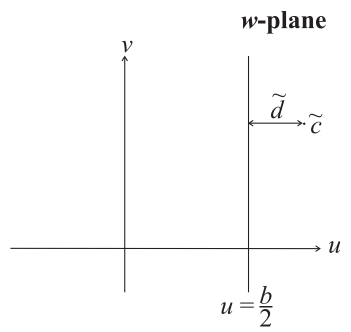
Solution to Homework 1

1.
$$\frac{w+z}{w-z} = \frac{Re^{i\phi}+re^{i\theta}}{Re^{i\phi}-re^{i\theta}} = \frac{Re^{i\phi}+re^{i\theta}}{Re^{i\phi}-re^{i\theta}} \cdot \frac{Re^{-i\phi}-re^{-i\theta}}{Re^{-i\phi}-re^{-i\theta}}$$
$$= \frac{R^2 - Rre^{i(\phi-\theta)} + Rre^{-i(\phi-\theta)} - r^2}{R^2 - Rre^{i(\phi-\theta)} - Rre^{-i(\phi-\theta)} + r^2}$$
$$= \frac{R^2 - r^2 - 2iRr\sin(\phi-\theta)}{R^2 - 2Rr\cos(\phi-\theta) + r^2}$$

so that

$$\mathrm{Re}\left(\frac{w+z}{w-z}\right) = \frac{R^2-r^2}{R^2-2Rr\cos(\phi-\theta)+r^2}.$$

2. Consider the mapping $w = \overline{a}z$. Note that this mapping is line-preserving, that is, a line is mapped to a line and the length of the image (line segment in the w-plane) is magnified by the factor $|a|$. The line $\overline{a}z + a\overline{z} = b$ becomes $2\mathrm{Re}\, w = b$ in the w-plane.



Let $\widetilde{c} = w(c) = \overline{a}c$, then

$$\widetilde{d} = \mathrm{Re}\,\widetilde{c} - \frac{b}{2} = \frac{\overline{a}c + a\overline{c}}{2} - \frac{b}{2}.$$

Hence, the distance d of the point c from the line $\overline{a}z + a\overline{z} = b$ is

$$d = \frac{\widetilde{d}}{|a|} = \frac{\overline{a}c + a\overline{c} - b}{2|a|}.$$

Alternative method

We express the given equation of line in normal form by dividing the equation throughout by $2|a|$. Write $a = |a|e^{i\theta}$ and $z = x+iy$, then the equation becomes $\dfrac{e^{-i\theta}z + e^{i\theta}\overline{z}}{2} = \dfrac{b}{2|a|}$ or $x\cos\theta + y\sin\theta = \dfrac{b}{2|a|}$. Let $c = \alpha + i\beta$, the distance from the point (α,β) to the straight line is $\alpha\cos\theta + \beta\sin\theta - \dfrac{b}{2|a|} = \dfrac{\overline{a}c + a\overline{c} - b}{2|a|}$.

3. Consider

$$1-\left|\frac{\alpha-\beta}{1-\overline{\alpha}\beta}\right|^2 = 1-\frac{(\alpha-\beta)(\overline{\alpha}-\overline{\beta})}{|1-\overline{\alpha}\beta|^2} = \frac{(1-\overline{\alpha}\beta)(1-\alpha\overline{\beta})-(\alpha-\beta)(\overline{\alpha}-\overline{\beta})}{|1-\overline{\alpha}\beta|^2}$$
$$= \frac{(1-|\alpha|^2)(1-|\beta|^2)}{|1-\overline{\alpha}\beta|^2} > 0 \quad \text{if} \quad |\alpha|<1 \quad \text{and} \quad |\beta|<1.$$

Hence, the result is obtained.

4. Since $|z_1| = |z_2|$, we write

$$z_1 = r(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r(\cos\theta_2 + i\sin\theta_2)$$

where $r = |z_1| = |z_2|$ and let θ_1 and θ_2 lie within $[0, 2\pi)$. We then have

$$z_1^{1/2} = \sqrt{r}\left(\cos\frac{\theta_1}{2} + i\sin\frac{\theta_1}{2}\right) \quad \text{and} \quad z_2^{1/2} = \sqrt{r}\left(\cos\frac{\theta_2}{2} + i\sin\frac{\theta_2}{2}\right),$$

where the imaginary parts of $z_1^{1/2}$ and $z_2^{1/2}$ are guaranteed to be positive. Now, we consider

