MATH304 — Complex Analysis

Solution to Homework 2

1. $f(z) = \dfrac{1}{z} = \dfrac{x - iy}{x^2 + y^2} \quad \Rightarrow \quad u = \dfrac{x}{x^2 + y^2}$ and $v = \dfrac{-y}{x^2 + y^2}$.

$$u = \alpha \iff \alpha(x^2 + y^2) = x \iff \alpha\left(x^2 - \frac{x}{\alpha} + \frac{1}{4\alpha^2}\right)^2 + \alpha y^2 = \frac{1}{4\alpha}$$
$$\iff \left(x - \frac{1}{2\alpha}\right)^2 + y^2 = \frac{1}{4\alpha^2}.$$

Hence, $u = \alpha$ and $\alpha \neq 0$ is a family of circles with radius $= \dfrac{1}{2\alpha}$ and centered at $\left(\dfrac{1}{2\alpha}, 0\right)$.

Similarly,

$$v = \beta \iff \beta\left(x^2 + \frac{y}{\beta}\right) + \beta y^2 = 0$$
$$\iff \beta\left(y^2 + \frac{y}{\beta} + \frac{1}{4\beta^2}\right) + \beta x^2 = \frac{1}{4\beta}$$
$$\iff x^2 + \left(y + \frac{1}{2\beta}\right)^2 = \frac{1}{4\beta^2}.$$

Hence, $v = \beta$ and $\beta \neq 0$ is a family of circles with radius $= \dfrac{1}{2\beta}$ and centered at $\left(0, \dfrac{-1}{2\beta}\right)$.

2. (a) $f(z) = \begin{cases} 0, & z = 0 \\ \dfrac{\operatorname{Re} z}{|z|}, & z \neq 0 \end{cases}.$

For $z \neq 0, f(z) = \dfrac{x}{\sqrt{x^2 + y^2}}$. Choose the path $y = x$ towards $(0, 0)$.

Then $\displaystyle\lim_{z\to 0} f(z) = \lim_{\substack{z\to 0 \\ (\text{along } y=x)}} f(z) = \lim_{\substack{z\to 0 \\ (\text{along } y=x)}} \frac{x}{\sqrt{2}|x|} = \frac{\operatorname{sgn}(x)}{\sqrt{2}} \neq 0$

where $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}.$

As $\lim_{z\to 0} f(z) \neq 0$, so $f(z)$ is not continuous at $(0, 0)$.

(b) $f(z) = \begin{cases} 0, & z = 0 \\ \dfrac{(\operatorname{Re} z)^2}{|z|}, & z \neq 0 \end{cases}.$

For $z \neq 0$, and write $z = re^{i\theta}, f(z) = u(r, \theta) = \dfrac{r^2 \cos^2\theta}{r}$. Hence, $\lim_{z\to 0} u(z) = \lim_{r\to 0} u(r, \theta) = 0$, so $\lim_{z\to 0} f(z) = f(0) = 0$. Hence, $f(z)$ is continuous at 0.

Remark

How to show $\displaystyle\lim_{z\to 0} \frac{(\operatorname{Re} z)^2}{|z|} = 0$ using $\varepsilon - \delta$ criterion? Write $z = re^{i\theta}$, and for any $\varepsilon > 0$, we take $\delta = \varepsilon$. When $|z| = r < \delta = \varepsilon$, we have

$$\left|\frac{(\operatorname{Re} z)^2}{|z|}\right| \leq |r\cos^2\theta| \leq r < \varepsilon.$$

3. (a)

$$\begin{aligned} z =& \ e^{-t}(2\sin t + i\cos t) \\ z' =& \text{ tangent vector } = -e^{-t}(2\sin t + i\cos t) + e^{-t}(2\cos t - i\sin t) \\ =& \ e^{-t}[2\cos t - 2\sin t - i(\sin t + \cos t)] \end{aligned}$$

$$\text{Unit tangent vector} = \frac{z'}{|z'|} = \frac{e^{-t}[2\cos t - 2\sin t - i(\sin t + \cos t)]}{e^{-t}\sqrt{4(\cos t - \sin t)^2 + (\sin t + \cos t)^2}}$$

$$\left.\frac{z'}{|z'|}\right|_{t=\pi/4} = \frac{-i\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)}{\left|\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right|} = -i.$$

(b) $z'(0) = 2 - i =$ velocity at $t = 0$

$|z'(0)| =$ speed at $t = 0 = \sqrt{5}$.

