MATH304 — Complex Analysis

Solution to Homework 3

1. (a) $\cosh^{-1} i = \log\left(i + (-2)^{\frac{1}{2}}\right) = \log((\pm\sqrt{2}+1)i) = \ln|1 \pm \sqrt{2}| + i\left(\frac{\pi}{2} + 2\pi k\right),\ k = 0, \pm 1, \pm 2, \cdots.$

   (b)
$$\begin{aligned}\sinh^{-1}(\log(-1)) &= \sinh^{-1}(i(\pi + 2\pi k)) \\ &= \log(mi + (1 + (mi)^2)^{\frac{1}{2}}) \quad [\text{write } m = (2k+1)\pi] \\ &= \log(mi \pm (1 - m^2)^{\frac{1}{2}}) = \log(mi \pm \sqrt{m^2 - 1}i) \\ &= \ln(m \pm \sqrt{m^2-1}) + i\left(\frac{\pi}{2} + 2\pi n\right), \quad k, n = 0, \pm 1, \pm 2, \cdots.\end{aligned}$$

   (c)
$$\begin{aligned}\tan^{-1}(2i) &= \frac{1}{2i}\log\left(\frac{1-2}{1+2}\right) = \frac{1}{2i}\log\left(-\frac{1}{3}\right) \\ &= \frac{1}{2i}\left[\ln\frac{1}{3} + i(\pi + 2\pi k)\right] \\ &= \frac{1}{2}(\pi + 2\pi k) + \frac{i}{2}\ln 3, \quad k = 0, \pm 1, \cdots.\end{aligned}$$

   (d) $\tanh^{-1} 2 = \frac{1}{2}\log\left(\frac{1+2}{1-2}\right) = \frac{1}{2}\log(-3) = \frac{1}{2}[\ln 3 + i(\pi + 2\pi k)], \quad k = 0, \pm 1, \cdots.$

2. $w = \sin^{-1} z \quad \Longleftrightarrow \quad z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}.$

   Let $z = x + iy$ and $w = \alpha + i\beta$, then

$$\begin{aligned}x + iy =& \frac{e^{i(\alpha+i\beta)} - e^{-i(\alpha+i\beta)}}{2i} = \frac{e^{-\beta}e^{i\alpha} - e^{\beta}e^{-i\alpha}}{2i} \\ \Leftrightarrow & -2y + i2x = e^{-\beta}(\cos\alpha + i\sin\alpha) - e^{\beta}(\cos\alpha - i\sin\alpha) \\ \Leftrightarrow & \begin{cases} -2y = e^{-\beta}\cos\alpha - e^{\beta}\cos\alpha & \Leftrightarrow \quad y = \sinh\beta\cos\alpha \\ 2x = e^{-\beta}\sin\alpha + e^{\beta}\sin\alpha & \Leftrightarrow \quad x = \cosh\beta\sin\alpha \end{cases}\end{aligned}$$

   Next, we consider

$$\begin{aligned}x^2 + y^2 + 2x + 1 &= \cosh^2\beta\sin^2\alpha + \sinh^2\beta\cos^2\alpha + 2\cosh\beta\sin\alpha + 1 \\ &= \cosh^2\beta\sin^2\alpha + (\cosh^2\beta - 1)\cos^2\alpha + 2\cosh\beta\sin\alpha + 1 \\ &= \cosh^2\beta - \cos^2\alpha + 2\cosh\beta\sin\alpha + \sin^2\alpha + \cos^2\alpha \\ &= (\cosh\beta + \sin\alpha)^2; \\ x^2 + y^2 - 2x + 1 &= \cosh^2\beta - 2\cosh\beta\sin\alpha + \sin^2\alpha = (\cosh\beta - \sin\alpha)^2.\end{aligned}$$