$$z_1^{1/2}z_2^{1/2} = r\left(\cos\frac{\theta_1+\theta_2}{2} + i\sin\frac{\theta_1+\theta_2}{2}\right)$$

and

$$\begin{aligned} z_1 - z_2 &= r[(\cos\theta_1 - \cos\theta_2) + i(\sin\theta_1 - \sin\theta_2)] \\ &= 2r\left[-\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_1+\theta_2}{2} + i\cos\frac{\theta_1+\theta_2}{2}\sin\frac{\theta_1-\theta_2}{2}\right] \\ &= 2ir\sin\frac{\theta_1-\theta_2}{2}\left(\cos\frac{\theta_1+\theta_2}{2} + i\sin\frac{\theta_1+\theta_2}{2}\right) \\ &= 2i\sin\frac{\theta_1-\theta_2}{2}z_1^{1/2}z_2^{1/2}. \end{aligned}$$

If we set

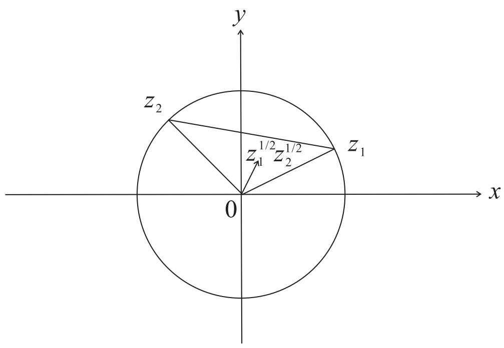
$$z_1 - z_2 = ipz_1^{1/2}z_2^{1/2},$$

then

$$p = 2\sin\frac{\theta_1-\theta_2}{2},$$

which is seen to be real.

Geometrical interpretation



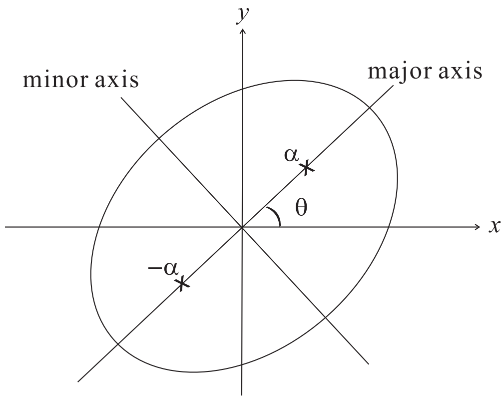
Note that $\operatorname{Arg}\left(z_1^{1/2}z_2^{1/2}\right) = \frac{\theta_1+\theta_2}{2}$. The vector represented by $z_1^{1/2}z_2^{1/2}$ lies along the direction perpendicular to the line segment joining z_1 and z_2, as deduced from the property of the isosceles triangle formed by $0, z_1$ and z_2. Multiplying $z_1^{1/2}z_2^{1/2}$ by i gives a vector perpendicular to $z_1^{1/2}z_2^{1/2}$, and such vector will be parallel to $z_1 - z_2$. Hence, $z_1 - z_2$ can be obtained by multiplying $iz^{1/2}z^{1/2}$ by a real multiple.

5. For $0 \le \theta \le \pi$, one root of $z^{1/2}$ has its real and imaginary parts both being positive and the other root has its real and imaginary part both being negative. Since $0 \le \frac{\theta}{2} \le \frac{\pi}{2}$, so $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$ assume the positive root $\sqrt{\frac{1+\cos\theta}{2}}$ and $\sqrt{\frac{1-\cos\theta}{2}}$, respectively. The two values of $z^{1/2}$ are
$$\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \quad \text{and} \quad -\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right),$$
which can be expressed as
$$\pm\sqrt{r}\left(\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\frac{1-\cos\theta}{2}}\right).$$
On the other hand, when $-\pi < \theta < 0$, this gives $-\frac{\pi}{2} < \frac{\theta}{2} < 0$ so that the real part and imaginary part of $z^{1/2}$ have opposite signs. In this case
$$z^{1/2} = \pm\sqrt{r}\left(\sqrt{\frac{1+\cos\theta}{2}} - i\sqrt{\frac{1-\cos\theta}{2}}\right).$$