$z'\left(\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}}(-2 - i)$

Speed at $t = \frac{\pi}{2} = \left|z'\left(\frac{\pi}{2}\right)\right| = \sqrt{5}e^{-\frac{\pi}{2}}$.

Acceleration $= z'' = -z' + e^{-t}[-2\sin t - 2\cos t - i(\cos t - \sin t)]$

$z''(0) = -z'(0) + (-2 - i) = -(2 - i) - 2 - i = -4$.

Magnitude of acceleration at $t = 0$ is $|z''(0)| = 4$.

$$z''\left(\frac{\pi}{2}\right) = -z'\left(\frac{\pi}{2}\right) + e^{-\frac{\pi}{2}}(-2 + i) = e^{-\frac{\pi}{2}}(2 + i) + e^{-\frac{\pi}{2}}(-2 + i) = e^{-\frac{\pi}{2}}(2i).$$

Magnitude of acceleration at $t = \frac{\pi}{2} = \left|z''\left(\frac{\pi}{2}\right)\right| = 2e^{-\frac{\pi}{2}}$.

4. (a)

$$\begin{aligned} &f(z) = xy^2 + ix^2y, \quad z = x + iy \\ &u = xy^2, \quad v = x^2y \quad \text{so that } u_x = y^2, u_y = 2xy, v_x = 2xy \text{ and } v_y = x^2; \\ &u_x = v_y \quad \Rightarrow \quad x = \pm y \\ &u_y = -v_x \quad \Rightarrow \quad xy = 0. \end{aligned}$$

Since $(x, y) = (0, 0)$ is the only possible solution, so the C-R equations are satisfied at $(0, 0)$ only.

(b) Since u_x, u_y, v_x and v_y are continuous everywhere and the C-R equations hold only at $(0, 0)$, so $f(z)$ is differentiable only at $(0, 0)$.

(c) $f(z)$ is *nowhere* analytic since $f(z)$ is differentiable only at the single point $(0, 0)$.

5. Let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$. From $u = v^2$, we have

$$u_x = 2vv_x \quad \text{and} \quad u_y = 2vv_y. \tag{1}$$

Since f is analytic in $\mathcal{D}$, we have

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \tag{2}$$

Combining Eqs. (1) and (2), we obtain

$$2vv_x = v_y \quad \text{and} \quad 2vv_y = -v_x$$

so that

$$v_x^2 + v_y^2 = 0.$$

This implies $v_x = v_y = 0$ so that $v(x,y) = \text{constant}$. From the Cauchy-Riemann relations, we deduce that $u(x,y) = \text{constant}$ also. Hence, f is constant in $\mathcal{D}$.

6. (a) $u(x,y) = y^3 - 3x^2y$

$$\begin{aligned}
u_x &= -6xy, \qquad u_y = 3y^2 - 3x^2 \\
v_y &= u_x = -6xy \quad \Longrightarrow \quad v = -3xy^2 + g(x) \\
-u_y &= v_x = -3y^2 + g'(x) \quad \Longrightarrow \quad g'(x) = 3x^2 \quad \Longrightarrow \quad g(x) = x^3 + C \\
f(i) &= 1 + i \quad \Longrightarrow \quad 1 = v(0,1) = C
\end{aligned}$$

