MATH 304 — Complex Analysis

Solution to Homework 5

1. Consider

$$\left|\frac{n}{1+in}+i\right| = \left|\frac{1}{1+in}\right| = \frac{1}{\sqrt{1+n^2}} < \frac{1}{n}, \quad n \text{ is a positive integer.}$$

Hence,

$$\left|\frac{n}{1+in}+i\right| < \frac{1}{n} < \frac{1}{N} \quad \text{whenever } n > N.$$

For any given $\epsilon > 0$, choose N such that $N > \frac{1}{\epsilon}$. We then have

$$\left|\frac{n}{1+in}+i\right| < \epsilon \quad \text{whenever } n > N,$$

as required.

2. Write $\alpha_n = a_n + ib_n$, and note that

$$|\alpha_n| = \frac{a_n}{\cos(\text{Arg }\alpha_n)} \le \frac{a_n}{\cos\left(\frac{\pi}{2}-\delta\right)} = \frac{a_n}{\sin\delta}.$$

Hence, when $\sum_{n=1}^{\infty}|\alpha_n|$ diverges, $\sum_{n=1}^{\infty}a_n$ diverges and so does $\sum_{n=1}^{\infty}\alpha_n$.

3. For $|z| > 0$, we have

$$\begin{aligned}
S_n(z) &= \sum_{k=0}^{n}\frac{z^2}{(1+|z|^2)^k} = z^2\sum_{k=0}^{n}\frac{1}{(1+|z|^2)^k} \\
&= z^2\left[\frac{1-\frac{1}{(1+|z|^2)^{n+1}}}{1-\frac{1}{1+|z|^2}}\right] \\
&= \frac{z^2}{|z|^2}\left[1+|z|^2-\frac{1}{(1+|z|^2)^n}\right]
\end{aligned}$$

$$S_n(z) \to S(z) = \frac{z^2}{|z|^2}(1+|z|^2).$$

For $z = 0$, obviously both $S_n(0)$ and $S(0)$ equals 0. Therefore,

$$|S_n(z)-S(z)| = \begin{cases} \left|\frac{z^2}{|z|^2}\frac{1}{(1+|z|^2)^n}\right| = \frac{1}{(1+|z|^2)^n} & \text{if } |z| > 0 \\ 0 & \text{if } z = 0 \end{cases}.$$

Note that for any positive integer n, we have

$$\lim_{|z|\to 0, z\neq 0}\frac{1}{(1+|z|^2)^2} = 1.$$

Now, for each $n, 0 < \epsilon < 1$, there exists z with $|z| > 0$ such that

$$|S_n(z)-S(z)| > \epsilon.$$

Thus it is not uniformly convergent.

4. For all z inside $1.01 < |1-z|$, we have

$$\left|\frac{1}{(1-z)^n}\right| = \frac{1}{|1-z|^n} \le r^n$$

where $r = \dfrac{1}{1.01} < 1$. Since $\displaystyle\sum_{n=1}^{\infty} r^n$ is convergent, for $r < 1$, we have uniform convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(1-z)^n}$ by virtue of the Weierstrass M-test.

5. (a) By the ratio test, $\dfrac{1}{R} = \displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!} = 0$, so the region of convergence is the whole complex plane. Indeed, it is the Taylor expansion of e^z.

   (b) Consider the limit of the ratio of the moduli of successive terms

$$\lim_{n\to\infty}\left|\frac{(z+2)^n}{(n+2)^3 4^{n+1}}\Big/\frac{(z+2)^{n-1}}{(n+1)^3 4^n}\right| = \lim_{n\to\infty}\left|\frac{(z+2)}{4}\frac{(n+1)^3}{(n+2)^3}\right| = \frac{|z+2|}{4};$$

   the series converges within the circle $\dfrac{|z+2|}{4} < 1$ i.e. $|z+2| < 4$.

   (c) We consider the evaluation of

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\frac{\left|\sin(n+1)\frac{\pi}{4}\right|}{2^{\frac{n+1}{2}}}}.$$

   For the subsequence where $n+1 = 2k$, we have

$$\lim_{k\to\infty}\sqrt[2k-1]{\frac{\left|\sin\frac{k\pi}{2}\right|}{2^{\frac{2k}{2}}}} = \frac{1}{\sqrt{2}};$$

   otherwise

$$\sqrt[n]{\frac{\left|\sin(n+1)\frac{\pi}{4}\right|}{2^{\frac{n+1}{2}}}} \le \frac{1}{2^{\frac{n+1}{2n}}} \le \frac{1}{\sqrt{2}}.$$

   Hence,

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\frac{\left|\sin(n+1)\frac{\pi}{4}\right|}{2^{\frac{n+1}{2}}}} = \frac{1}{\sqrt{2}},$$

   so the series converges for $|z-1| < \sqrt{2}$. This is not surprising since the distance from $z = 1$ to the nearest singularity of $\dfrac{1}{1+z^2}$ is $\sqrt{2}$.