6. Suppose there exists z such that
$$|z-\alpha| + |z+\alpha| = 2|\beta|, \tag{1}$$
then by the Triangle Inequality
$$|z-\alpha| + |z+\alpha| \ge |\alpha - z + z + \alpha| = 2|\alpha|.$$
Hence, we obtain $|\alpha| \le |\beta|$. On the other hand, suppose $|\alpha| \le |\beta|$, we take $z = \frac{|\beta|}{|\alpha|}\alpha$. Since $\frac{|\beta|}{|\alpha|} > 1$, we have
$$|z-\alpha| + |z+\alpha| = \left(\frac{|\beta|}{|\alpha|} - 1\right)|\alpha| + \left(\frac{|\beta|}{|\alpha|} + 1\right)|\alpha| = 2|\beta|.$$
Hence, there exists z such that eq (1) is satisfied.
It is well known that eq (1) represents an ellipse with foci at α and $-\alpha$. The major axis lies along the line joining the two foci. The two points on the ellipse that lie along the major axis

are $\pm\frac{|\beta|}{|\alpha|}\alpha$, and they are most distant from the center of the ellipse. Since the center of the ellipse is the origin, hence

$$\max|z| = \frac{|\beta}{|\alpha|}|\alpha| = |\beta|.$$



The points on the ellipse that are closest to the center of the ellipse lie on the minor axis. The distance from any one of these two points to either foci is $|\beta|$. Hence, the minimum value of $|z|$ is $\sqrt{|\beta|^2 - |\alpha|^2}$.

Without the knowledge that eq (i) represents the equation of an ellipse, one may find the maximum and minimum value of $|z|$ by showing that the maximum value is attained by a point lying along the line joining α and $-\alpha$ and the minimum value is attained by a point lying perpendicular to the line joining α and $-\alpha$.

Let $\alpha = |\alpha|e^{i\theta}$ and $z = re^{i\phi}$, eq (1) can be expressed as

$$\sqrt{r^2 - 2|\alpha|r\cos(\theta-\phi) + |\alpha|^2} + \sqrt{r^2 + 2|\alpha|r\cos(\theta-\phi) + |\alpha|^2} = 2|\beta|.$$

One can show by simple calculus that r attains its maximum value when $\phi = \theta$ or $\phi = \theta + \pi$ (may be expressed as $\theta - \pi$) and minimum value when $\phi = \theta \pm \frac{\pi}{2}$.

7. (a) connected set, domain. (b) connected set, not a domain.

8. (a) $z = 0$ is a boundary point, but it does not belong to the set.
   (b) Boundary points lie on the circle $|z - i| = 3$ and they are not in the set. Other boundary points lie on $|z - i| = 2$ and they belong to the set.

9. The point $z = 0$ is a limit point. For any deleted neighborhood of $z = 0$, say, $0 < |z| < \epsilon$, we can find an integer n such that $n > \frac{1}{\epsilon}$. The point $x = \frac{1}{n}, y = 0$ is a member of the given point set since $\sin\frac{\pi}{x} = 0$ and $y = 0$ are satisfied. The point is contained inside the deleted neighborhood since $\left|\frac{1}{n}\right| < \varepsilon$. Hence, any deleted neighborhood contains at least one point in the point set.

10. Suppose P and P' are the two end points of a diameter of the Riemann sphere, then $\xi' = -\xi, \eta' = -\eta$ and $\zeta' = 1 - \zeta$. Recall

$$z = \frac{\xi + i\eta}{1-\zeta} \quad \text{and} \quad z' = \frac{\xi' + i\eta'}{1-\zeta'},$$

then

$$\begin{aligned} z\overline{z}' + 1 &= \frac{\xi + i\eta}{1-\zeta}\frac{-\xi + i\eta}{\zeta} + 1 \\ &= \frac{-\xi^2 - \eta^2 - \zeta^2 + \zeta}{(1-\zeta)\zeta} = 0. \end{aligned}$$

On the other hand, suppose $z' = -\dfrac{1}{\overline{z}}$, then

$$\begin{aligned} \xi' &= \frac{1}{2}\frac{z' + \overline{z}'}{z'\overline{z}' + 1} = -\frac{1}{2}\frac{z + \overline{z}}{z\overline{z} + 1} = -\xi \\ \eta' &= \frac{1}{2i}\frac{z' - \overline{z}'}{z'\overline{z}' + 1} = -\frac{1}{2i}\frac{z - \overline{z}}{z\overline{z} + 1} = -\eta \\ \zeta' &= \frac{z'\overline{z}'}{z'\overline{z}' + 1} = \frac{1}{z\overline{z} + 1} = 1 - \frac{z\overline{z}}{z\overline{z} + 1} = 1 - \zeta. \end{aligned}$$