So, $f(z) = y^3 - 3x^2y + i(x^3 - 3xy^2 + 1) = iz^3 + i.$

(b) $u(x,y) = \dfrac{y}{x^2+y^2}$

$$\begin{aligned}
u_x &= \frac{-2xy}{x^2+y^2}, \qquad u_y = \frac{x^2+y^2-y\cdot 2y}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \\
v_y &= u_x = \frac{-2xy}{x^2+y^2} \quad \Longrightarrow \quad v = \frac{x}{x^2+y^2} + g(x) \\
-u_y &= v_x = \frac{x^2+y^2-x\cdot 2x}{x^2+y^2} + g' = \frac{y^2-x^2}{x^2+y^2} + g' \quad \Longrightarrow \quad g' = 0 \quad \Longrightarrow \quad g(x) = C \\
f(1) &= 0 \quad \Longrightarrow \quad 0 = v(1,0) = 1 + C
\end{aligned}$$

So, $f(z) = \dfrac{y}{x^2+y^2} + i\left(\dfrac{x}{x^2+y^2} - 1\right) = i\left(\dfrac{1}{z} - 1\right),$

(c) $u(x,y) = (x-y)(x^2+4xy+y^2)$

$$\begin{aligned}
u_x &= (x^2+4xy+y^2) + (x-y)(2x+4y) = 3x^2+6xy-3y^2 \\
u_y &= -(x^2+4xy+y^2) + (x-y)(4y+2y) = 3x^2-6xy-3y^2 \\
v_y &= u_x = 3x^2+6xy-3y^2 \quad \Longrightarrow \quad v = 3x^2y+3xy^2-y^3+g(x) \\
-u_y &= v_x = 6xy+3y^2+g' \quad \Longrightarrow \quad g' = -3x^2 \quad \Longrightarrow \quad g(x) = -x^3+C \\
v(x,y) &= -x^3+3x^2y+3xy^2-y^3+C = -(x+y)(x^2-4xy+y^2)+C
\end{aligned}$$

So, $f(z) = (x-y)(x^2+4xy+y^2) - i\left[(x+y)(x^2-4xy+y^2) - C\right] = (1-i)z^3 + iC, C$ is any real constant.

7. Two families of curves are orthogonal if one is a harmonic conjugate of the other.

(a) $\phi(x,y) = x^3y - xy^3 = \alpha$

$$\begin{aligned}
\phi_x &= 3x^2y - y^3, \qquad \phi_y = x^3 - 3xy^2 \\
\psi_y &= \phi_x = 3x^2y - y^3 \quad \Longrightarrow \quad \psi = \frac{3}{2}x^2y^2 - \frac{y^4}{4} + g(x) \\
-\phi_y &= \psi_x = 3xy^2 + g' \quad \Longrightarrow \quad g' = -x^3 \quad \Longrightarrow \quad g(x) = -\frac{x^4}{4}.
\end{aligned}$$

So, $\psi(x,y) = -\frac{1}{4}(x^4 - 6x^2y^2 + y^4) = \beta$

(b) $\phi(x,y) = 2e^{-x}\sin y + x^2 - y^2 = \alpha$

$$\begin{aligned}
\phi_x &= -2e^{-x}\sin y + 2x, \qquad \phi_y = 2e^{-x}\cos y - 2y \\
\psi_y = \phi_x &= -2e^{-x}\sin y + 2x \quad \Longrightarrow \quad \psi = 2e^{-x}\cos y + 2xy + g(x) \\
-\phi_y = \psi_x &= -2e^{-x}\cos y + 2y + g' \quad \Longrightarrow \quad g' = 0
\end{aligned}$$

So, $\psi(x,y) = 2e^{-x}\cos y + 2xy = \beta$

(c) $\phi(x,y) = \dfrac{(r^2+1)\cos\theta}{r} = \alpha$. Rewrite $\phi(x,y) = \dfrac{(1+r^2)r\cos\theta}{r^2} = \dfrac{x}{x^2+y^2} + x$