   Since $\cosh\beta \geq 1$ for all $\beta \in \mathbb{R}$, so
   $\sqrt{x^2+y^2+2x+1} - \sqrt{x^2+y^2-2x+1} = (\cosh\beta + \sin\alpha) - (\cosh\beta - \sin\alpha) = 2\sin\alpha.$
   Hence, $\alpha = \mathrm{Re}(\sin^{-1} z) = \sin^{-1}\left(\frac{1}{2}\left\{\sqrt{x^2+y^2+2x+1} - \sqrt{x^2+y^2-2x+1}\right\}\right).$

3. Let $w = \dfrac{ia-1}{ia+1} = |w|e^{i\arg w}$ and note that $|w| = 1$. We then have

$$\left(\frac{ia-1}{ia+1}\right)^{ib} = e^{-b(\arg w)} = e^{-b\arg\left(\frac{ia-1}{ia+1}\right)} = e^{-b\arg\left(\frac{a^2-1+2ai}{a^2+1}\right)} = e^{-b\tan^{-1}\left(\frac{2a}{a^2-1}\right)}.$$

Using the double angle formula for $\tan\alpha$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}, \text{ so that } \tan(\pi - 2\alpha) = \frac{2\tan\alpha}{\tan^2\alpha - 1}.$$

Let $a = \tan\alpha$ so that $\dfrac{\pi}{2} - \alpha = \cot^{-1} a$. We then have

$$\tan^{-1}\left(\frac{2a}{a^2-1}\right) = 2\left(\frac{\pi}{2} - \alpha\right) = 2\cot^{-1} a.$$

4. $w = \cosh z = \cosh x \cos y + i \sinh x \sin y$. For $x \geq 0$ and $0 \leq y \leq \pi/2$,

$$\frac{e^x + e^{-x}}{2} \geq 0, \quad \frac{e^x - e^{-x}}{2} \geq 0 \quad \text{and} \quad 0 \leq \cos y, \sin y \leq 1.$$

Since $\cosh x, \sinh x \to \infty$ as $x \to \infty$ so $0 \leq \cosh x \cos y < \infty$ and $0 \leq \sin y \sinh x < \infty$. The function $w = \cosh z$ maps the semi-infinite strip onto the first quadrant of the w-plane.

5. (a) $w^3 = z = r_0 e^{i\theta_0} \quad \Rightarrow \quad w = \sqrt[3]{r_0}\, e^{i\left(\frac{\theta_0 + 2\pi k}{3}\right)}, \quad k = 0, 1, 2.$

   If $z = 1$ (choose $\theta_0 = 0$), the 3 roots are $1, e^{i2\pi/3}$, and $e^{i4\pi/3}$.

   If $z = e^{i2\pi}$ (after traversing around the origin once), then $w = e^{i(\theta_0 + 2\pi)/3} = e^{i2\pi/3} = w_1$.

   (b) If z traverses around the origin twice, the image point is $w_2 = e^{i4\pi/3}$. Note that all image points lie in different branches of the triple valued function $w = z^{\frac{1}{3}}$. In general, z traverses around the origin n times, we have

$$w_n = e^{i\frac{2\pi n}{3}} \quad \text{(with period 3 such that } w_n = w_{n-3}, \forall n).$$

   If the closed circuit in the z-plane does not enclose the point $z = 0$, the value of $w = z^{1/3}$ remains unchanged.

6. From $z^c = e^{c\log z}$, we need to evaluate $\dfrac{2}{3} z^{2/3}/z$ at $-8i$. Using the principal branch, we have

$$\frac{2}{3}\frac{e^{\frac{2}{3}\mathrm{Log}\,(-8i)}}{-8i} = \frac{2}{3}\frac{e^{\frac{2}{3}\left(\ln 8 - i\frac{\pi}{2}\right)}}{-8i} = \frac{1}{3}e^{i\pi/6}.$$

7. With $r = |z|$ and $\theta = \arg z$, we have

$$z^{1/2} = \sqrt{r}e^{i\left(\frac{\theta}{2} + k\pi\right)}, \quad k = 0, 1.$$

The choice of the branch corresponding to $4^{1/2} = 2$ requires that $k = 0$. With $k = 0$,

$$\left[9\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right]^{1/2} = \sqrt{9}e^{i/2(-2\pi/3)} = 3\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right).$$