6. The primitive function of $\dfrac{1}{1-z}$ is $\text{Log}\,(1-z)$ and $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $|z| < 1$. For $z \in D = \{z : |z| < 1\}$, we have

$$-\text{Log}\,(1-z) = \int_C \frac{d\xi}{1-\xi} = \int_C \sum_{n=0}^{\infty} \xi^n \, d\xi = \sum_{n=0}^{\infty} \int_C \xi^n \, d\zeta = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1},$$

where C is any simple curve joining 0 and z and lying completely inside D. Note that termwise integration is valid in D. Also, z^n is entire so the integral $\int_C \xi^n \, d\xi$ is path independent and hence $\int_C \xi^n \, d\xi = \dfrac{z^{n+1}}{n+1}$.

We then have

$$-\text{Log}\,(1-z_0) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} \sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{z_0^n}{n}, \quad \forall z_0 \neq 1, z_0 \text{ lies on } |z| = 1.$$

Hence, the series expansion $-\text{Log}\,(1-z) = \sum_{n=1}^{\infty} \dfrac{z^n}{n}$ is valid for all z lying on $|z| = 1$ except at the point $z = 1$.

For the series $\sum_{n=1}^{\infty} \dfrac{e^{in\theta}}{n}$, we observe that $\sum_{n=1}^{\infty} \left| \dfrac{e^{in\theta}}{n} \right| = \sum_{n=1}^{\infty} \dfrac{1}{n}$. The last series is well known to be divergent. Therefore, $\sum_{n=1}^{\infty} \dfrac{e^{in\theta}}{n}, \theta \neq 0$, is only conditionally convergent.

(a) $-\text{Log}\,(1 - e^{i\theta}) = -[\ln|1-e^{i\theta}| + i\text{Arg}\,(1-e^{i\theta})]$

$|1-e^{i\theta}|^2 = (1-\cos\theta)^2 + \sin^2\theta = 2 - 2\cos\theta = 4\sin^2\dfrac{\theta}{2}$

so $|1-e^{i\theta}| = 2\sin\dfrac{\theta}{2}$ if $0 < \theta < 2\pi$. Therefore,

$$\text{Re}\left(\sum_{n=1}^{n} \frac{e^{in\theta}}{n}\right) = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\ln\left(2\sin\frac{\theta}{2}\right), \quad 0 < \theta < 2\pi.$$

(b) Next, we consider the imaginary part of $-\text{Log}\,(1-e^{i\theta})$. $\text{Arg}\,(1-e^{i\theta}) = \text{Arg}\,(1 - \cos\theta + i(-\sin\theta)) = \tan^{-1}\left(\dfrac{-\sin\theta}{1-\cos\theta}\right)$,

provided that $-\pi < \tan^{-1}\left(\dfrac{-\sin\theta}{1-\cos\theta}\right) \leq \pi$. Note that

$$\frac{-\sin\theta}{1-\cos\theta} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2}.$$

Hence,

$$\text{Im}\left(\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}\right) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = -\tan^{-1}\left(-\cot\frac{\theta}{2}\right) = \frac{\pi-\theta}{2}, \quad 0 < \theta < 2\pi.$$

Note that $0 < \theta < 2\pi$ implies $-\pi < \tan^{-1}\left(-\cot\frac{\theta}{2}\right) \leq \pi$.

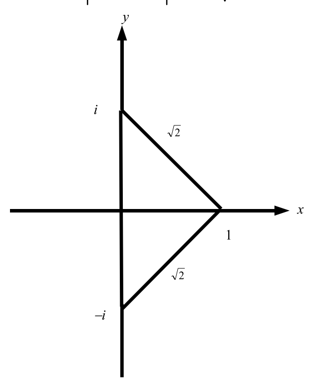
7. (a) $\cos z = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(z-\pi/2)^{2n+1}}{(2n+1)!}, \quad |z-\pi/2| < \infty;$

(b) $\frac{1}{1+z} = \frac{1}{2\left(1+\frac{z+1}{2}\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2} \left(\frac{z-1}{2}\right)^n, \quad |z-1| < 2.$