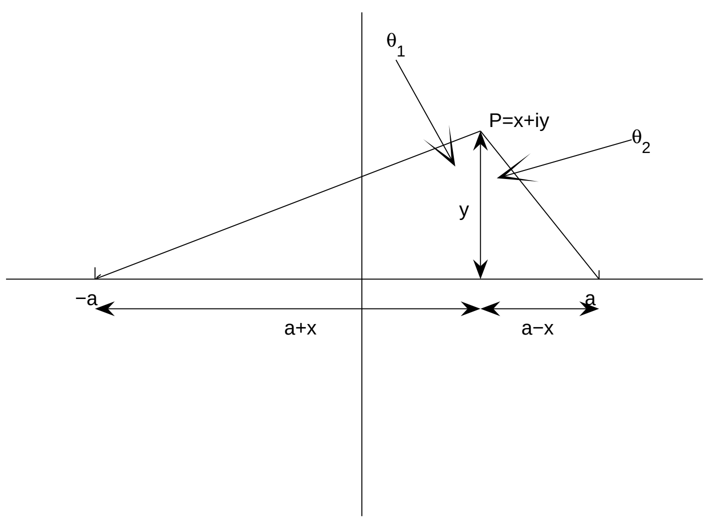
$$\begin{aligned}
\phi_x &= \frac{x^2+y^2 - x\cdot 2x}{(x^2+y^2)^2} + 1 = \frac{y^2-x^2}{(x^2+y^2)^2} + 1, \qquad \phi_y = \frac{-2xy}{(x^2+y^2)^2} \\
\psi_y = \phi_x &= \frac{y^2-x^2}{(x^2+y^2)^2} + 1 \quad \Longrightarrow \quad \psi = -\frac{y}{x^2+y^2} + y + g(x) \\
-\phi_y = \psi_x &= \frac{2xy}{(x^2+y^2)^2} + g' \quad \Longrightarrow \quad g' = 0
\end{aligned}$$

So, $\psi(x,y) = -\dfrac{y}{x^2+y^2} + y = \beta$. Similar to ϕ, we can rewrite ψ:

$$\psi = -\frac{r\sin\theta}{r^2} + r\sin\theta = \frac{1}{r}(-\sin\theta + r^2\sin\theta) = \frac{1}{r}(r^2-1)\sin\theta.$$

So the curve is given by $(r^2-1)\sin\theta = \beta r$.

8.



We can easily see that $\theta_1 = \tan^{-1}\dfrac{a+x}{y}$, $\qquad \theta_2 = \tan^{-1}\dfrac{a-x}{y}$

Let $s = \dfrac{a+x}{y}$, $t = \dfrac{a-x}{y}$. By inspection, we have $x = \dfrac{a(s-t)}{s+t}$ and $y = \dfrac{2a}{s+t}$.

Therefore, we can write our function as

$$\theta = \tan^{-1} s + \tan^{-1} t.$$

Take all corresponding partial derivatives:

$$\begin{aligned}
s_x &= \frac{1}{y}, & s_y &= -\frac{a+x}{y^2} = -\frac{s}{y}, & s_{xx} &= 0, & s_{yy} &= \frac{2(a+x)}{y^3} = \frac{2s}{y^2}, \\
t_x &= -\frac{1}{y}, & t_y &= -\frac{a-x}{y^2} = -\frac{t}{y}, & t_{xx} &= 0, & t_{yy} &= \frac{2(a-x)}{y^3} = \frac{2t}{y^2}, \\
\theta_s &= \frac{1}{1+s^2}, & \theta_{ss} &= \frac{-2s}{(1+s^2)^2}, & \theta_t &= \frac{1}{1+t^2}, & \theta_{tt} &= -\frac{2t}{(1+t^2)^2}.
\end{aligned}$$

