MATH304 — Complex Analysis

Solution to Homework 4

1. Consider $\int_C (z^2+3z)\,dz$ and $C = \left\{ z = 2e^{i\theta} : \quad 0 \le \theta \le \frac{\pi}{2} \right\}$

$$\begin{aligned}\int_C (z^2+3z)\,dz &= \int_0^{\frac{\pi}{2}} (4e^{2i\theta}+6e^{i\theta})i2e^{i\theta}\,d\theta \\ &= 8i\left(\frac{e^{3i\theta}}{3i}\bigg|_0^{\frac{\pi}{2}} \right) + 12i\left(\frac{e^{2i\theta}}{2i}\bigg|_0^{\frac{\pi}{2}} \right) \\ &= \frac{8}{3}\left(e^{i\frac{3\pi}{2}}-1\right) + 6(e^{i\pi}-1) \\ &= -\frac{8}{3}i - \frac{44}{3}.\end{aligned}$$

Alternative solution

Since z^2+3z is entire, so the integral is path independent. By finding the primitive function of the integrand, we have

$$\int_C (z^2+3z)\,dz = \frac{z^3}{3} + \frac{3z^2}{2}\bigg|_2^{2e^{\frac{i\pi}{2}}} = -\frac{8}{3}i - \frac{44}{3}.$$

2. (a) $\displaystyle\int_{|z|=1} \overline{z}^2\,dz = \int_0^{2\pi} (e^{-i\theta})^2 ie^{i\theta}\,d\theta = \int_0^{2\pi} ie^{-i\theta}\,d\theta = -e^{-i\theta}\Big|_0^{2\pi} = 0.$

(b) $$\begin{aligned}\int_{|z-1|=1} \overline{z}^2\,dz &= \int_0^{2\pi} (1+e^{-i\theta})^2 ie^{i\theta}\,d\theta = \int_0^{2\pi} (1+2e^{-i\theta}+e^{-i2\theta})ie^{i\theta}\,d\theta \\ &= \int_0^{2\pi} (ie^{i\theta}+2i+ie^{-i\theta})\,d\theta = e^{i\theta}+2i\theta-e^{-i\theta}\Big|_0^{2\pi} \\ &= 1+2\pi i-1-(1+0-1) = 2\pi i.\end{aligned}$$

3. Let $P(x,y) = -y$ and $Q(x,y) = x$. Clearly, P and Q have continuous first order partial derivatives. By Green's Theorem,

$$\begin{aligned}\int_C P(x,y)\,dx + Q(x,y)\,dy &= \iint_C (Q_x - P_y)\,dxdy \\ &= \iint_C 1+1\,dxdy \\ &= 2\ \text{Area}(\Omega).\end{aligned}$$

Hence, $A = \frac{1}{2}\oint_C x\,dy - y\,dx$.

For the given ellipse: $x = a\cos\theta, y = b\sin\theta, \quad 0 \le \theta < 2\pi$.

$$\begin{aligned}\text{Area of ellipse } &= \frac{1}{2}\int_C x\,dy - y\,dx \\ &= \frac{1}{2}\int_0^{2\pi} a\cos\theta(b\cos\theta\,d\theta) - \int_0^{2\pi} (y\sin\theta)(-a\sin\theta)\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} ab\,d\theta = \pi ab.\end{aligned}$$

4. $\left|\int_C \frac{dz}{z^2-1}\right| \leq \int_C \frac{|dz|}{|z^2-1|} \leq \int_C \frac{|dz|}{||z|^2-1|} = \int_C \frac{|dz|}{4-1} = \frac{\pi}{3}.$

5. (a) Since $R > 1$, by the triangle inequality, we have

$$R^2 - 1 = |R^2-1| \leq |z^2-1| = |f_1(z)| \leq |z|^2 + 1 = R^2 + 1$$
$$\frac{1}{R^2+1} \leq \left|\frac{1}{z^2+1}\right| = |f_2(z)| \leq \frac{1}{R^2-1}.$$

(b) $\left|\int_C f_1(z)\,dz\right| \leq \int_C |f_1(z)|\,|dz| = (R^2+1)\pi R;$

$\left|\int_C f_2(z)\,dz\right| \leq \int_C |f_2(z)\,|dz| = \frac{\pi R}{R^2-1}.$

(c) $\left|\int_C f_1(z)f_2(z)\,dz\right| \leq \int_C |f_1(z)f_2(z)|\,|dz| \leq \frac{R^2+1}{R^2-1}(\pi R).$

6. Let C_1 and C_2 be a small circle centered at i and $-i$, respectively, such that C_1 and C_2 do not overlap. Then

$$\oint_C \frac{e^z}{(z^2+1)}\,dz = \oint_{C_1} \frac{e^z}{(z^2+1)^2}\,dz + \oint_{C_2} \frac{e^z}{(z^2+1)^2}\,dz.$$