8. (a) Consider

$$\begin{aligned} & e^x \cos x = \text{Re } e^z = \text{Re } e^{(1+i)x}, \quad z = (1+i)x \\ = & \text{ Re } \sum_{n=0}^{\infty} \frac{(1+i)^n}{n!} = \sum_{n=0}^{\infty} 2^{n/2} \cos \frac{n\pi}{4} \frac{x^n}{n!}. \end{aligned}$$

(b) The function is analytic inside $|z-1| < \sqrt{2}$.



$$\begin{aligned} \frac{1}{1+z^2} &= \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right] \\ &= \frac{1}{2i}\left[\frac{1}{1-i}\frac{1}{1+\frac{z-1}{1-i}} - \frac{1}{i+1}\frac{1}{1+\frac{z-1}{i+1}}\right] \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2i}\left[\frac{1}{(1-i)^{n+1}} - \frac{1}{(1+i)^{n+1}}\right](z-1)^n. \end{aligned}$$

Since $1-i = \sqrt{2}e^{-i\pi/4}$ and $1+i = \sqrt{2}e^{i\pi/4}$, we then have

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-1)^n \frac{\sin\left((n+1)\frac{\pi}{4}\right)}{2^{(n+1)/2}}(z-1)^n.$$

9. (a) Since $\frac{1}{z^2+4}$ has singularities at $z = \pm 2i$ and $\sin z$ is entire, so the Taylor expansion at $z = 0$ has a radius of convergence equals 2, and the circle of convergence is $|z| < 2$.

(b) The function has singularities at $z = 1$ and $z = 4$. The distance from $z = 2$ to the nearest singularity is 1, so the radius of convergence is 1. The region of convergence is $|z-2| < 1$.

(c) The function has singularities at $z = 0$ and $z = 1$. Note that $z = 4i$ is closer to $z = 0$. Hence, the region of convergence is $|z-4i| < 4$.

10. First, we write

$$\frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z},$$

and observe

$$\frac{1}{z-1} = \begin{cases} -\sum_{n=0}^{\infty} z^n, & |z| < 1 \\ \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}, & |z| > 1 \end{cases}$$

$$\frac{2}{2-z} = \begin{cases} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n, & |z| < 2 \\ -\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^{n+1}, & |z| > 2 \end{cases}$$

(a) For $|z| < 1$, $\frac{z}{(z-1)(2-z)} = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2^n} - 1\right) z^n.$

(b) For $1 < |z| < 2$, $\frac{z}{(z-1)(2-z)} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n$

(c) For $|z| > 2$, $\frac{z}{(z-1)(2-z)} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{1-2^{n+1}}{z^{n+1}}$

(d) For $|z-1| > 1$,

$$\frac{2}{2-z} = \frac{2}{1-(z-1)} = \frac{-2}{(z-1)\left(1-\frac{1}{z-1}\right)} = \frac{-2}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n$$

$$\frac{z}{(z-1)(2-z)} = \frac{1}{z-1} - \frac{2}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n = \frac{1}{z-1} - 2\sum_{n=1}^{\infty}\frac{1}{(z-1)^{n+1}}.$$

11. (i) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \Rightarrow \quad \frac{e^z - 1}{z} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}, \quad z \neq 0.$

Define $f(0) = 1$ with $f(z) = \frac{e^z - 1}{z}$. The singularity of $f(z)$ at $z = 0$ is thus removed. Hence, $z = 0$ is removable singularity of $f(z)$.

(ii) The roots of $e^z - 1$ are $z = 2k\pi i$, where k is any integer. The roots of $e^{2z} - 1$ are $z = k\pi i$, where k is any integer.

(a) For $z = (2\ell + 1)\pi i$, ℓ is any integer, it is a simple root of $e^{2z} - 1$ but not a root of $e^z - 1$. In this case

$$\lim_{z \to (2\ell+1)\pi i} [z - (2\ell + 1)\pi i]\frac{e^z - 1}{e^{2z} - 1} = \frac{e^{(2\ell+1)\pi i} - 1}{2e^{(4\ell+2)\pi i}} = -1.$$

Hence, $z = (2\ell + 1)\pi i$ is a simple pole of $\frac{e^z - 1}{e^{2z} - 1}$.

(b) For $z = 2\ell\pi i$, ℓ is any integer, it is a simple pole of both $e^z - 1$ and $e^{2z} - 1$. Consider

$$\lim_{z \to 2\ell\pi i} \frac{e^z - 1}{e^{2z} - 1} = \frac{1}{2}\frac{e^z}{e^{2z}}\bigg|_{z=2\ell\pi i} = \frac{1}{2}.$$

Hence, it is a removable singularity.