Then by chain rule,

$$\begin{aligned}
\theta_x &= \theta_s s_x + \theta_t t_x = \frac{1}{1+s^2}\frac{1}{y} + \frac{1}{1+t^2}\left(-\frac{1}{y}\right) = \frac{1}{y}\left[\frac{1}{1+s^2} - \frac{1}{1+t^2}\right] \\
\theta_y &= \theta_s s_y + \theta_t t_y = \frac{1}{1+s^2}\left(-\frac{s}{y}\right) + \frac{1}{1+t^2}\left(-\frac{t}{y}\right) = -\frac{1}{y}\left[\frac{s}{1+s^2} + \frac{t}{1+t^2}\right] \\
\theta_{xx} &= \theta_{ss}(s_x)^2 + \theta_s s_{xx} + \theta_{tt}(t_x)^2 + \theta_t t_{xx} = \frac{-2s}{(1+s^2)^2}\frac{1}{y^2} + \frac{-2t}{(1+t^2)^2}\frac{1}{y^2} \\
&= -\frac{2}{y^2}\left[\frac{s}{(1+s^2)^2} + \frac{t}{(1+t^2)^2}\right] \\
\theta_{yy} &= \theta_{ss}(s_y)^2 + \theta_s s_{yy} + \theta_{tt}(t_y)^2 + \theta_t t_{yy} = \frac{-2s}{(1+s^2)^2}\frac{s^2}{y^2} + \frac{1}{1+s^2}\frac{2s}{y^2} + \frac{-2t}{(1+t^2)^2}\frac{t^2}{y^2} + \frac{1}{1+t^2}\frac{2t}{y^2} \\
&= \frac{2s}{(1+s^2)^2 y^2}\left[-s^2 + (1+s^2)\right] + \frac{2t}{(1+t^2)^2 y^2}\left[-t^2 + (1+t^2)\right] = \frac{2}{y^2}\left[\frac{s}{(1+s^2)^2} + \frac{t}{(1+t^2)^2}\right].
\end{aligned}$$

Therefore, we can see that θ is harmonic, since $\theta_{xx} + \theta_{yy} = 0$.

Now that θ is harmonic, its harmonic conjugate is simply $v(x,y) = \int -\theta_y \, dx + \int \theta_x \, dy$. To reexpress the partial derivatives in terms of x and y, we have

$$\begin{aligned}
-\theta_y &= \frac{1}{y}\left[\frac{s}{1+s^2} + \frac{t}{1+t^2}\right] = \frac{1}{y}\left[\frac{(a+x)/y}{1+(a+x)^2/y^2} + \frac{(a-x)/y}{1+(a-x)^2/y^2}\right] \\
&= \frac{a+x}{(a+x)^2+y^2} + \frac{a-x}{(a-x)^2+y^2} \\
\theta_x &= \frac{1}{y}\left[\frac{1}{1+s^2} - \frac{1}{1+t^2}\right] = \frac{1}{y}\left[\frac{1}{1+(a+x)^2/y^2} - \frac{1}{1+(a-x)^2/y^2}\right] \\
&= \frac{y}{(a+x)^2+y^2} - \frac{y}{(a-x)^2+y^2}
\end{aligned}$$

Set the initial point to be $(x_0, y_0) = (0, y_0)$ since it is the symmetric point. Then the integral of the above terms can be easily computed:

$$\begin{aligned}
\int_0^x -\theta_y(x, y_0)\, dx &= \frac{1}{2}\ln\left[(a+x)^2 + y_0^2\right] - \frac{1}{2}\ln\left[(a-x)^2 + y_0^2\right]\Big|_0^x = \frac{1}{2}\ln\left[\frac{(a+x)^2+y_0^2}{(a-x)^2+y_0^2}\right] \\
\int_{y_0}^y \theta_x(x,y)\, dy &= \frac{1}{2}\ln\left[(a+x)^2 + y^2\right] - \frac{1}{2}\ln\left[(a-x)^2 + y^2\right]\Big|_{y_0}^y = \frac{1}{2}\ln\left[\frac{(a+x)^2+y^2}{(a-x)^2+y^2}\right]\Big|_{y_0}^y
\end{aligned}$$

Therefore, $v(x,y) = \int_0^x -\theta_y(x,y_0)\,dx + \int_{y_0}^y \theta_x(x,y)\,dy = \frac{1}{2}\ln\left[\frac{(a+x)^2+y^2}{(a-x)^2+y^2}\right]$.

