MATH304

Homework 4

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1. Evaluate $\int_c (z^2 + 3z)\ dz$ along the circle $|z| = 2$ from $(2, 0)$ to $(0, 2)$ in a counterclockwise direction.

2. Evaluate $\oint_C \overline{z}^2\ dz$ around the circles (a) $|z| = 1$, (b) $|z - 1| = 1$.

3. Let C be any simple closed curve bounding a region having area A. Prove that

$$A = \frac{1}{2} \oint_C x\ dy - y\ dx.$$

   Use the result to find the area bounded by the ellipse: $x = a\cos\theta, y = b\sin\theta, 0 \le \theta < 2\pi$.

4. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Without evaluating the integral, show that

$$\left| \int_C \frac{1}{z^2 - 1}\ dz \right| \le \frac{\pi}{3}.$$

5. Let C represent a semi-circle of radius R, with center at the origin, where $R > 1$, and consider the functions

$$f_1(z) = z^2 - 1, \qquad f_2(z) = \frac{1}{z^2 + 1}.$$

   (a) Using the triangle inequalities, show that when z assumes values on C

$$R^2 - 1 \le |f_1(z)| \le R^2 + 1, \quad \frac{1}{R^2 + 1} \le |f_2(z)| \le \frac{1}{R^2 - 1}.$$

   (b) Deduce that

$$\left| \int_C f_1(z)\ dz \right| \le \pi R(R^2 + 1), \quad \left| \int_C f_2(z)\ dz \right| \le \frac{\pi R}{R^2 - 1}.$$

   (c) Hence show that

$$\left| \int_C f_1(z) f_2(z) dz \right| \le \pi R \frac{R^2 + 1}{R^2 - 1}.$$

6. Let C be the circle: $|z| = r > 1$.
   Evaluate
$$\oint_C \frac{e^z}{(z^2+1)^2}\, dz.$$

7. By evaluating $\oint_C e^z\, dz$ around the circle $|z| = 1$, show that
$$\int_0^{2\pi} e^{\cos\theta} \cos(\theta + \sin\theta)\, d\theta = \int_0^{2\pi} e^{\cos\theta} \sin(\theta + \sin\theta)\, d\theta = 0.$$

8. Let
$$g(z) = \oint_{|\zeta|=2} \frac{2\zeta^2 - \zeta + 1}{\zeta - z}\, d\zeta.$$
   Compute (a) $g(1)$; (b) $g(z_0), |z_0| > 2$. Can we evaluate $g(2)$?

9. If $f(z)$ is an nth-degree polynomial with non-zero leading coefficient,
$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n,$$
   and C is a simple closed contour enclosing all the zeros of $f(z)$, show that
$$\frac{1}{2\pi i} \oint_C \frac{z f'(z)}{f(z)}\, dz = -\frac{a_1}{a_0}.$$

10. (a) Let $f(z)$ be analytic inside and on a simple closed curve C. Prove that if $f(z) \neq 0$ inside C, then $|f(z)|$ must assume its minimum value on C.
    (b) Give an example to show that if $f(z)$ is analytic inside and on a simple closed curve C and $f(z) = 0$ at some point inside C, then $|f(z)|$ need not assume its minimum value on C.