Note that

$$\oint_{C_1} \frac{e^z}{(z^2+1)^2}\,dz = \oint_{C_1} \frac{\frac{e^z}{(z+i)^2}}{(z-i)^2}\,dz$$
$$= \frac{2\pi i}{(2-1)!}\frac{d}{dz}\left[\frac{e^z}{(z+i)^2}\right]\Bigg|_{z=i} = \frac{1-i}{2}e^i\pi;$$
$$\oint_{C_2} \frac{e^z}{(z^2+1)^2}\,dz = \oint_{C_2} \frac{\frac{e^z}{(z-i)^2}}{(z+i)^2}\,dz$$
$$= \frac{2\pi i}{(2-1)!}\frac{d}{dz}\left[\frac{e^z}{(z-i)^2}\right]\Bigg|_{z=-i} = -\frac{(1+i)}{2}e^{-i}\pi.$$

Hence,

$$\oint_C \frac{e^z}{(z^2+1)^2}\,dz = \frac{\pi}{2}(1-i)(e^i - ie^{-i}) = \frac{\pi}{2}(1-i)^2(\cos 1 - \sin 1) = \sqrt{2}\pi \sin\left(1 - \frac{\pi}{4}\right) i.$$

7. Since e^z is entire, so $\oint_C e^z\,dz = 0$ for all simple closed curve.

On the other hand, we have

$$\oint_C e^z\,dz = \int_0^{2\pi} e^{(\cos\theta + i\sin\theta)} ie^{i\theta}\,d\theta = \int_0^{2\pi} e^{\cos\theta + i(\sin\theta+\theta)} i\,d\theta$$
$$= \int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta+\theta) + i\sin(\sin\theta+\theta)]i\,d\theta$$

$$\Longrightarrow \quad \int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta+\theta)\,d\theta = \int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta+\theta)\,d\theta = 0.$$

8. $g(z) = \oint_{|\xi|=2} \frac{2\xi^2-\xi+1}{\xi-z}\,d\xi.$

   (a) By the Cauchy integral formula, since $z=1$ lies inside $|\xi|\le 2$,

   $$g(1) = 2\pi i[2(1)^2-1+1] = 4\pi i.$$

   (b) For $|z_0|>2$, the integrand is analytic inside $|\xi|=2$. By the Cauchy Theorem, $g(z_0)=0$.

   (c) $g(2)$ does not exist since the integrand is not defined at 2. It can be shown that the principal value of the integral exists (see Topic 7).

9. Let $\widehat{f}(z) = z^n + b_1 z^{n-1} + \cdots + b_n$ and C is a simple closed curve enclosing all zeros of $\widehat{f}(z)$. Suppose $\widehat{f}(z)$ has a zero of order k_1 at $z=\beta_1$, we have

   $$\widehat{f}(z) = (z-\beta_1)^{k_1}Q(z),$$

   where $Q(z)$ is a polynomial of degree $n-k_1$.

   Using logarithmic differentiation $\left(\text{recall the fact that } \frac{d}{dz}\operatorname{Log} z = \frac{1}{z}\right)$, we obtain

   $$\frac{\widehat{f}'(z)}{\widehat{f}(z)} = \frac{k_1}{z-\beta_1} + \frac{Q'(z)}{Q(z)}$$

   so that

   $$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{z\widehat{f}'(z)}{\widehat{f}(z)}dz &= \frac{1}{2\pi i}\oint_C \frac{k_1 z}{z-\beta_1}\,dz + \frac{1}{2\pi i}\oint_C \frac{zQ'(z)}{Q(z)}\,dz \\ &= k_1\beta_1 + \frac{1}{2\pi i}\oint_C \frac{zQ'(z)}{Q(z)}\,dz.\end{aligned}$$

   Repeating the above argument, we get

   $$\frac{1}{2\pi i}\oint_C \frac{z\widehat{f}'(z)}{\widehat{f}(z)}\,dz = \sum_{i=1}^m k_i\beta_i = \text{ sum of roots (counting multiplicities).}$$

   Recall that if $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_0$, then the sum of all roots is $-\frac{a_1}{a_0}$. Therefore,

   $$\frac{1}{2\pi i}\oint_C \frac{zf'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_C \frac{z\widehat{f}'(z)}{\widehat{f}(z)}\,dz = -b_1 = -\frac{a_1}{a_0}.$$

10. (a) Since $f(z)$ is analytic inside and on C and since $f(z)\neq 0$ inside C, it follows that $1/f(z)$ is analytic inside C. By the maximum modulus theorem, it follows that $1/|f(z)|$ cannot assume this maximum value inside C and so $|f(z)|$ cannot assume its minimum value inside C. Given that $|f(z)|$ has a minimum, this minimum must be attained on C.

    (b) Let $f(z)=z$ for $|z|\leqq 1$, so that C is a circle with center at the origin and radius one. We have $f(z)=0$ at $z=0$. If $z=re^{i\theta}$, then $|f(z)|=r$ and it is clear that the minimum value of $|f(z)|$ does not occur on C but occurs inside C where $r=0$, i.e. at $z=0$.