Alternative solution

The included angle $\theta(x,y) = \theta_1 + \theta_2 = \operatorname{Arg}(z-a) - \operatorname{Arg}(z+a)$. Note that $\operatorname{Arg}(z-a)$ is the imaginary part of $\log(z-a)$ so that a harmonic conjugate of $\operatorname{Arg}(z-a)$ is

$$-\operatorname{Re}\log(z-a) = -\frac{1}{2}\ln\left[(z-a)^2+y^2\right].$$

Similarly, a harmonic conjugate of $\operatorname{Arg}(z+a)$ is $-\frac{1}{2}\ln\left[(z+a)^2+y^2\right]$. Combining the results, we obtain

$$v(x,y) = \frac{1}{2}\ln\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right].$$

9. Write $f = U(x,y) + iV(x,y)$, where $U = u_y - v_x$ and $V = u_x + v_y$. Then we have

$$\begin{aligned} U_x &= u_{yx} - v_{xx}, \qquad & U_y &= u_{yy} - v_{xy} \\ V_y &= u_{xy} + v_{yy}, \qquad & V_x &= u_{xx} - v_{yx} \end{aligned}$$

Since u and v are harmonic, they both have continuous derivatives up to the second order. Therefore, the first partials of U and V are continuous. Furthermore, notice that the Cauchy-Riemann conditions are also satisfied:

$$U_x - V_y = -v_{xx} - v_{yy} = 0, \qquad U_y + V_x = u_{yy} + u_{xx} = 0$$

Therefore, f is analytic.

10. Suppose the isothermal lines are given by the family $x^2+y^2 = \alpha > 0$. If $T(x,y) = \alpha = x^2+y^2$, then $T_xx = 2, T_yy = 2$, so T cannot be harmonic. Instead, we let $T(x,y) = f(t)$, where $t = x^2+y^2$. Then

$$\begin{aligned} T_{xx} &= f''(t)\left(\frac{\partial t}{\partial x}\right)^2 + f'(t)\frac{\partial^2 t}{\partial x^2} = f''\cdot 4x^2 + f'\cdot 2 \\ T_{yy} &= f''(t)\left(\frac{\partial t}{\partial y}\right)^2 + f'(t)\frac{\partial^2 t}{\partial y^2} = f''\cdot 4y^2 + f'\cdot 2 \end{aligned}$$

So T is harmonic if and only if $f''(t)4t + 4f' = 0$, or $\frac{f''}{f'} = -\frac{1}{t}$. To solve this,

$$\begin{aligned} (\ln(f'))' &= \frac{f''}{f'} = -\frac{1}{t} \\ \ln(f') &= -\ln(t) + C_1 \\ f' &= At^{-1}, \qquad A = e^{C_1} \geq 0 \\ f(t) &= A\ln(t) + B, \qquad B \in \mathcal{R} \end{aligned}$$

Therefore, the temperature function is given by:

$$T(x,y) = A\ln(x^2+y^2) + B, \qquad \text{where } A \geq 0, B \in \mathcal{R}.$$

In terms of z, we have

$$T(z) = A\ln(r^2) + B = 2A\ln|z| + B$$

To find the family of flux lines, which are given by the harmonic conjugates of $T(x, y)$. We have

$$T_x = \frac{2Ax}{x^2+y^2} \qquad , \qquad T_y = \frac{2Ay}{x^2+y^2}$$

$$F_y = T_x = \frac{2Ax}{x^2+y^2} \Longrightarrow F(x,y) = \int \frac{2A/x}{1+(y/x)^2}dy = 2A\tan^{-1}\frac{y}{x} + g(x)$$

$$-T_y = F_x = \frac{2A}{1+(y/x)^2}(-\frac{y}{x^2}) + g'(x) = -\frac{2Ay}{x^2+y^2} + g'(x) \Longrightarrow g'(x) = 0.$$

So finally, we have $F(x, y) = 2A\tan^{-1}\frac{y}{x} + C = 2A \text{ Arg } z + C$, where $A \geq 0, C \in \mathcal{R}$.
So the flux lines are given by $2A \text{ Arg } z + C = \theta$. We can easily see that the isothermal lines are circles centered at origin, and the flux lines are straight lines through the origin. By inspection, they are orthogonal to each other.